

# COMMON FIXED POINT THEOREMS FOR ASYMPTOTIC MAPPINGS IN COMPLETE METRIC SPACES

TOSHIKAZU WATANABE

ABSTRACT. In this paper we consider an asymptotic version of  $\alpha$ - $\psi$  contractive mappings in vector metric spaces taking value in Riesz spaces. and prove fixed point theorem on this spaces.

## 1. INTRODUCTION

In [31], Samet and Vetro-Vetro introduced the notion of  $\alpha$ - $\psi$  mapping and consider the fixed point theorem. In [34], we introduced the notion of  $\alpha$ - $\psi_n$  mapping which is a generalization of the  $\alpha$ - $\psi$  mapping and consider the fixed point theorem. It is a generalization of the the mapping in Caccioppoli's fixed point theorem and also the generalization of the (c)-comparison operator. In this paper we consider the common fixed point theorem for the  $\alpha$ - $\psi$  and  $\alpha$ - $\psi_n$  mappings under the ordered vector metric spaces settings. For the mappings of the metric spaces and vector metric spaces, a lot of authors study common fixed theorem. For instance in [4, 11], Altun and Cevik introduce an vector metric spaces and proved Banach contraction theorem. In [30], Rahimi generalized fixed point theorem and they prove some common fixed point theorems for four mappings in ordered vector metric spaces. They also extend and generalize well-known comparable results in the literature.

## 2. PRELIMINARIES

Throughout this paper we denote by  $\mathbb{N}$  the set of all positive integers and  $\mathbb{R}$  the real number. In this section we give several preliminaries for the ordered vector metric spaces settings,  $\alpha$ - $\psi$  and  $\alpha$ - $\psi_n$  mappings, etc.

Let  $E$  be a non-empty set. A relation  $\leq$  on  $E$  is called:

- (i) reflexive if  $x \leq x$  for all  $x \in E$
- (ii) transitive if  $x \leq y$   $y \leq z$  imply  $x \leq z$
- (iii) antisymmetric if  $x \leq y$  and  $y \leq x$  imply  $x = y$
- (iv) preorder if it is reflexive and transitive. A preorder is called a partial order if it is antisymmetric.
- (v) translation invariant if  $x \leq y$  implies  $(x + z) \leq (y + z)$  for any  $z \in E$
- (vi) scale invariant if  $x \leq y$  implies  $(\lambda x) \leq (\lambda y)$  for any  $\lambda > 0$ . A preorder  $\leq$  is called partial order or an order relation if it is antisymmetric.

Given a partially ordered set  $(E, \leq)$ , that is, the set  $E$  equipped with a partial order  $\leq$ , the notation  $x < y$  stands for  $x \leq y$  and  $x \neq y$ . An order interval  $[x, y]$  in  $E$  is the set  $\{z \in E : x \leq z \leq y\}$ . A real linear space  $E$  equipped with an order relation  $\leq$  on  $E$  which is compatible with the algebraic structure of  $E$  is called an

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ordered linear space or ordered vector space. The ordered vector space  $(E, \leq)$  is called a Riesz space (vector lattice or linear lattice) if for every  $x, y \in E$ , there exist  $x \wedge y = \inf\{x, y\}$  and  $x \vee y = \sup\{x, y\}$ . If we denote  $x_+ = 0 \vee x$ ,  $x_- = 0 \wedge (-x)$  and  $|x| = x \vee (-x)$ , then  $x = x_+ - x_-$  and  $|x| = x_+ + x_-$ .

A closed convex set  $K$  in  $E$  is called closed convex cone, if for any  $\lambda > 0$  and  $x \in K$ ,  $\lambda x \in K$ . A convex cone  $K$  is called pointed if  $K \cap (-K) = \{0\}$ . From now on we shall call a closed convex pointed cone simply cone.

Let  $E$  be a Riesz space. The cone  $\{x \in E : x \geq 0\}$  of nonnegative elements in an ordered vector space  $E$  is denoted by  $E_+$ .  $E$  is said to be Archimedean if  $\frac{a}{n} \downarrow 0$  holds for every  $a \in E_+$ .

Let  $K$  be a cone in  $E$ . Denote  $x \leq y$  if  $y - x \in K$ . Then, order  $\leq$  defines a partial order on  $E$  called the order induced by  $K$ . Conversely, if order  $\leq$  is a partial order on  $E$ , then  $E$  is called ordered vector space and the set  $K = \{x \in E : 0 \leq x\}$  is a cone called the positive cone of  $E$ . In this case it is easy to see that  $x < y$  if and only if  $y - x \in E$ . Note that if  $a \leq ha$  where  $a \in K$  and  $h \in (0, 1)$ , then  $a = 0$ .

**Definition 1.** A sequence of vectors  $\{x_n\}$  in  $E$  is said to: (i) decrease to an element  $x \in E$  if  $x_{n+1} \leq x_n$  for every  $n \in N$  (set of natural numbers) and  $x = \inf\{x_n : n \in N\} = \bigwedge_{n \in N} x_n$ . We denote it by  $x_n \downarrow x$ . (ii) increase to an element  $x \in E$  if  $x_n \leq x_{n+1}$  for every  $n \in N$  and  $x = \sup\{x_n : n \in N\} = \bigvee_{n \in N} x_n$ . We denote it by  $x_n \uparrow x$ .

