Improvement of eigen vector approximation method in DC programming

新潟大学理学部　山田修司
Syuuji YAMADA
Faculty of Science, Niigata University

Abstract

In this paper,we propose a global optimization algorithm based on a procedure for listing KKT points to solve a quadratic canonical dc programming problem (QDC) whose feasible set is expressed as the area excluded the interior of a convex set from another convex set. We can obtain an approximate solution of (QDC) by combining our algorithm with a parametric optimization method and branch-and-bound procedure.

1 Introduction

In this paper, we propose a procedure for listing KKT (Karush-Kuhn-Tucker) points of a quadratic canonical dc programming problem (QDC) whose feasible set is expressed as the area excluded the interior of a convex set from another convex set. It is known that many global optimization problems can be transformed into such a problem (see, e.g., [2]). Iterative solution methods for solving (QDC) have been proposed by many other researchers. Since it is difficult to solve (QDC), we transform (QDC) into a parametric quadratic programming problem. In order to solve such a quadratic programming problem for each parameter, we introduce an algorithm for listing KKT points. Moreover, we propose an global optimization algorithm for (QDC) by incorporating our KKT listing algorithm into a parametric optimization method and a branch-and-bound procedure.

Throughout this paper, we use the following notation: $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of all real numbers and an n-dimensional Euclidean space. The origin of $\mathbb{R}^n$ is denoted by $\mathbf{0}_n$. Given a vector $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a}^\top$ denotes the transposed vector of $\boldsymbol{a}$. For given real numbers α and β ($\alpha < \beta$), we set $[\alpha, \beta] := \{x \in \mathbb{R} : \alpha \leq x \leq \beta\}$, $]\alpha, \beta[:= \{x \in \mathbb{R} : \alpha < x < \beta\}$, $]\alpha, \beta] := \{x \in \mathbb{R} : \alpha < x \leq \beta\}$ and $[\alpha, \beta[:= \{x \in \mathbb{R} : \alpha \leq x < \beta\}$. The sets of all nonnegative real numbers and all nonnegative vectors are denoted by $\mathbb{R}_+$ and $\mathbb{R}^n_+$ respectively, that is, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}^n_+ := \{\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n : x_i \geq 0 \ i = 1, \ldots, n\}$. Given a vector $\boldsymbol{a} \in \mathbb{R}^n$, $\|\boldsymbol{a}\|$ denotes the Euclidean norm, that is, $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}^\top \boldsymbol{a}}$. Given a vector $\boldsymbol{a} \in \mathbb{R}^n$ and a positive real number $r > 0$, $B^n_<(\boldsymbol{a}, r) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| < r\}$ and $B^n_\leq(\boldsymbol{a}, r) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| \leq r\}$. Given a subset $X \subset \mathbb{R}^n$, $\dim X$ denotes the dimension of X. For a subset $X \subset \mathbb{R}^n$, $\operatorname{int} X$, $\operatorname{ri} X$ $\operatorname{cl} X$, $\operatorname{bd} X$ and $\operatorname{co} X$ denote the interior, the relative interior, the closure, the boundary and the convex hull of X, respectively. For a subset $X \subset \mathbb{R}^n$, $\operatorname{diam} X$ denotes the diameter of X, that is, $\operatorname{diam} X := \max_{x', x'' \in X} \|\boldsymbol{x}' - \boldsymbol{x}''\|$. The

$n \times n$ unit matrix is denoted by E_n. Given real numbers $a_1, \dots, a_n$, diag $\{a_1, \dots, a_n\}$ denotes the $n \times n$ diagonal matrix whose diagonal elements are $a_1, \dots, a_n$. For a given differentiable function $f : \mathbb{R} \to \mathbb{R}$, $\frac{d}{dx}f(\bar{x})$ and $\frac{d^2}{dx^2}f(\bar{x})$ denote the differential and the second order differential of f at $\bar{x} \in \mathbb{R}$, respectively. Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\partial f(\boldsymbol{x})$ denotes the subdifferential of f at $\boldsymbol{x}$, that is, $\partial f(\boldsymbol{x}) := \{\boldsymbol{a} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{a}^\top(\boldsymbol{y} - \boldsymbol{x}),\ \boldsymbol{y} \in \mathbb{R}^n\}$. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(\boldsymbol{x})$ denotes the gradient vector of f at $\boldsymbol{x} \in \mathbb{R}^n$.

