# Scattering by the local perturbation of an open periodic waveguide in the half plane

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## 1 Introduction

Let $k > 0$ be the wave number, and let $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$ be the upper half plane, and let $W := \mathbb{R} \times (0, h)$ be the waveguide in $\mathbb{R}^2_+$. We denote by $\Gamma_a := \mathbb{R} \times \{a\}$ for $a > 0$. Let $n \in L^\infty(\mathbb{R}^2_+)$ be real value, $2\pi$-periodic with respect to $x_1$ (that is, $n(x_1 + 2\pi, x_2) = n(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$), and equal to one for $x_2 > h$. We assume that there exists a constant $n_0 > 0$ such that $n \geq n_0$ in $\mathbb{R}^2_+$. Let $q \in L^\infty(\mathbb{R}^2_+)$ be real valued with the compact support supp $q$ in $W$. We denote by $Q :=$ supp $q$. In this paper, we consider the following scattering problem: For fixed $y \in \mathbb{R}^2_+ \setminus \overline{W}$, determine the scattered field $u^s \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u^s + k^2(1 + q)nu^s = -k^2 qnu^i(\cdot, y) \text{ in } \mathbb{R}^2_+, \tag{1.1}$$

$$u^s = 0 \text{ on } \Gamma_0, \tag{1.2}$$

Here, the incident field $u^i$ is given by $u^i(x, y) = G_n(x, y)$, where $G_n$ is the Dirichlet Green's function in the upper half plane $\mathbb{R}^2_+$ for $\Delta + k^2 n$, that is,

$$G_n(x, y) := G(x, y) + \tilde{u}^s(x, y), \tag{1.3}$$

where $G(x, y) := \Phi_k(x, y) - \Phi_k(x, y^*)$ is the Dirichlet Green's function in $\mathbb{R}^2_+$ for $\Delta + k^2$, and $y^* = (y_1, -y_2)$ is the reflected point of $y$ at $\mathbb{R} \times \{0\}$. Here, $\Phi_k(x, y)$ is the fundamental solution to Helmholtz equation in $\mathbb{R}^2$, that is,

$$\Phi_k(x, y) := \frac{i}{4} H^{(1)}_0(k|x - y|), \ x \neq y, \tag{1.4}$$

where $H^{(1)}_0$ is the Hankel function of the first kind of order one. $\tilde{u}^s$ is the scattered field of the unperturbed problem by the incident field $G(x, y)$, that is, $\tilde{u}^s$ vanishes for $x_2 = 0$ and solves

$$\Delta \tilde{u}^s + k^2 n \tilde{u}^s = k^2(1 - n)G(\cdot, y) \text{ in } \mathbb{R}^2_+. \tag{1.5}$$

If we impose a suitable radiation condition introduced in [8], the unperturbed solution $\tilde{u}^s$ is uniquely determined. Later, we will explain the exact definition of this radiation condition (see Definition 2.4).

In order to show the well-posedness of the perturbed scattering problem (1.1)–(1.2), we make the following assumption.

**Assumption 1.1.** *We assume that $k^2$ is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ in $H_0^1(\mathbb{R}^2_+)$, that is, every $v \in H^1(\mathbb{R}^2_+)$ which satisfies*

$$\Delta v + k^2(1+q)nv = 0 \text{ in } \mathbb{R}^2_+, \tag{1.6}$$

$$v = 0 \text{ on } \Gamma_0, \tag{1.7}$$

*has to vanish for $x_2 > 0$.*

If we assume that $q$ and $n$ satisfy in addition that $\partial_2\big((1+q)n\big) \geq 0$ in $W$, then $v$ which satisfies (1.6)–(1.7) vanishes, that is, under this assumption all of $k^2$ is not the point spectrum of $\frac{1}{(1+q)n}\Delta$ (see Section 6). Our aim in this paper is to show the following theorem.

**Theorem 1.2.** *Let Assumptions 1.1 and 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3 and let $f \in L^2(\mathbb{R}^2_+)$ such that $\mathrm{supp} f = Q$. Then, there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that*

$$\Delta u + k^2(1+q)nu = f \text{ in } \mathbb{R}^2_+, \tag{1.8}$$

$$u = 0 \text{ on } \Gamma_0, \tag{1.9}$$

*and $u$ satisfies the radiation condition in the sense of Definition 2.4.*

Roughly speaking, the radiation condition of Definition 2.4 requires that we have a decomposition of the solution $u$ into $u^{(1)}$ which decays in the direction of $x_1$, and a finite combination $u^{(2)}$ of *propagative modes* which does not decay, but it exponentially decays in the direction of $x_2$.