**Definition 2.** A sequence of vectors  $\{x_n\}$  in  $E$  is said to be order convergent to  $x \in E$  if there exist sequences  $\{y_n\}$  and  $\{z_n\}$  in  $E$  such that  $y_n \downarrow x$ ,  $z_n \uparrow x$  and  $z_n \leq x_n \leq y_n$ . We denote this by  $x = o\text{-}\lim_{n \rightarrow \infty} x_n$ . If the sequence is order convergent, then its order limit is unique.

**Definition 3.** A sequence of vectors  $\{x_n\}$  in  $E$  is said to be order Cauchy sequence in  $E$  if the sequence  $\{x_m - x_n\}$  in cone  $K$  is order convergent to 0.

**Definition 4.** Let  $E$  and  $F$  be two Riesz spaces. A mapping  $f : E \rightarrow F$  is called order continuous at  $x_0$  in  $E$  if for any sequence  $\{x_n\}$  in  $E$  such that  $x = o\text{-}\lim_{n \rightarrow \infty} x_n$ , we have  $f(x) = o\text{-}\lim_{n \rightarrow \infty} f(x_n)$ .

**Remark 5.** If  $x_n \in K$  for every  $n \in N$  and  $x = o\text{-}\lim_{n \rightarrow \infty} x_n$ , then  $x \in K$ . Also, if  $x_n \in K$  for every  $n \in N$  and  $\{y_n\}$  is any sequence for which  $y_n - x_n \in K$  with  $o\text{-}\lim_{n \rightarrow \infty} y_n = 0$ , then  $o\text{-}\lim_{n \rightarrow \infty} x_n = 0$ .

**Definition 6.** A cone  $K \subset E$  is called regular if every decreasing sequence of elements in  $K$  is convergent.

**Definition 7.** Riesz space  $E$  is complete if there exists  $\sup A$  and  $\inf A$  for each bounded countable subset  $A$  of  $E$ . For more details on Riesz space, order convergence, and order continuity, we refer to [25] and references mentioned therein.

**Definition 8.** If  $(E, \leq)$  is a Riesz space and  $f : E \rightarrow E$  is such that  $f(x) \leq f(y)$  whenever  $x, y \in E$  and  $x \leq y$ , then  $f$  is said to be nondecreasing.

**Definition 9.** Let  $(E, \leq)$  be a Riesz space. The set  $(UF)f = \{x \in E : x - f(x) \in K\}$  is called upper fixed point set of  $f$ ,  $(LF)f = \{x \in E : f(x) - x \in K\}$  is called lower fixed point set of  $f$  and  $(F)f = \{x \in E : f(x) = x\}$  is called the set of all fixed points of  $f$ .

**Definition 10.** Let  $(E, \leq)$  be a Riesz space. The self map  $f$  on  $E$  is called: (i) dominated on  $E$  if  $(UF)f = E$ . (ii) dominating on  $E$  if  $(LF)f = E$ .

**Example 1.** Let  $E = [0, 1]$  be endowed with the usual ordering. Let  $f : E \rightarrow E$  be defined by  $f(x) = x$ . Then  $(LF)f = E$ .

**Example 2.** Let  $E = [0, \infty)$  be endowed with the usual ordering. Define  $f : E \rightarrow E$  by

$$f(x) = \begin{cases} x^{1/n} & \text{for } x \in [0, 1), \\ x^n & \text{for } x \in [1, \infty), \end{cases}$$

$n \in \mathbb{N}$ , then  $(LF)f = E$ .

**Definition 11.** Let  $(E, \leq)$  be a Riesz space. Two mappings  $f, g : E \rightarrow E$  are said to be mutually dominated if  $f(x) \in (UF)g$  and  $g(x) \in (UF)f$  for all  $x \in E$ . That is,  $f(x) \geq g(f(x))$  and  $g(x) \geq f(g(x))$  for all  $x \in E$ .

**Definition 12.** Let  $(E, \leq)$  be a Riesz space. Two mappings  $f, g : E \rightarrow E$  are said to be mutually dominating if  $f(x) \in (LF)g$  and  $g(x) \in (LF)f$  for all  $x \in E$ . That is,  $f(x) \leq g(f(x))$  and  $g(x) \leq f(g(x))$  for all  $x \in E$ . The following two examples show that there exist discontinuous and mutually dominating mappings which are not nondecreasing mappings

The examples of mutually dominating maps which are not non decreasing maps and mutually dominating maps but not non decreasing. These examples are given in [27, example 15,16].

**Definition 13.** Let  $(E, \leq)$  be a Riesz space and  $K$  be its positive cone. A monotone increasing mapping  $\Phi : K \rightarrow K$  is called comparison operator if  $\lim_{n \rightarrow \infty} \Phi_n(t) = 0$  for each  $t \in K$ .

**Definition 14.** Let  $(E, \leq)$  be a Riesz space and  $\{x_n\}$  be a sequence in  $E$ . If  $o\text{-}\lim_{j \rightarrow \infty} x_j$  exists, then we say that the series  $\sum_{n=1}^{\infty} x_n$  is order convergent.

Next we consider the extension of  $\alpha\text{-}\psi_n$  and  $\alpha\text{-}\psi$  mappings.