2 A quadratic canonical dc programming problem

Let us consider the following quadratic canonical dc programming problem:

$$\text{(QDC)} \begin{cases} \text{minimize} & \boldsymbol{w}^\top \boldsymbol{x} \\ \text{subject to} & g_i(\boldsymbol{x}) := \boldsymbol{x}^\top A_i \boldsymbol{x} - (\boldsymbol{b}^i)^\top \boldsymbol{x} - c_i \leq 0,\ i = 1, \dots, m, \\ & h(\boldsymbol{x}) := \boldsymbol{x}^\top \boldsymbol{x} - r^2 \geq 0,\ \boldsymbol{x} \in \mathbb{R}^n, \end{cases}$$

where $A_i \in \mathbb{R}^{n \times n}$ $(i = 1, \dots, m)$ are real positive definite symmetric matrices, $\boldsymbol{b}^1, \dots, \boldsymbol{b}^m, \boldsymbol{w} \in \mathbb{R}^n$ $(\|\boldsymbol{w}\| = 1)$ and $c_1, \dots, c_m, r$ are real values $(r > 0)$. Let $G := \{\boldsymbol{x} \in \mathbb{R}^n : g_i(\boldsymbol{x}) \leq 0,\ i = 1, \dots, m\}$ and $H := \{\boldsymbol{x} \in \mathbb{R}^n : h(\boldsymbol{x}) \leq 0\}$. From the definition of A_i $(i = 1, \dots, m)$, g_i $(i = 1, \dots, m)$ are strictly convex functions. Hence, G and H are compact convex sets. Then, $G \backslash \operatorname{int} H$ denotes the feasible set of (QDC). It is well known that quadratic dc programming problems can be transformed into the (QDC).

For (QDC), we suppose the following statements.

(A1) The feasible set of (QDC) is nonempty, that is, $G \backslash \operatorname{int} H \neq \emptyset$.

(A2) The reverse convex constraint of (QDC) is essential, that is, $\arg\min\{\boldsymbol{w}^\top \boldsymbol{x} : \boldsymbol{x} \in G\} \subset \operatorname{int} H$.

(A3) $n \geq 2$.

From assumption (A2), (QDC) has globally optimal solutions. We notice that (QDC) is a convex programming problem if the reverse convex constraint of (QDC) is not essential. Moreover, by assumption (A2), we note that

$$-r \leq \alpha_0 := \min\{\boldsymbol{w}^\top \boldsymbol{x} : \boldsymbol{x} \in G\} < \min(\text{QDC}) \leq r$$

because $\|\boldsymbol{w}\| = 1$, where min(QDC) denotes the optimal value of (QDC). From the following proposition, we note that all globally optimal solution of (QDC) are contained in the intersection of the boundaries of G and H under assumption (A3).

Proposition 2.1 (See Proposition 2.1 in [3]) *Assume that* $n \geq 2$ *and assumption* (A1) *holds. Then, all locally optimal solutions of* (QDC) *are contained in* $(\operatorname{bd} G) \cap (\operatorname{bd} H)$.

3 Optimality conditions

In this section, we introduce optimality conditions for (QDC).

Now, we consider the following parametric programming problem for each $\alpha \in [\alpha_0, r]$, because it is hard to solve (QDC) directly.

$$\begin{cases} \text{minimize} & g(\boldsymbol{x}) \\ \text{subject to} & h(\boldsymbol{x}) = 0,\ \boldsymbol{w}^\top \boldsymbol{x} = \alpha. \end{cases} \tag{1}$$

From the definition of α_0 and assumptions (A1) and (A2), we note that the feasible set of problem (1) is nonempty for each $\alpha \in [\alpha_0, r]$. Let $D \in \mathbb{R}^{n\times(n-1)}$ be a matrix satisfying the followings.

- $D = (\boldsymbol{d}^1, \ldots, \boldsymbol{d}^{n-1})$ ($\boldsymbol{d}^i \in \mathbb{R}^n$, $i = 1, \ldots, n-1$)
- $\|\boldsymbol{d}^i\| = 1$ for all $i = 1, \ldots, n-1$
- $\boldsymbol{w}^\top \boldsymbol{d}^i = 0$ for all $i = 1, \ldots, n-1$

By replacing $\boldsymbol{x}$ by $D\boldsymbol{y} + \alpha\boldsymbol{w}$ ($\boldsymbol{y} \in \mathbb{R}^{n-1}$), problem (1) can be transformed into the following problem.

$$(\text{QP}(\alpha)) \begin{cases} \text{minimize} & \bar{g}(\boldsymbol{y}; \alpha) \\ \text{subject to} & \bar{h}(\boldsymbol{y}; \alpha) = 0, \end{cases}$$

where

$$\begin{aligned}
&\bar{g}(\boldsymbol{y}; \alpha) := \max\{\bar{g}_i(\boldsymbol{y}; \alpha) : i = 1, \ldots, m\}, \\
&\bar{g}_i(\boldsymbol{y} : \alpha) := \boldsymbol{y}\bar{A}_i\boldsymbol{y} - \left(\bar{\boldsymbol{b}}(\alpha)^i\right)^\top \boldsymbol{y} - \bar{c}_i(\alpha),\ i = 1, \ldots, m, \\
&\bar{h}(\boldsymbol{y}; \alpha) := \boldsymbol{y}^\top\boldsymbol{y} - r(\alpha)^2, \\
&\bar{A}_i := D^\top A_i D,\ i = 1, \ldots, m, \\
&\bar{\boldsymbol{b}}(\alpha)^i := D^\top \boldsymbol{b}^i - 2\alpha D^\top A_i \boldsymbol{w},\ i = 1, \ldots, m, \\
&\bar{c}_i(\alpha) := c_i - \alpha^2 \boldsymbol{w}^\top A_i \boldsymbol{w} + \alpha \left(\boldsymbol{b}^1\right)^\top \boldsymbol{w},\ i = 1, \ldots, m, \\
&r(\alpha) := \sqrt{r^2 - \alpha^2}.
\end{aligned}$$

Then, we have the following theorem.