This paper is organized as follows. In Section 2, we briefly recall a radiation condition introduced in [8]. Under the radiation condition in the sense of Definition 2.4, we show the uniqueness of $u^{(2)}$ and $u^{(1)}$ in Section 3 and 4, respectively. In Section 5, we show the existence of $u$. In Section 6, we give an example of $n$ and $q$ with respect to Assumption 1.1.

## 2 A radiation condition

In Section 2, we briefly recall a radiation condition introduced in [8]. Let $f \in L^2(\mathbb{R}^2_+)$ have the compact support in $W$. First, we consider the following problem: Find $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that

$$\Delta u + k^2 nu = f \text{ in } \mathbb{R}^2_+, \tag{2.1}$$

$$u = 0 \text{ on } \Gamma_0. \tag{2.2}$$

(2.1) is understood in the variational sense, that is,

$$\int_{\mathbb{R}^2_+} \left[\nabla u \cdot \nabla \overline{\varphi} - k^2 nu\overline{\varphi}\right] dx = -\int_W f\overline{\varphi} dx, \tag{2.3}$$

for all $\varphi \in H^1(\mathbb{R}^2_+)$, with compact support. In such a problem, it is natural to impose the *upward propagating radiation condition*, that is, $u(\cdot, h) \in L^\infty(\mathbb{R})$ and

$$u(x) = 2\int_{\Gamma_h} u(y)\frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y) = 0, \ x_2 > h. \tag{2.4}$$

However, even with this condition we can not expect the uniqueness of this problem. (see Example 2.3 of [8].) In order to introduce a *suitable radiation condition*, [8] discussed limiting absorption solution of this problem, that is, the limit of the solution $u_\epsilon$ of $\Delta u_\epsilon + (k+i\epsilon)^2 n u_\epsilon = f$ as $\epsilon \to 0$. For the details of an introduction of this radiation condition, we refer to [5, 6, 7, 8].

Let us prepare for the exact definition of the radiation condition. First we recall that the *Floquet Bloch transform* $T_{per} : L^2(\mathbb{R}) \to L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ is defined by

$$T_{per}f(t,\alpha) = \tilde{f}_\alpha(t) := \sum_{m\in\mathbb{Z}} f(t+2\pi m)e^{-i\alpha(t+2\pi m)}, \tag{2.5}$$

for $(t,\alpha)\in(0,2\pi)\times(-1/2,1/2)$. The inverse transform is given by

$$T_{per}^{-1}g(t) = \int_{-1/2}^{1/2} g(t,\alpha)e^{i\alpha t}d\alpha,\ t\in\mathbb{R}. \tag{2.6}$$

By taking the Floquet Bloch transform with respect to $x_1$ in (2.1)–(2.2), we have for $\alpha\in(-1/2,1/2]$

$$\Delta\tilde{u}_\alpha + 2i\alpha\frac{\partial\tilde{u}_\alpha}{\partial x_1} + (k^2 n - \alpha^2)\tilde{u}_\alpha = \tilde{f}_\alpha \text{ in } (0,2\pi)\times(0,\infty). \tag{2.7}$$

$$\tilde{u}_\alpha = 0 \text{ on } (0,2\pi)\times\{0\}. \tag{2.8}$$

By taking the Floquet Bloch transform with respect to $x_1$ in (2.4), $\tilde{u}_\alpha$ satisfies the *Rayleigh expansion* of the form

$$\tilde{u}_\alpha(x) = \sum_{n\in\mathbb{Z}} u_n(\alpha)e^{inx_1 + i\sqrt{k^2-(n+\alpha)^2}(x_2-h)},\ x_2 > h, \tag{2.9}$$

where $u_n(\alpha) := (2\pi)^{-1}\int_0^{2\pi} u_\alpha(x_1,h)e^{-inx_1}dx_1$ are the Fourier coefficients of $u_\alpha(\cdot,h)$, and $\sqrt{k^2-(n+\alpha)^2} = i\sqrt{(n+\alpha)^2-k^2}$ if $n+\alpha > k$.