**Definition 15.** [31] Let  $(X, d)$  be a metric space. We say that mapping  $T : X \rightarrow X$  is  $\alpha\text{-}\psi$  contractive if there exist a mapping  $\alpha : X \times X \rightarrow [0, 1)$  and a sequence of nondecreasing mappings  $\psi$  of  $[0, 1)$  into itself such that the series  $\sum_{n=1}^{\infty} \psi^n(t)$  converges for all  $t > 0$  and for any  $x, y \in X, n \in \mathbb{N}$ , we have

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)).$$

**Definition 16.** [34] Let  $(X, d)$  be a metric space. We say that mapping  $T : X \rightarrow X$  is  $\alpha\text{-}\psi_n$  contractive if there exist a mapping  $\alpha : X \times X \rightarrow [0, 1)$  and a sequence of nondecreasing mappings  $\psi_n$  of  $[0, 1)$  into itself such that the series  $\sum_{n=1}^{\infty} \psi_n(t)$  converges for all  $t > 0$  and for any  $x, y \in X, n \in \mathbb{N}$ , we have

$$\alpha(x, y)d(T^n x, T^n y) \leq \psi_n(d(x, y)).$$

**Definition 17.** Let  $(X, d)$  be a metric space. We say that two mappings  $f, g : X \rightarrow X$  are said to mutually  $\alpha\text{-}\psi$  contractive if there exist a mapping  $\alpha : X \times X \rightarrow [0, 1)$  and a sequence of nondecreasing mappings  $\psi$  of  $[0, 1)$  into itself such that the series  $\sum_{n=1}^{\infty} \psi^n(t)$  converges for all  $t > 0$  and for any  $x, y \in X, n \in \mathbb{N}$ , we have

$$\alpha(x, y)d(f(x), f(y)) \leq \psi(d(g(x), g(y))).$$

And we say that two mappings  $f, g : X \rightarrow X$  are said to mutually  $\alpha\text{-}\psi_n$  contractive if there exist a mapping  $\alpha : X \times X \rightarrow [0, 1)$  and a sequence of nondecreasing

mappings  $\psi_n$  of  $[0, 1)$  into itself such that the series  $\sum_{n=1}^{\infty} \psi_n(t)$  converges for all  $t > 0$  and for any  $x, y \in X, n \in N$ , we have

$$\alpha(x, y)d(f^n(x), f^n(y)) \leq \psi_n(d(g(x), g(y))).$$

We also give a definition of  $\alpha$ -admissible mapping.

**Definition 18.** [31] We say mapping  $f$  is  $\alpha$ -admissible if

$$\alpha(x, y) > 1 \text{ implies } \alpha(f(x), f(y)) > 1.$$

**Example 3.**  $E = (C([0, 1], R^2))$

$$f(x_1, x_2) = \left( \frac{3x_1}{5}, \frac{3x_2}{5} \right), g(x_1, x_2) = \left( \frac{4x_1}{5}, \frac{4x_2}{5} \right)$$

Define order  $x = (x_1, x_2) \succ y = (y_1, y_2)$  iff  $x_1 \geq y_1$  and  $x_2 \geq y_2$ . Define  $\alpha : X \times X \rightarrow [0, \infty)$  by the following.

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \succeq y, \\ 0 & \text{if } x \prec y. \end{cases}$$

Let  $x, y \in R^2$  such that  $y = (y_1, y_2) \preceq x = (x_1, x_2)$ . Then,  $y_1 \leq x_1$  and  $y_2 \leq x_2$ . In this case since

$$f(x) - f(y) = \left( \frac{3}{5}(x_1 - y_1), \frac{3}{5}(x_2 - y_2) \right) \succeq 0,$$

Then we have  $\alpha(f(x), f(y)) \geq 1$  Thus  $f$  is  $\alpha$ -admissible. Moreover if we take  $x_0 = (1, 1)$ , then  $f(x_0) = (\frac{3}{5}, \frac{3}{5})$  and  $x_0 - f(x_0) = (\frac{2}{5}, \frac{2}{5})$ . thus  $\alpha(f(x_0), f(x_0)) \geq 1$ .  $d(x, y) = \|x - y\| = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{1/2}$ .  $\psi_n(t) = (\frac{3}{4})^n t$ . Then

$$\begin{aligned} \alpha(x, y)d(f^n(x), f^n(y)) &= \left( \frac{3}{5} \right)^n \|x - y\| \leq \left( \frac{3}{4} \right)^n \left( \left( \frac{4}{5} \right)^n \|x - y\| \right) \\ &= \psi_n(d(g(x), g(y))). \end{aligned}$$

**Definition 19.** If  $(E, \leq)$  is a Riesz space and  $f : E \rightarrow E$  is such that  $f(x) \leq f(y)$  whenever  $x, y \in E$  and  $x \leq y$ , then  $f$  is said to be nondecreasing.

**Definition 20.** Let  $(E, \leq)$  be a Riesz space. The set  $(UF)f = \{x \in E : x - f(x) \in K\}$  is called upper fixed point set of  $f$ ,  $(LF)f = \{x \in E : f(x) - x \in K\}$  is called lower fixed point set of  $f$  and  $(F)f = \{x \in E : f(x) = x\}$  is called the set of all fixed points of  $f$ .