Theorem 3.1 *Let $\bar{\alpha} \in [\alpha_0, r]$ satisfy the following conditions, and let $\bar{\boldsymbol{y}}$ be an optimal solution of* (QP(α))*.*

(i) $\min(\text{QP}(\bar{\alpha})) = 0$

(ii) $\min(\text{QP}(\alpha)) > 0$ *for each* $\alpha \in]-r, \bar{\alpha}[$

Then, $\bar{\alpha}$ and $D\bar{\boldsymbol{y}} + \bar{\alpha}\boldsymbol{w}$ are the optimal value and an optimal solution of (QDC)*, respectively.*

Let $\alpha \in [\alpha_0, r[$. If $\bar{\boldsymbol{y}} \in \mathbb{R}^n$ satisfies the following conditions (KKT1) and (KKT2) with a Lagrangian multiplier $\mu \in \mathbb{R}$, then $\bar{\boldsymbol{y}}$ is called a KKT point of (QP(α)).

(KKT1) $\boldsymbol{\xi} - \mu\nabla h(\bar{\boldsymbol{x}}) = \boldsymbol{0}_n$ for some $\boldsymbol{\xi} \in \partial_y \bar{g}(\bar{\boldsymbol{y}}; \alpha)$, that is,
$2\bar{A}(\boldsymbol{s})\bar{\boldsymbol{y}} - \bar{\boldsymbol{b}}(\boldsymbol{s}, \alpha) - 2\mu\bar{\boldsymbol{y}} = 0$ for some $\boldsymbol{s} \in S$,

(KKT2) $\bar{h}(\bar{\boldsymbol{y}}; \alpha) = 0$

where $\boldsymbol{\xi} = 2\bar{A}(\boldsymbol{s})\bar{\boldsymbol{y}} - \bar{\boldsymbol{b}}(\boldsymbol{s}, \alpha)$, $\bar{A}(\boldsymbol{s}) := \sum_{i=1}^{m} s_i \bar{A}_i$, $\bar{\boldsymbol{b}}(\bar{\boldsymbol{s}}, \alpha) := \sum_{i=1}^{m} s_i \bar{\boldsymbol{b}}^i(\alpha)$ and

$$S := \left\{ \boldsymbol{s} \in \mathbb{R}^m : \sum_{i=1}^{m} s_i = 1, \ s_1, \ldots, s_m \geq 0 \right\}.$$

Then, we have the following theorems.

Theorem 3.2 (See, e.g., Theorem 4.2.8 in [1]) *Let $\alpha \in [\alpha_L, r]$. Each locally optimal solution of* (QP(α)) *satisfies* (KKT1) *and* (KKT2).

Since $\bar{A}_i$ $(i = 1, \ldots, m)$ are symmetric positive definite matrices, $\bar{A}(\boldsymbol{s})$ is an $n \times n$ symmetric positive definite matrix for each $\boldsymbol{s} \in S$. Let $\lambda_i(\boldsymbol{s}) \in \mathbb{R}$ $(i = 1, \ldots, n-1)$ and $\boldsymbol{p}^i(\boldsymbol{s})$ $(i = 1, \ldots, n-1)$ satisfy the following conditions for each $\boldsymbol{s} \in S$.

$$\begin{aligned}
&\bar{A}(\boldsymbol{s})\boldsymbol{p}^i(\boldsymbol{s}) = \lambda_i(\boldsymbol{s})\boldsymbol{p}^i(\boldsymbol{s}), \ i = 1, \ldots, n-1, \\
&\|\boldsymbol{p}^i(\boldsymbol{s})\| = 1, \ i = 1, \ldots, n-1, \\
&\left(\boldsymbol{p}^i(\boldsymbol{s})\right)^\top \boldsymbol{p}^j(\boldsymbol{s}) = 0, \ i, j \in \{1, \ldots, n-1\} \ (i \neq j), \\
&0 < \lambda_1(\boldsymbol{s}) \leq \lambda_2(\boldsymbol{s}) \leq \cdots \leq \lambda_{n-1}(\boldsymbol{s}),
\end{aligned}$$