We denote by $C_R := (0,2\pi)\times(0,R)$ for $R\in(0,\infty]$, and $H^1_{per}(C_R)$ the subspace of the $2\pi$-periodic function in $H^1(C_R)$. We also denote by $H^1_{0,per}(C_R) := \{u\in H^1_{per}(C_R) : u = 0 \text{ on } (0,2\pi)\times\{0\}\}$ that is equipped with $H^1(C_R)$ norm. The space $H^1_{0,per}(C_R)$ has the inner product of the form

$$\langle u,v\rangle_* = \int_{C_h}\nabla u\cdot\nabla\overline{v}dx + 2\pi\sum_{n\in\mathbb{Z}}\sqrt{n^2+1}u_n\overline{v_n}, \tag{2.10}$$

where $u_n = (2\pi)^{-1}\int_0^{2\pi} u(x_1,R)e^{-inx_1}dx_1$. The problem (2.7)–(2.9) is equivalent to the following operator equation (see section 3 in [8]),

$$\tilde{u}_\alpha - K_\alpha\tilde{u}_\alpha = \tilde{f}_\alpha \text{ in } H^1_{0,per}(C_h), \tag{2.11}$$

where the operator $K_\alpha : H^1_{0,per}(C_h)\to H^1_{0,per}(C_h)$ is defined by

$$\begin{aligned}\langle K_\alpha u, v\rangle_* \;&=\; -\int_{C_h}\left[i\alpha\left(u\frac{\partial\overline{v}}{\partial x_1} - \overline{v}\frac{\partial\overline{u}}{\partial x_1}\right) + (\alpha^2 - k^2 n)u\overline{v}\right]dx \\ &+\; 2\pi i\sum_{|n+\alpha|\le k} u_n\overline{v_n}\big(\sqrt{k^2-(n+\alpha)^2} - i\sqrt{n^2+1}\big) \\ &+\; 2\pi\sum_{|n+\alpha|>k} u_n\overline{v_n}\big(\sqrt{n^2+1} - \sqrt{(n+\alpha)^2-k^2}\big).\end{aligned} \tag{2.12}$$

For several $\alpha \in (-1/2, 1/2]$, the uniqueness of this problem fails. We call these $\alpha$ *exceptional values* if the operator $I - K_\alpha$ fails to be injective. For the difficulty of treatment of $\alpha$ such that $|\alpha + l| = k$ for some $l \in \mathbb{Z}$ in an periodic scattering problem, we set $A_k := \{\alpha \in (-1/2, 1/2] : \exists l \in \mathbb{Z}\ s.t.\ |\alpha + l| = k\}$, and make the following assumption:

**Assumption 2.1.** *For every $\alpha \in A_k$, $I - K_\alpha$ has to be injective.*

The following properties of exceptional values was shown in Lemmas 4.2 and 5.6 of [8].

**Lemma 2.2.** *Let Assumption 2.1 hold. Then, there exists only finitely many exceptional values $\alpha \in (-1/2, 1/2]$. Furthermore, if $\alpha$ is an exceptional value, then so is $-\alpha$. Therefore, the set of exceptional values can be described by $\{\alpha_j : j \in J\}$ where some $J \subset \mathbb{Z}$ is finite and $\alpha_{-j} = -\alpha_j$ for $j \in J$. For each exceptional value $\alpha_j$ we define*

$$X_j := \left\{ \phi \in H^1_{loc}(\mathbb{R}^2_+) : \begin{array}{c} \Delta\phi + 2i\alpha_j \frac{\partial \phi}{\partial x_1} + (k^2 n - \alpha^2)\phi = 0 \text{ in } \mathbb{R}^2_+, \\ \phi = 0 \text{ for } x_2 = 0, \quad \phi \text{ is } 2\pi\text{−periodic for } x_1, \\ \phi \text{ satisfies the Rayleigh expansion (2.9)} \end{array} \right\}$$

*Then, $X_j$ are finite dimensional. We set $m_j = \mathrm{dim} X_j$. Furthermore, $\phi \in X_j$ is evanescent, that is, there exists $c > 0$ and $\delta > 0$ such that $|\phi(x)|,\ |\nabla\phi(x)| \leq c e^{-\delta |x_2|}$ for all $x \in \mathbb{R}^2_+$.*