**Definition 21.** Let  $X$  be a nonempty set and  $E$  a Riesz space. A mapping  $d : X \times X \rightarrow E$  is said to be a vector metric or  $E$ -metric if it satisfies the following conditions:

- (E 1)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (E 2)  $d(x, y) \leq d(x, z) + d(y, z)$ ; for all  $x, y, z \in X$ .

We call  $(X, d, E)$  a vector metric space.

**Definition 22.** (See [24]). Let  $f, g : X \rightarrow X$  be mappings of a set  $X$ . If  $f(w) = g(w) = z$  for some  $z \in X$ , then  $w$  is called a coincidence point of  $f$  and  $g$ , and  $z$  is called a point of coincidence of  $f$  and  $g$ .

**Definition 23.** (See [24]). Paire of self-mappings  $(f, g)$  on an ordered vector metric space  $(X, d, E)$  is said to be compatible if, for arbitrary sequence  $(x_n) \subset X$ , such that  $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} g(x_n) \in X$ , and for arbitrary  $c \in E$ , there exists  $n_0 \in \mathbb{N}$  such that

$$d(fg(x_n), gf(x_n)) \leq c,$$

whenever  $n > n_0$ . It is said to be weakly compatible if  $f(x) = g(x)$  implies  $fg(x) = gf(x)$ . It is clear that, as in the case of metric space, the pair  $(f, i_X)$  ( $i_X$  is the identity mapping) is both compatible and weakly compatible, for each self-map  $f$ .

**Lemma 24.** (See [2]). Let  $f$  and  $g$  be weakly compatible self-maps of a set  $X$ . If  $f$  and  $g$  have a unique point of coincidence  $z = f(w) = g(w)$ , then  $z$  is the unique common fixed point of  $f$  and  $g$ .

For arbitrary elements  $x, y, z$  and  $w$  of a vector metric space, the following holds true:

(Em 1)  $0 \leq d(x, y)$ ;

(Em 2)  $d(x, y) = d(y, x)$ ;

(Em 3)  $|d(x, z) - d(y, z)| \leq d(x, y)$ ;

(Em 4)  $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$ .

A Riesz space  $E$  is a vector metric space with  $d : E \times E \rightarrow E$  defined by  $d(x, y) = |x - y|$ . This vector metric is called absolute valued metric on  $E$ .

**Definition 25.** (See [6, 11]).

- (i) A sequence  $\{x_n\}$  in  $X$  is vectorial converges or  $E$ -converges to some  $x \in X$  (we write  $x_n \rightarrow^{d, E} x$ ), if there is a sequence  $\{a_n\}$  in  $E$  satisfying  $a_n \downarrow 0$  and  $d(x_n, x) \leq a_n$  for all  $n$ ;
- (ii) A sequence  $(x_n)$  is called  $E$ -Cauchy sequence if there exists a sequence  $(a_n)$  in  $E$  such that  $a_n \downarrow 0$  and  $d(x_n, x_{n+p}) \leq a_n$  holds for all  $n$  and  $p$ ;
- (iii) A vector metric space  $X$  is called  $E$ -complete if each  $E$ -Cauchy sequence in  $X$   $E$ -converges to a limit in  $X$ .

**Lemma 26.** (See [6, 11]). We have following properties in vector metric space  $X$ :

- (a) The limit  $x$  is unique;
- (b) Every subsequence of  $(x_n)$   $E$ -converges to  $x$ ;
- (c) If  $x_n \rightarrow^{d, E} x$  and  $y_n \rightarrow^{d, E} y$ , then  $d(x_n, y_n) \rightarrow^o d(x, y)$ .

**Definition 27.** (See [6, 11]) An ordered set is  $\sigma$ -complete if  $\sup A$  and  $\inf A$  exists in numerable subset  $A$  in  $X$ .

### 3. MAIN RESULTS

A fixed point problem is to find some  $x \in E$  such that  $f(x) = x$  and we denote it by  $FP(f, E)$ . Let  $f, g : E \rightarrow E$ . A common fixed point problem is to find some  $x \in E$  such that  $x = f(x) = g(x)$  and we denote it by  $CFP(f, g, E)$ . The equation  $f(x) = g(x)$  ( $f(x) = g(x) = x$ ) is called coincidence point equation (resp. common fixed point equation).

In [4, Corollary 1], the following results obtained, see also [1].

**Theorem 28.** Let  $(X, d)$  be a complete metric space. Suppose mappings  $f, g : X \rightarrow X$  satisfy

$$(1) \quad d(f(x), f(y)) \leq kd(g(x), g(y)), \text{ for all } x, y \in X,$$

where  $k \in [0, 1)$  is a constant. If the range of  $g$  contains the range of  $f$  and  $g(X)$  is a complete subspace of  $X$ , then  $f$  and  $g$  have a unique point of coincidence in  $X$ . Moreover if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point.

Inspiring the above result, we give a fixed point theorem versions of  $\alpha$ - $\psi$  and  $\alpha$ - $\psi_n$  contractive mappings.

**Definition 29.** Let  $f, g$  be mappings such that the range of  $f$  is contained in the range of  $g$ . We say that  $f$  is  $g$ -continuous at  $x_0 \in X$  if  $g(x) \rightarrow g(x_0)$  implies  $f(x) \rightarrow f(x_0)$ .