We note that $\lambda_i(\boldsymbol{s})$ and $\boldsymbol{p}^i(\boldsymbol{s})$ are an eigen value and an eigen vector of $\bar{A}(\boldsymbol{s})$ respectively, for each $i \in \{1, \ldots, n-1\}$. Let $P(\boldsymbol{s}) := \left(\boldsymbol{p}^1(\boldsymbol{s}), \ldots, \boldsymbol{p}^{n-1}(\boldsymbol{s})\right) \in \mathbb{R}^{(n-1)\times(n-1)}$. Then, $P(\boldsymbol{s})$ is an orthogonal matrix and satisfies the following.

$$P(\boldsymbol{s})^\top \bar{A}(\boldsymbol{s}) P(\boldsymbol{s}) = \operatorname{diag}\left(\lambda_1(\boldsymbol{s}), \ldots, \lambda_{n-1}(\boldsymbol{s})\right) =: \Lambda(\boldsymbol{s})$$

By fixing $\boldsymbol{s} \in S$ and replacing $\boldsymbol{y}$ by $P(\boldsymbol{s})\boldsymbol{z}$ $(\boldsymbol{s} \in \mathbb{R}^{n-1})$, (KKT1) and (KKT2) can be rewritten as follows.

(KKT1) $2\Lambda(\boldsymbol{s})\boldsymbol{z} - \hat{\boldsymbol{b}}(\boldsymbol{s}, \alpha) - 2\mu\boldsymbol{z} = \boldsymbol{0}_{n-1}$,

(KKT2) $\boldsymbol{z}^\top\boldsymbol{z} - r(\alpha)^2 = 0$

where $\hat{\boldsymbol{b}}(\boldsymbol{s}, \alpha) = P(\boldsymbol{s})^\top \bar{\boldsymbol{b}}(\boldsymbol{s}, \alpha)$.

We note that $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is a globally optimal solution of (QDC) if and only if there exists $\bar{\boldsymbol{z}} \in \mathbb{R}^{n-1}$ satisfying

- $\bar{\boldsymbol{x}} = DP(\boldsymbol{s})\bar{\boldsymbol{z}} + \boldsymbol{w}^\top \bar{\boldsymbol{x}} \boldsymbol{w}$
- $\bar{\boldsymbol{z}}$ satisfies (KKT1) and (KKT2) for some $\boldsymbol{s} \in S$, where $\alpha = \boldsymbol{w}^\top \bar{\boldsymbol{x}}$

To find an approximate solution of (QDC), we propose an algorithm for listing $\bar{\boldsymbol{z}}$ satisfying (KKT1) and (KKT2) for any $\alpha \in [\alpha_0, r]$ and $\boldsymbol{s} \in S$.

4 Procedures for listing KKT points

For given $\alpha \in [\alpha_0, r]$ and $\boldsymbol{s} \in S$, we define $\boldsymbol{z}(\mu; \boldsymbol{s}, \alpha) : \mathbb{R} \to \mathbb{R}^{n-1}$ and $\psi(\mu; \boldsymbol{s}, \alpha) : \mathbb{R} \to \mathbb{R}$ as follows.

$$\begin{aligned}
\boldsymbol{z}(\mu; \boldsymbol{s}, \alpha) &:= \frac{1}{2}(\Lambda(\boldsymbol{s}) - I_{n-1})^{-1}\hat{\boldsymbol{b}}(\boldsymbol{s}, \alpha), \\
\psi(\mu; \boldsymbol{s}, \alpha) &:= \boldsymbol{z}(\mu; \boldsymbol{s}, \alpha)^\top \boldsymbol{z}(\mu; \boldsymbol{s}, \alpha) - r(\alpha)^2 \\
&= \frac{1}{4}\sum_{i=1}^{n-1} \frac{\hat{b}_i(\boldsymbol{s}, \alpha)^2}{(\lambda_i(\boldsymbol{s}) - \mu)^2} - r(\alpha)^2
\end{aligned}$$

For each $\mu \in \mathbb{R}$, $\boldsymbol{z}(\mu; \boldsymbol{s}, \alpha)$ satisfies (KKT1). Moreover, if $\psi(\mu; \boldsymbol{s}, \alpha) = 0$ holds, then $\boldsymbol{z}(\mu; \boldsymbol{s}, \alpha)$ satisfies (KKT2). On $\mathbb{R}\backslash\{\lambda_1(\boldsymbol{s}), \dots, \lambda_{n-1}(\boldsymbol{s})\}$, we obtain the derivative $\frac{d}{d\mu}\psi(\mu; \boldsymbol{s}, \alpha)$ and the second derivative $\frac{d^2}{d\mu^2}\psi(\mu; \boldsymbol{s}, \alpha)$ as follows.

