Convergence theorems for families of monotone nonexpansive mappings

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Abstract

In this paper, we show nonlinear mean convergence theorems for monotone nonexpansive semigroups in uniformly convex Banach spaces endowed with a partial order.

1 Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of T, i.e.,

$$F(T) = \{z \in C : Tz = z\}.$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. The fixed point theory for such mappings is rich and varied. It finds many applications in nonlinear functional analysis. The existence of fixed points for nonexpansive mappings in Banach and metric spaces has been investigated since the early 1960s (For example, see [7, 8, 9, 11, 14]). Among other things, in 1975, Baillon [5] proved the following first nonlinear mean convergence theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$,

$$\{S_n x\} = \left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\}$$

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converges weakly to a fixed point of T (see also [21]).

In recent years, a new direction has been very active essentially after the publication of Ran and Reurings results [18]. They proved an analogue of the classical Banach contraction principle [6] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [22, 23, 17, 13]). Bin Dehaish and Khamsi [12] proved a weak convergence theorem of Mann's type [16] for monotone nonexpansive mappings in Banach spaces endowed with a partial order (see also [16, 19]). Shukla and Wiśnicki [20] obtained a nonlinear mean convergence theorem for monotone nonexpansive mappings in such Banach spaces.

In this paper, we show nonlinear mean convergence theorems for monotone nonexpansive semigroups in uniformly convex Banach spaces endowed with a partial order.

2 Preliminaries and notations

Throughout this paper, we assume that E is a real Banach space with norm $||\cdot||$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x+z \preceq y+z,$$

$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x,y,z \in E$ and $\lambda \geq 0$. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x,\rightarrow] = \{z \in E : x \preceq z\}$ and $[\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x,\rightarrow]$ and $[\leftarrow,y]$ are closed. Recall that an order interval is any of the subsets

$$[a,\rightarrow] = \{x \in X; a \preceq x\} \quad \text{or} \quad [\leftarrow,a] = \{x \in X; x \preceq a\}.$$

for any $a \in E$. As a direct consequence of this, the subset

$$[a,b] = \{x \in X; a \preceq x \preceq b\} = [a,\rightarrow] \cap [\leftarrow,b]$$

is also closed and convex for each $a,b \in E$.

Let E be a real Banach space with norm $||\cdot||$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E. Let C be a nonempty subset of E. A mapping $T : C \to C$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x-y||$$

for all $x,y \in C$. A mapping $T : C \to C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x,y \in C$ such that $x \preceq y$. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or $\text{w-}\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|\,(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \;:\; \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Let $S_E = \{x \in E : \|x\| = 1\}$ be a unit sphere in a Banach space E. Let C be a bounded closed convex subset of a Banach space E. A family $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$of mappings of C into itself satisfying is called a one-parameter monotone nonexpansive semigroup on C; if it satisfies the following conditions:

1. For each $t \in \mathbb{R}^+$, $T(t)$ is monotone and nonexpansive;
2. $T(0) = I$;
3. $T(t+s) = T(t)T(s)$ for all t;
4. $s \in \mathbb{R}^+$ and $T(t)x$ is continuous in $t \in \mathbb{R}^+$ for each $x \in C$.

3 Nonlinear mean convergence theorems for semigroups

In this section, we study approximate fixed point sequences and monotone sequences. And we show nonlinear mean convergence theorems for one-parameter monotone nonexpansive semigroups. Let C be a nonempty subset of E and let T be a mapping of C into E. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping T of C into itself if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(see also [15, 21]). Let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter monotone nonexpansive semigroup on C. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* for $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$if

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$$

for each $t \in \mathbb{R}^+$ (see also [15, 21]). A sequence $\{x_n\}$ in E is said to be *monotone* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [12]).

The following lemma was obtained by the author and Takahashi [4] (see also [2].

Lemma 3.1 ([4]). *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E. Let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$be a monotone nonexpansive semigroup on C. Then, for each $s \in S$,*

$$\lim_{t\to\infty} \sup_{x\in C} \sup_{h\in S} \left\| \frac{1}{t} \int_0^t T(t+h)x - T(s) \left(\frac{1}{t} \int_0^t T(t+h)x \right) \right\| = 0$$

We can prove the following result which is crucial in this paper.

Theorem 3.2 ([2]). *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E. Let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$be a monotone nonexpansive semigroup on C such thta $F(\mathscr{S}) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C which is a monotone, and approximate fixed point sequence of $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$, i.e.,*

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$$

for each $t \in S$. Then, then the sequence $\{x_n\}$ converges weakly to a point of $F(\mathscr{S})$.

We can prove nonlinear mean convergence theorems for monotone nonexpansive semigroups.

Theorem 3.3 ([2]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter monotone nonexpansive semigroup on C such that $F(\mathscr{S}) \neq \emptyset$. Assume $y \preceq T(t)y$ for $y \in C$, $t \in S$. Let $\{t_n\} \subset \mathbb{R}^+$ be a sequence in $\mathbb{R}^+$ such that $t_n \leq t_{n+1}$ for each $n \in \mathbb{N}$, and $t_n \to \infty$. Then, $\left\{ \frac{1}{t_n} \int_0^{t_n} T(t+h)xdt \right\}$ converges weakly to a point of $F(\mathscr{S})$ uniformly in $h \in \mathbb{R}^+$.*

Theorem 3.4 ([2]). *Let C be a nonempty compact convex subset of an ordered strictly convex Banach space E and let $\mathscr{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter monotone nonexpansive semigroup on C. Assume $y \preceq T(t)y$ for $y \in C$, $t \in S$. Let $\{t_n\} \subset \mathbb{R}^+$ be a sequence in $\mathbb{R}^+$ such that $t_n \leq t_{n+1}$ for each $n \in \mathbb{N}$, and $t_n \to \infty$. Then, $\left\{ \frac{1}{t_n} \int_0^{t_n} T(t+h)x dt \right\}$ converges strongly to a point of $F(\mathscr{S})$ uniformly in $h \in \mathbb{R}^+$.*

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