**Proposition 30.** Let  $f$  and  $g$  be weakly compatible self maps of a set  $X$ . If  $f$  and  $g$  have a unique point of coincidence  $w = f(x) = g(x)$ , then  $w$  is the unique common fixed point of  $f$  and  $g$ .

*Proof.* Since  $w = f(x) = g(x)$  and  $f$  and  $g$  are weakly compatible, we have  $f(w) = fg(x) = gf(x) = g(w)$ : i.e.,  $f(w) = g(w)$  is a point of coincidence of  $f$  and  $g$ . However  $w$  is the only point of coincidence of  $f$  and  $g$ , so  $w = f(w) = g(w)$ . Moreover if  $z = f(z) = g(z)$ , then  $z$  is a point of coincidence of  $f$  and  $g$ , and therefore  $z = w$  by uniqueness. Thus  $w$  is a unique common fixed point of  $f$  and  $g$ .  $\square$

**Theorem 31.** Let  $(X, d, E)$  be an  $E$ -complete vector metric space and we assume that  $E$  is Archimedean. Let  $f, g : X \rightarrow X$  be  $\alpha$ - $\psi_n$  mappings satisfying the following conditions;

- (i)  $f$  is  $\alpha$ -admissible;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, fx_0) \geq 1$ ;
- (iii) The range of  $g$  contains the range of  $f$  and  $g(X)$  is a  $E$ -complete subspace of  $X$ .
- (iv)  $f$  is  $g$ -continuous.
- (v)  $f$  and  $g$  are weakly compatible.

Then the common fixed point problem  $CFP(f, g, E)$  has a solution.

*Proof.* Let  $x_0 \in X$  be a arbitrary point in  $X$ . Choose a point  $x_1$  in  $X$  such that  $\alpha(x_0, x_1) \geq 1$  and  $f(x_0) = g(x_1)$ . Also for  $x_n$  there exists  $x_{n+1}$  such that  $f(x_n) = g(x_{n+1})$ . This can be done since the range of  $g$  contains the range of  $f$ . Then

$$\begin{aligned} d(g(x_n), g(x_{n+1})) &= d(f(x_{n-1}), f(x_n)) = d(f^n(x_0), f^n(x_1)) \\ &\leq \alpha(x_1, x_0) d(f^n(x_1), f^n(x_0)) \leq \psi_n(d(g(x_1), g(x_0))) \end{aligned}$$

In this case we have

$$d(g(x_m), g(x_n)) \leq \sum_{k=n}^m \psi_k(d(g(x_1), g(x_0))).$$

Then there exists  $n_0$  such that  $\sum_{k=n_0}^{\infty} \psi_k(d(g(x_0), g(x_1))) \rightarrow^o 0$ . Since  $E$  is Archimedean, for any  $m, n$  with  $m > n > n_0$ ,  $d(g(x_m), g(x_n)) \rightarrow^o 0$ ,  $d(g(x_m), g(x_n))$  is  $E$ -Cauchy in  $E$ . Hence  $(g(x_n))$  is a  $E$ -Cauchy sequence in  $g(X)$ . Since  $g(X)$  is  $E$ -complete, there exists  $q \in g(X)$  such that  $g(x_n) \rightarrow^{d,E} q$  as  $n \rightarrow \infty$ . Then there exists  $p \in X$  such that  $g(p) = q$ . Since  $f$  is  $g$ -continuous,  $g(x_n) \rightarrow^{d,E} g(p)$  implies  $f(x_n) \rightarrow^{d,E} f(p)$ . Since

$$d(g(x_{n+1}), f(p)) = d(f(x_n), f(p)),$$

we have  $g(x_n) \xrightarrow{d,E} f(p)$ . The uniqueness of limit in an ordered metric space  $E$ , we have  $f(p) = g(p)$ . From Proposition 30 and (vi),  $f$  and  $g$  have a common fixed point. □

**Corollary 32.** *Let  $(X, d, E)$  be a  $E$ -complete vector metric space with  $E$  is Archimedean. Let  $f, g : X \rightarrow X$  be two  $\alpha$ - $\psi$  mappings satisfying the same conditions in theorem 31. Then, the common fixed point problem  $CFP(f, g, E)$  has a solution.*

*Proof.* Put  $\psi_n = \psi^n$ , then  $\psi^n$  satisfies the condition of Theorem 33 and also the rest of proof is same. □

Now we give the another versions of theorem.

**Theorem 33.** *Let  $(X, d, E)$  be a  $E$ -complete vector metric space and we assume that  $E$  is Archimedean. Let  $f, g : X \rightarrow X$  be mutually  $\alpha$ - $\psi_n$  mappings satisfying the following conditions;*

- (i)  $f$  is  $\alpha$ -admissible;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, f x_0) \geq 1$ ;
- (iii) If  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in N$  and  $x_n \rightarrow x \in X$  as  $n \rightarrow \infty$ , then there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that

$$\alpha(x_{n_k}, x) \geq 1 \text{ for all } k \in N.$$

- (iv) For any  $t > 0$ ,  $\psi_1(t) < t$ .
- (v) The range of  $g$  contains the range of  $f$  and  $g(X)$  is a complete subspace of  $X$ .
- (vi)  $f$  and  $g$  are weakly compatible.