$$\begin{aligned}
\frac{d}{d\mu}\psi(\mu; \boldsymbol{s}, \alpha) &= \frac{1}{2}\sum_{i=1}^{n-1} \frac{\hat{b}_i(\boldsymbol{s}, \alpha)^2}{(\lambda_i(\boldsymbol{s}) - \mu)^3}, \\
\frac{d^2}{d\mu^2}\psi(\mu; \boldsymbol{s}, \alpha) &= \frac{3}{2}\sum_{i=1}^{n-1} \frac{\hat{b}_i(\boldsymbol{s}, \alpha)^2}{(\lambda_i(\boldsymbol{s}) - \mu)^4}
\end{aligned} \tag{2}$$

Let $T_i(\boldsymbol{s})$ $(i = 1, \dots, n(\boldsymbol{s}))$ be line segments defined as follows.

$$\begin{aligned}
&T_1(\boldsymbol{s}) :=]-\infty, \hat{\lambda}_1(\boldsymbol{s})[, \\
&T_i(\boldsymbol{s}) :=]\hat{\lambda}_{i-1}(\boldsymbol{s}), \hat{\lambda}_i(\boldsymbol{s})[, \ i = 2, \dots, n(s) - 1, \\
&T_{n(s)}(\boldsymbol{s}) :=]\lambda_{n-1}(\boldsymbol{s}), +\infty[.
\end{aligned}$$

Here, $\hat{\lambda}_1(\boldsymbol{s}), \dots, \hat{\lambda}_{n(s)-1}$ satisfy the followings.

- For each $i \in \{1, \dots, n-1\}$, there exists $j \in \{1, \dots, n(\boldsymbol{s}) - 1\}$ such that $\lambda_i(\boldsymbol{s}) = \hat{\lambda}_j(\boldsymbol{s})$.
- $0 < \hat{\lambda}_1(\boldsymbol{s}) < \hat{\lambda}_2(\boldsymbol{s}) < \cdots < \hat{\lambda}_{n(s)-1}(\boldsymbol{s})$.

From the definition of $T_i(\boldsymbol{s})$ and $\hat{\lambda}_i(\boldsymbol{s})$, $T_i(\boldsymbol{s})$ is nonempty for each $i \in \{1, \dots, n(\boldsymbol{s})\}$. Moreover, by (2), since $\frac{d^2}{d\mu^2}\psi(\mu; \boldsymbol{s}, \alpha) > 0$, $\psi(\mu; \boldsymbol{s}, \alpha)$ is a strictly convex function with respect to μ on each $T_i(\boldsymbol{s})$ $(i = 1, \dots, n(\boldsymbol{s}))$. Therefore, we can list KKT points of (QP(α)) by utilizing a standard algorithm for solving nonlinear equations (e.g., Newton method).

5 Procedure for updating a parameter for the parametric programming problem

By Assumption (A2), we have $\min(\text{QP}(\alpha_0)) > 0$.

For each $\alpha \in [\alpha_0, r[$, we define $L(\alpha)$ as follows.

$$L(\alpha) := \{(\boldsymbol{z}^\top, \alpha)^\top : \|\boldsymbol{z}\|^2 = r^2 - \alpha^2\}.$$

Then, the following theorem holds.

Theorem 5.1 *For each* $\alpha, \beta \in [\alpha_0, r[$ *and* $\boldsymbol{z}_\alpha \in L(\alpha)$*, there exists* $\boldsymbol{z}_\beta \in L(\beta)$ *satisfying*

$$\|(\boldsymbol{z}_\beta^\top, \beta)^\top - (\boldsymbol{z}_\alpha^\top, \alpha)^\top\|^2 = 2r^2 - 2\alpha\beta + 2\sqrt{r^2 - \beta^2}\sqrt{r^2 - \alpha^2} =: \phi(\beta, \alpha).$$

For each $\alpha \in [\alpha_0, r[$ and $\eta \in [\alpha_0 - \alpha, r - \alpha[$, we have the followings.

$$\begin{aligned}
&\phi(\alpha + \eta, \alpha) = 2r^2 - 2\alpha^2 - 2\alpha\eta - 2\sqrt{r^2 - (\alpha + \eta)^2}\sqrt{r^2 - \alpha^2},\\
&\frac{\partial}{\partial \eta}\phi(\alpha + \eta, \alpha) = -2\alpha + \frac{2(\alpha + \eta)\sqrt{r^2 - \alpha^2}}{\sqrt{r^2 - (\alpha + \eta)^2}},\\
&\frac{\partial^2}{\partial \eta^2}\phi(\alpha + \eta, \alpha) = \frac{2r^2\sqrt{r^2 - \alpha^2}}{(\sqrt{r^2 - (\alpha + \eta)^2})^3} > 0.
\end{aligned}$$

Hence, $\phi(\alpha+\eta, \alpha)$ is a strictly convex function with respect to η on $[\alpha_0 - \alpha, r - \alpha[$. Moreover, since $\phi(\alpha, \alpha) = 0$ and $\frac{\partial}{\partial \eta}\phi(\alpha, \alpha) = 0$, we have $\phi(\alpha + \eta, \alpha) > 0$ for each $\eta \in [\alpha_0 - \alpha, r - \alpha[$ $(\eta \neq 0)$. From the following theorem, we obtain a Lipschitz constant of $\bar{g}$.

Theorem 5.2 *For each* $\alpha, \beta \in] - r, r[$*,* $(\boldsymbol{z}_\alpha^\top, \alpha)^\top \in L(\alpha)$ *and* $(\boldsymbol{z}_\beta^\top, \beta)^\top \in L(\beta)$*, the following inequality hold.*

$$|\bar{g}(P(\boldsymbol{s})\boldsymbol{z}_\beta; \beta) - \bar{g}(P(\boldsymbol{s})\boldsymbol{z}_\alpha; \alpha)| \leq (2r\lambda_*(\boldsymbol{s}) + \|\hat{\boldsymbol{b}}(\boldsymbol{s}, \alpha)\|)\|(\boldsymbol{z}_\alpha^\top, \alpha)^\top - (\boldsymbol{z}_\beta^\top, \beta)^\top\|,$$

where $\lambda_*(\boldsymbol{s})$ *is the maximal eigen value of* $A(\boldsymbol{s})$*.*

From the strict convexity of ϕ with respect to η, Theorems 5.1 and 5.2, we have the following theorem.

Theorem 5.3 *Assume that* $\alpha \in] - r, r[$ *and* $\bar{\eta} \in]0, r - \alpha[$ *satisfy the following inequalities.*

$$\begin{aligned}
&\bar{g}(P(\boldsymbol{s})\bar{\boldsymbol{z}}(\alpha); \alpha) > 0,\\
&\phi(\alpha + \bar{\eta}, \alpha) \leq \frac{\bar{g}(P(\boldsymbol{s})\bar{\boldsymbol{z}}(\alpha); \alpha)}{2r\lambda_*(\boldsymbol{s}) + \|\hat{\boldsymbol{b}}(\boldsymbol{s}, \alpha)\|},
\end{aligned}$$

where $P(\boldsymbol{s})\bar{\boldsymbol{z}}(\alpha) \in \arg\min\{g(P(\boldsymbol{s})\boldsymbol{z} : (\boldsymbol{z}, \alpha) \in L(\alpha)\}$*. Then, for each* $\eta \in]0, \bar{\eta}]$*,* $\bar{g}(P(\boldsymbol{s})\bar{\boldsymbol{z}}(\alpha + \eta); \alpha + \eta) > 0$*, where* $P(\boldsymbol{s})\bar{\boldsymbol{z}}(\alpha + \eta) \in \arg\min\{g(P(\boldsymbol{s})\boldsymbol{z} : (\boldsymbol{z}, \alpha + \eta) \in L(\alpha + \eta)\}$*.*

By Theorem 5.3, for given $\boldsymbol{s} \in S$, we propose the following algorithm LKKT for listing KKT points.

Algorithm LKKT

Step 0: Set a tolerance $\delta \geq 0$ $k := 1$. Calculate an optimal solution of (QP(α_0)). Set $k := 1$ and go to Step 1.

Step 1: Find $\eta_k \in]0, r - \alpha_k[$ satisfying

$$\phi(\alpha_k + \eta_k, \alpha_k) = \frac{\bar{g}(P\boldsymbol{z}(\alpha_k + \eta_k); \boldsymbol{s}, \alpha_k)}{2r\lambda_*(\boldsymbol{s}) + \|\hat{\boldsymbol{b}}(\boldsymbol{s}, \alpha_k)\|}.$$

Go to Step 2.

Step 2: Calculate $\bar{\boldsymbol{z}}(\alpha_k + \eta_k + \delta)$ by executing Newton method. Go to Step 3.

Step 3: If $\bar{g}(P\bar{\boldsymbol{z}}(\alpha_k + \eta_k + \delta); \boldsymbol{s}, \alpha_k) \leq 0$, then stop; $(D, \boldsymbol{w})((P\bar{\boldsymbol{z}}(\alpha_k + \eta_k + \delta))^\top, \alpha_k + \eta_k + \delta)^\top$ is an approximate solution of (QRC). Otherwise, set $\alpha_{k+1} := \alpha_k + \eta_k + \delta$, $k \leftarrow k + 1$, and return to Step 1.

6 Branch and Bound Procedure

In this section, we propose a branch and bound procedure to execute Algorithm LKKT throughout S.

6.1 Subdivision Process

In order to calculate Lagrangian multiplier vector $\boldsymbol{s} \in S$, we utilize the bisection which is one of the classical subdivision processes.