*Then, the common fixed point problem  $CFP(f, g, E)$  has a solution.*

*Proof.* Following the proof of Theorem 31, we know that the sequence  $\{g(x_n)\}$  defined in Theorem 31  $E$ -converges to some  $q \in X$  and there exists  $p \in X$  such that  $g(p) = q$ . In this case by (iii) and (iv)

$$\begin{aligned} d(g(x_{n+1}), f(p)) &= d(f(x_n), f(p)) \\ &\leq \alpha(n_k, p) d(f(x_{n_k}), f(p)) \\ &\leq \psi_1(d(g(x_{n_k}), g(p))) \end{aligned}$$

Thus  $g(x_{n+1}) \xrightarrow{d,E} f(p)$  as  $n \rightarrow \infty$ , and  $f(x_n) \xrightarrow{d,E} f(p)$  as  $n \rightarrow \infty$ . The uniqueness of a limit in an ordered metric space implies that  $f(p) = g(p)$ . From Proposition 30 and (vi),  $f$  and  $g$  have a common fixed point. □

**Corollary 34.** *Let  $(X, d, E)$  be a  $E$ -complete vector metric space with  $E$  is Archimedean. Let  $f, g : X \rightarrow X$  be two  $\alpha$ - $\psi$  mappings satisfying the same conditions in theorem 33. Then, the common fixed point problem  $CFP(f, g, E)$  has a solution.*

*Proof.* Put  $\psi_n = \psi^n$ , then  $\psi^n$  satisfies the condition of Theorem 33 and also the rest of proof is same. □

In order to take an uniqueness of coincidence point, we give the following condition.

We also give the following assumption.

- (i) Condition (H): For all  $x, y \in X$ , there exists  $z \in X$  such that  $\alpha(x, z) \geq 1$  and  $\alpha(y, z) \geq 1$ .

Then we have the following theorem.

**Theorem 35.** *Adding to the settings of Theorem 31 or Theorem 33 we assume  $\psi_1$  satisfies  $\psi_1(t)M < t$  for all  $t > 0$  and satisfies condition (H), then the coincidence point of  $f$  and  $g$  has unique common fixed point.*

*Proof.* Suppose that  $u$  and  $v$  are two different points of coincidence of  $f$  and  $g$ . From (H), there exists  $z \in X$  such that

$$(2) \quad \alpha(u, z) \geq 1 \text{ and } \alpha(v, z) \geq 1.$$

Since  $f$  is  $\alpha$ -admissible, from (2), we get

$$(3) \quad \alpha(f(u), f(z)) \geq 1 \text{ and } \alpha(f(v), f(z)) \geq 1.$$

Since  $\psi_1$  is (c)-comparison,  $\psi_1(t) < t$  for all  $t > 0$ , we have

$$\begin{aligned} d(g(u), g(z)) &= d(f(u), f(z)) \\ &\leq \alpha(u, z)d(f(u), f(z)) \leq \psi_1(d(g(u), g(z))) < d(g(u), g(z)) \end{aligned}$$

which is a contradiction. Thus  $g(u) = g(v)$ . From Proposition 30 and (ii),  $f$  and  $g$  have a unique common fixed point.  $\square$

**Remark 36.** *For the condition (iv) of theorem 33, we consider the following property.*

**Definition 37.** *A mapping  $\psi : R_+ \rightarrow R_+$  is said to be a comparison mapping if  $\psi$  satisfies:*

- (a)  $\psi$  is monotone increasing, that is,  $t_1 \leq t_2$  implies  $\psi(t_1) \leq \psi(t_2)$ .
- (b)  $(\psi^n(t))$  converges to 0 as  $n \rightarrow \infty$  for every  $t \in R_+$ .

If we replace (b) by (b')

- (b')  $\sum_{n=0}^{\infty} \psi^n(t)$  converges for all  $t \in R_+$ ,

then  $\psi$  is said to be a (c)-comparison mapping.

If a mapping  $\psi_1 : R_+ \rightarrow R_+$  is (c)-comparison, then  $\psi_1(t) < t$  for any  $t > 0$  holds.

**Example 4.** ([31, Example 2.4.]). Let  $X = [0, \infty)$  and  $E = (C([0, 1], R))$ . Define  $d : X \times X \rightarrow E$  by  $d(x, y)(t) = (|x - y|)e^t$ , where  $e^t \in E$ . Consider  $f, g : X \rightarrow X$  defined by

$$f(x) = \begin{cases} 2x - \frac{5}{3}, & \text{if } x > 1 \\ \frac{x}{3}, & \text{if } x \in [0, 1] \\ 0, & \text{if } x < 0 \end{cases}$$

$$g(x) = \begin{cases} 2x - \frac{4}{3}, & \text{if } x > 1 \\ \frac{2x}{3}, & \text{if } x \in [0, 1] \\ 0, & \text{if } x < 0 \end{cases}$$

Define

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$



Then

$$\alpha(x, y)d(f^n(x), f^n(y)) = \left(\frac{1}{3}\right)^n (x - y)e^t \leq \frac{1}{2} \left(\frac{1}{3}\right)^{n-1} \frac{2}{3}(x - y)e^t$$

Then

$$\psi_n(u) = \begin{cases} \frac{1}{2} & \text{if } n = 1, \\ \frac{1}{2} \left(\frac{1}{3}\right)^{n-1} & \text{if } n \geq 2. \end{cases}$$

Clearly  $f(X) \subseteq g(X)$  and  $f(X) \supseteq g(X)$  and  $f$  is  $g$ -continuous. Next let  $x, y \in R$  with  $\alpha(x, y) = 1$ . This implies  $x, y \in [0, 1]$ . In this case  $\frac{x}{3}, \frac{y}{3} \in [0, 1]$  thus  $\alpha(f(x), f(y)) \geq 1$ . Therefore  $f$  is  $\alpha$ -admissible.