Let $S_1 := S$ and $\tilde{\mathcal{S}}_1 := \{S_1\}$. Moreover, for each $k > 0$, we set S_k and $\tilde{\mathcal{S}}_{k+1}$ as follows.

$$S_k \in \arg\max\{\operatorname{diam} S : S \in \mathcal{S}_k\} \tag{3}$$

$$\tilde{\mathcal{S}}_{k+1} := (\mathcal{S}_k \cup \{S', S''\}) \backslash \{S_k\} \tag{4}$$

Here

$$\begin{aligned}
&S' := \operatorname{co}(V(S_k) \cup \{\hat{\boldsymbol{v}}\}) \backslash \{\boldsymbol{v}''\},\\
&S'' := \operatorname{co}(V(S_k) \cup \{\hat{\boldsymbol{v}}\}) \backslash \{\boldsymbol{v}'\},\\
&\hat{\boldsymbol{v}} := \frac{\boldsymbol{v}' + \boldsymbol{v}''}{2},\\
&\boldsymbol{v}' \text{ and } \boldsymbol{v}'' \in V(S_k) \text{ satisfy } \|\boldsymbol{v}' - \boldsymbol{v}''\| = \operatorname{diam} S_k,\\
&V(S_k) \text{ is the vertex set of } S_k.
\end{aligned}$$

Since S_1 is an $(m-1)$-simplex, all elements of $\tilde{\mathcal{S}}_k$ are $(m-1)$-simplices for each $k > 0$. Moreover, we have the following proposition and theorem.

Proposition 6.1 (See [2], Proposition IV.2) *Assume that the sequences* $\{S_k\}$ *and* $\{\tilde{\mathcal{S}}_k\}$ *are generated based on* (3) *and* (4)*, respectively. Let an infinite subsequence* $\{S_{k_q}\} \subset \{S_k\}$ *satisfy* $S_{k_{q+1}} \subset S_{k_q}$ *for each* $q > 0$*. Then, the following statements hold.*

(i) $\operatorname{diam} S_{k_{q+m}} \leq \frac{\sqrt{3}}{2} \operatorname{diam} S_{k_q}$ *for each* $q > 0$

(ii) $\lim_{q \to +\infty} \operatorname{diam} S_{k_q} = 0$

Theorem 6.1 *Assume that the sequences* $\{S_k\}$ *and* $\{\tilde{\mathcal{S}}_k\}$ *are generated based on* (3) *and* (4)*, respectively. Then,* $\lim_{k \to +\infty} \operatorname{diam} S_k = 0$.

From Theorem 6.1, we notice that $\tilde{\mathcal{S}}_{\hat{k}}$ is empty for some $\hat{k} > 0$ by (4) by the following.

$$\tilde{\mathcal{S}}_{k+1} = (\tilde{S}_k \backslash \{S_k\}) \cup \{S \in \{S', S''\} : \operatorname{diam} S > \tau\}. \tag{5}$$

Here, τ is a positive real number as a tolerance. Then, by utilizing the following stopping condition, the branch-and-bound procedure proposed in this section terminates within a finite number of iterations.

(SC) If $\tilde{\mathcal{S}} = \emptyset$, then stop.

6.2 Lower Bound

The following theorem holds.

Lemma 6.1 *Let* $\tilde{\boldsymbol{s}}^1, \tilde{\boldsymbol{s}}^2 \in S$*,* $i, j \in \{1, \ldots, n-1\}$ *satisfy* $\lambda_j(\tilde{\boldsymbol{s}}^1) = \lambda_i(\tilde{\boldsymbol{s}}^1)$*. Then, the following inequality holds.*

$$|\lambda_j(\tilde{\boldsymbol{s}}^2) - \lambda_i(\tilde{\boldsymbol{s}}^1)| \leq \lambda_{\max} \|\tilde{\boldsymbol{s}}^2 - \tilde{\boldsymbol{s}}^1\|$$

Here,

$$\lambda_{\max} := \max\{\lambda_n^q : q = 1, \ldots, m\},$$
$$\lambda_1^q, \ldots, \lambda_{n-1}^q : \text{ all eigen values of } A_q \text{ satisfying } 0 < \lambda_1^q \leq \lambda_2^q \leq \cdots \leq \lambda_{n-1}^q.$$

Then, there exists $\delta > 0$ *such that* $|\lambda_j(\boldsymbol{s}) - \lambda_i(\tilde{\boldsymbol{s}}_1)| < \varepsilon$ *for each* $j \in \{1, \ldots, n-1\}$ *satisfying* $\lambda_j(\tilde{\boldsymbol{s}}_1) = \lambda_i(\tilde{\boldsymbol{s}}_1)$*, and* $\boldsymbol{s} \in S \cap B_<^m(\tilde{\boldsymbol{s}}_1, \delta)$*.*

Theorem 6.2 *Let* $\boldsymbol{s}^1, \boldsymbol{s}^2 \in S$*,* $\{t_k\} \subset]0, 1[$ *a sequence satisfying* $t_k \to 0$ *as* $k \to +\infty$ *and* $\boldsymbol{s}(k) := (1 - t_k)\boldsymbol{s}^1 + t_k \boldsymbol{s}^2$ *for each* k*. Then,* $\lambda_i(\boldsymbol{s}(k)) \to \lambda_i(\boldsymbol{s}^1)$ *and* $A(\boldsymbol{s}(k))\boldsymbol{p}^i(\boldsymbol{s}^1) \to \lambda_i(\boldsymbol{s}^1)\boldsymbol{p}^i(\boldsymbol{s}^1)$ *as* $k \to +\infty$ *for each* $i \in \{1, \ldots, n-1\}$*.*

6.3 Algorithm

In this section, we propose a branch and bound procedure for calculating a globally optimal solution of (QDC).