Moreover  $x_0 = \frac{1}{2}$ , then  $f(x_0) = \frac{1}{6}$ . Thus  $\alpha(x_0, f(x_0)) \geq 1$ . Therefore the conditions of Theorem 31 are satisfied.

For more details on  $(c)$ -comparison operators, we refer to [7] and references mentioned therein.

#### 4. APPLICATIONS

We shall study sufficient condition for the existence of common solution of the following integral equations([1, 14]) in the framework of  $E$ -metric spaces.

We consider the implicit integral equation

$$(4) \quad p(t, x(t)) = \int_0^1 q(t, s, x(s))ds, t, s \in [0, 1],$$

where  $x \in L^p[0, 1]$ ,  $1 < p < \infty$ . Integral equations like (4) were introduced by Feckan [14] and could occur in the study of nonlinear boundary value problems of ordinary differential equations.

For  $E = R$  and  $X = L^1([0, 1])$ , its norm is defined by  $\|x\| = \int_0^1 |x(t)|dt$  for any  $x \in X$  and we define  $d : X \times X \rightarrow R$  by

$$d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|.$$

for any  $x, y \in X$ . Then  $d$  is a  $E$ -metric on  $X$ . Suppose that the following conditions holds:

- (i) For all  $t \in [0, 1]$ ,  $n \in N$  and  $x, y \in X$ , there exists  $\psi : [0, 1] \rightarrow R$  such that

$$\sup_{t \in [0, 1]} |p(t, x(t)) - p(t, y(t))| \leq \psi \left( \sup_{t \in [0, 1]} \left| \int_0^1 q(t, s, x(s)) - q(t, s, y(s)) \right| ds \right),$$

$$\int_0^1 \sup_{t \in [0, 1]} |q(t, s, x(s)) - q(t, s, y(s))| ds \leq \psi \left( \sup_{t \in [0, 1]} |x(t) - y(t)| \right)$$

and  $\psi$  is  $(c)$ -comparison. For instane, take  $\psi(t) = rt$ , where  $0 \leq r < 1$ .

- (ii)  $p(t, x(t)) \leq \int_0^1 q(t, s, x(s))ds \leq x(t)$  for all  $t \in [0, 1]$ .  
(iii)  $p(t, \int_0^1 q(t, s, x(s))ds) \leq \int_0^1 q(t, s, x(s))ds$  for all  $t \in [0, 1]$ .  
(iv)  $x(t) \leq y(t)$  implies  $p(t, x(t)) \leq p(t, y(t))$  for all  $t \in [0, 1]$ .

Then, the implicit integral equation (4) has a solution in  $L^1[0, 1]$ .

*Proof.* Take  $f(x(t)) = p(t, x(t))$  and  $g(x(t)) = \int_0^1 q(t, s, x(s))ds$ . Let  $r_0$  be a bounded positive number and  $M$  be the closed subset of  $L^1[0, 1]$  defined by

$$M = B_{r_0} = \{x \in L^1[0, 1] \mid \|x\| \leq r_0\}.$$

For each  $x \in M$ , since  $\int_0^1 q(t, s, x(s))ds \leq p(t, x(t)) \leq x(t)$  for all  $t \in [0, 1]$ , the range of  $f$  is contained in that of  $g$ . Also since  $\int_0^1 q(t, s, x(s))ds \leq r_0$ ,  $g(M) \subset M$  and  $g(M)$  is closed set, then  $g(M)$  is complete metric spaces by the  $L^1$ -norm. Thus we have the condition (iv) of Theorem 31. By the condition (i), we have

$$\begin{aligned} d(f(x), f(y)) &= \sup_{t \in [0, 1]} |p(t, x(t)) - p(t, y(t))| \\ &\leq \psi \left( \sup_{t \in [0, 1]} \left| \int_0^1 (q(t, s, x(s)) - q(t, s, y(s)))ds \right| \right) \\ &\leq \psi \left( \int_0^1 \sup_{t \in [0, 1]} |q(t, s, x(s)) - q(t, s, y(s))| ds \right) \\ &\leq \psi(d(g(x), g(y))) \end{aligned}$$

Also by (i), we have

$$\begin{aligned} d(f^2(x), f^2(y)) &= \sup_{t \in [0, 1]} |p(t, p(t, x(t))) - p(t, p(t, y(t)))| \\ &\leq \psi \left( \left| \int_0^1 \sup_{t \in [0, 1]} q(t, s, p(s, x(s))) ds - \int_0^1 \sup_{t \in [0, 1]} q(t, s, p(s, y(s))) ds \right| \right) \\ &\leq \psi \left( \int_0^1 \sup_{t \in [0, 1]} |q(t, s, p(s, x(s))) - q(t, s, p(s, y(s)))| ds \right) \\ &\leq \psi \left( \psi \left( \sup_{t \in [0, 1]} |p(t, x(t)) - p(t, y(t))| \right) \right) \\ &\leq \psi^2(d(g(x), g(y))) \end{aligned}$$

Then by induction, we have

$$d(f^n(x), f^n(y)) \leq \psi^n(d(g(x), g(y)))$$

Put  $\psi_n = \psi^n$ , then we have

$$d(f^n(x), f^n(y)) \leq \psi_n(d(g(x), g(y)))$$

By the definition of  $\psi$ , it satisfies (c)-comparison,  $\psi_n$  is also so.