From the following theorem, we notice that at least one feasible solution can be calculated over each maximal connected subset of $G \backslash \operatorname{int} H$ by executing algorithm LKKT throughout S.

Theorem 6.3 *For each maximal connected subset of* $G \backslash \operatorname{int} H$*, there exists a KKT point for* (QDC)*.*

In order to execute Algorithm LKKT throughout S, we propose a branch and bound procedure as follows.

Algorithm BBP

Step 0: Set tolerances $\tau, \rho \geq 0$, $\mathcal{S}_1 = \{S\}$, $\boldsymbol{x}^1 = \alpha_0 \boldsymbol{w}$, $k = 1$, Go to Step 1.

Step 1: If $\mathcal{S}_k = \emptyset$, then stop; $\boldsymbol{x}^k$ is an approximate solution of (QDC). Otherwise, go to Step 2.

Step 2. Choose $S_k \in \mathcal{S}_k$ satisfying $\operatorname{diam} S_k = \max_{S \in \mathcal{S}_k} \operatorname{diam} S$. Set $\boldsymbol{s}_k$ as follows.

$$\boldsymbol{s}_k := \frac{1}{m} \sum_{i=1,\ldots,m} \boldsymbol{\kappa}^i,$$

where $\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m$ are all vertices of S_k. Go to Step 3.

Step 3: Execute Algorithm LKKT with $\boldsymbol{s}^k$ selected at Step 2. Go to Step 4.

Step 4: If $\tilde{\boldsymbol{x}}$ calculated by executing Algorithm LKKT satisfies $\tilde{\boldsymbol{x}} \in G \backslash \operatorname{int} H$ and $\boldsymbol{w}^\top \tilde{\boldsymbol{x}} < \boldsymbol{w}^\top \boldsymbol{x}^k$, then $\boldsymbol{x}^{k+1} := \tilde{\boldsymbol{x}}$. Otherwise, $\boldsymbol{x}^{k+1} := \boldsymbol{x}^k$. Go to Step 5.

Step 5: Choose $\boldsymbol{\kappa}', \boldsymbol{\kappa}'' \in \{\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m\}$ satisfying $\|\boldsymbol{\kappa}' - \boldsymbol{\kappa}''\| = \operatorname{diam} S_k$. Update $\mathcal{S}_{k+1}$ as follows.

$$\mathcal{S}_{k+1} := \begin{cases} (\mathcal{S}_k \cup \{S', S''\}) \backslash \{S_k\}, & \text{if } \operatorname{diam} S' \geq \rho \text{ and } \operatorname{diam} S'' \geq \rho, \\ (\mathcal{S}_k \cup \{S'\}) \backslash \{S_k\}, & \text{if } \operatorname{diam} S' \geq \rho \text{ and } \operatorname{diam} S'' < \rho, \\ (\mathcal{S}_k \cup \{S''\}) \backslash \{S_k\}, & \text{if } \operatorname{diam} S' < \rho \text{ and } \operatorname{diam} S'' \geq \rho, \\ \mathcal{S}_k \backslash \{S_k\}, & \text{if } \operatorname{diam} S' < \rho \text{ and } \operatorname{diam} S'' < \rho, \end{cases}$$

where $S' := \operatorname{co}\left(\{\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m, \check{\boldsymbol{\kappa}}\} \backslash \{\boldsymbol{\kappa}''\}\right)$, $S'' := \operatorname{co}\left(\{\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m, \check{\boldsymbol{\kappa}}\} \backslash \{\boldsymbol{\kappa}'\}\right)$, and $\check{\boldsymbol{\kappa}} := \dfrac{\boldsymbol{\kappa}' - \boldsymbol{\kappa}''}{2}$. Set $k \leftarrow k+1$ and return to Step 1.

Since S_k is bisected at Step 5 of Algorithm BBP, by setting a tolerance ρ to a positive number, the routine between Step 1 and Step 5 is terminates within a finite number of iterations (see, e.g., Theorem IV.1 and Proposition IV.2 in [2]).

7 Conclusions

In this paper, we propose Algorithm LKKT for listing KKT points of (QP(α)). Moreover by combining Algorithm LKKT with a parametric optimization method and a branch-and-bound procedure, we present Algorithm BBP for (QDC).

References

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