Next let

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \geq y, \\ 0 & \text{if } x < y. \end{cases}$$

Let  $x(t) \geq y(t)$  for all  $t \in [0, 1]$ . In this case by (iv), for any  $t \in [0, 1]$ ,  $f(x(t)) \geq f(y(t))$ . Thus  $f$  is  $\alpha$ -admissible. Moreover by (ii), there exists  $x_0(t)$  such that  $x_0(t) - f(x_0(t)) = x_0(t) - p(t, x_0(t)) \geq 0$ , thus  $\alpha(x_0(t), f(x_0(t))) \geq 1$ .

By by (ii) and (iii),  $f, g$  are mutually dominated. In fact

$$f(g(x(t))) = p\left(t, \int_0^1 q(t, s, x(s))ds\right) \leq \int_0^1 q(t, s, x(s))ds = g(x(t)),$$

$$g(f(x(t))) = \int_0^1 q(t, s, p(s, x(s)))ds \leq p(t, x(t)) = f(x(t)).$$

We define a sequence  $\{x_n\}$  with  $x_{2n+1} = f(x_{2n})$  and  $x_{2n} = g(x_{2n+1})$ . Then we have

$$x_1 = f(x_0) \geq gf(x_0) = g(x_1) = x_2 = g(x_1) \geq fg(x_1) = f(x_2) = x_3 \geq \dots,$$

repeaing this arguments, we have

$$x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq \dots$$

Then the sequence  $\{x_n\}$  is decreasing and  $X$  is complete, there exists  $x \in X$  and we have  $x_n \rightarrow x$  as  $n \rightarrow \infty$ . Then for all  $n \geq 1$ , we have  $\alpha(x_n, x_{n+1}) \geq 1$ . Consider the subsequence  $x_{n_k} = x_{2n}$ , then we have  $\alpha(x_{n_k}, x) \geq 1$ . Thus the condition (iv) of Theorem 31 is satisfied.

Finally let  $f(x(t)) = g(x(t))$  for all  $t \in [0, 1]$ , then we have

$$g(f(x(t))) = \int_0^1 q(t, s, f(x(s)))ds = p(t, f(x(t))) = f(f(x(t))) = f(g(x(t))).$$

So,  $f$  and  $g$  are weakly compartible.

By using Theorem 29, the common fixed point problem  $CFP(f, g, K)$  has a solution which in turn solves the integral equation (I). □

**Remark 38.** *In examples, the decision of  $r_0$  is obtained by the following assumptions, see [3].*

*We assume that the function  $q(s, t, x(t))$  is given by*

$$q(s, t, x(t)) = \xi(s, t)h(s, y(s))$$

*where  $\xi : [0, 1] \times [0, 1] \rightarrow R$  is strongly measurable and  $\int_0^1 \xi(\cdot, s)y(s)ds \in L^1[0, 1]$  whenever  $y \in L^1[0, 1]$  and there exists a function  $\theta : [0, 1] \rightarrow R$  belonging to  $L^\infty[0, 1]$  such that  $0 \leq \xi(t, s) \leq \theta(t)$  for all  $t, s \in [0, 1] \times [0, 1]$ . The function  $f : [0, 1] \times R \rightarrow R$  is a Caratheodory function and there exist a constant  $b > 0$  and a function  $a(\cdot) \in L^1[0, 1]$  such that*

$$|h(t, u)| \leq a(t) + b|u|$$

*for all  $t \in [0, 1]$  and  $u \in R$ . Moreover,  $h(t, x(t)) \geq 0$  whenever  $x \in L_+^1[0, 1]$ , where  $L_+^1[0, 1] = \{x \in L^1[0, 1] \mid x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ ;*

In this case  $r_0$  is defined by  $r_0 = \frac{\|\theta\|_\infty \|a\|}{1 - \|b\| \|\theta\|}$ . In fact for each  $x \in M$ ,

$$\begin{aligned} \int_0^1 \|q(t, s, x(s))\| ds &= \int_0^1 \|\xi(s, t)h(s, x(s))\| ds \\ &\leq \theta(t) \int_0^1 (a(s) + bx(x)) ds \\ &\leq \|\theta\|_\infty (\|a\| + br_0) = r_0. \end{aligned}$$

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(Toshikazu Watanabe) TOKYO UNIVERSITY OF INFORMATION SCIENCES 4-1 ONARIDAI, WAKABA-KU, CHIBA, 265-8501 JAPAN

*Email address:* twatana@edu.tuis.ac.jp