Next, we consider the following eigenvalue problem in $X_j$: Determine $d \in \mathbb{R}$ and $\phi \in X_j$ such that

$$\int_{C_\infty} \left[ -i \frac{\partial \phi}{\partial x_1} + \alpha_j \phi \right] \overline{\psi} dx = dk \int_{C_\infty} n \phi \overline{\psi} dx, \tag{2.13}$$

for all $\psi \in X_j$. We denote by the eigenvalues $d_{l,j}$ and the eigenfunction $\phi_{l,j}$ of this problem, that is,

$$\int_{C_\infty} \left[ -i \frac{\partial \phi_{l,j}}{\partial x_1} + \alpha_j \phi_{l,j} \right] \overline{\psi} dx = d_{l,j} k \int_{C_\infty} n \phi_{l,j} \overline{\psi} dx, \tag{2.14}$$

for every $l = 1, ..., m_j$ and $j \in J$. We normalize the eigenfunction $\{\phi_{l,j} : l = 1, ..., m_j\}$ such that

$$k \int_{C_\infty} n \phi_{l,j} \overline{\phi_{l',j}} dx = \delta_{l,l'}, \tag{2.15}$$

for all $l, l'$. We will assume that the wave number $k > 0$ is *regular* in the following sense.

**Definition 2.3.** $k > 0$ is *regular* if $d_{l,j} \neq 0$ for all $l = 1, ...m_j$ and $j \in J$.

Now we are ready to define the radiation condition.

**Definition 2.4.** Let Assumptions 2.1 hold, and let $k > 0$ be regular in the sense of Definition 2.3. We set

$$\psi^{\pm}(x_1) := \frac{1}{2} \left[ 1 \pm \frac{2}{\pi} \int_0^{x_1/2} \frac{sint}{t} dt \right],\ x_1 \in \mathbb{R}. \tag{2.16}$$

Then, $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies the *radiation condition* if $u$ satisfies the upward propagating radiation condition (2.4), and has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}|_{\mathbb{R}\times(0,R)} \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$, and $u^{(2)} \in L^\infty(\mathbb{R}^2_+)$ has the following form

$$u^{(2)}(x) = \psi^+(x_1) \sum_{j \in J} \sum_{d_{l,j} > 0} a_{l,j} \phi_{l,j}(x) + \psi^-(x_1) \sum_{j \in J} \sum_{d_{l,j} < 0} a_{l,j} \phi_{l,j}(x) \tag{2.17}$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (2.8).

**Remark 2.5.** We can replace $\psi^{\pm}$ by any smooth functions $\tilde{\psi}^{\pm}$ such that $\left|\psi^{\pm}(x_1) - \tilde{\psi}^{\pm}(x_1)\right| \to 0$, and $\left|\frac{d}{dx_1}\psi^{\pm}(x_1) - \frac{d}{dx_1}\tilde{\psi}^{\pm}(x_1)\right| \to 0$ as $|x_1| \to \infty$ because (2.12) is of the form

$$u^{(2)}(x) = \tilde{\psi}^{+}(x_1)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \tilde{\psi}^{-}(x_1)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x)$$
$$+ \Big(\psi^{+}(x_1) - \tilde{\psi}^{+}(x_1)\Big)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \Big(\psi^{-}(x_1) - \tilde{\psi}^{-}(x_1)\Big)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x), \tag{2.18}$$

where the second term in the right-hand side of (2.13) is a $H^1$-function, which is the role of $u^{(1)}$.

The following was shown in Theorems 2.2, 6.6, and 6.8 of [8].

**Theorem 2.6.** *Let Assumptions 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3. For every $f \in L^2(\mathbb{R}^2_+)$ with the compact support in $W$, there exists a unique solution $u_{k+i\epsilon} \in H^1(\mathbb{R}^2_+)$ of the problem (2.1)–(2.2) replacing $k$ by $k+i\epsilon$. Furthermore, $u_{k+i\epsilon}$ converge as $\epsilon \to +0$ in $H^1_{loc}(\mathbb{R}^2_+)$ to some $u \in H^1_{loc}(\mathbb{R}^2_+)$ which satisfy (2.1)–(2.2) and the radiation condition in the sense of Definition 2.4. Furthermore, the solution $u$ of this problem is uniquely determined.*

Finally in this section, we will show the following integral representation.

**Lemma 2.7.** *Let $f \in L^2(\mathbb{R}^2_+)$ have a compact support in $W$, and let $u$ be a solution of (2.1)–(2.2) which satisfying the radiation condition in the sense of Definition 2.4. Then, $u$ has an integral representation of the form*

$$u(x) = k^2 \int_W (n(y) - 1)u(y)G(x, y)dy - \int_W f(y)G(x, y)dy, \quad x \in \mathbb{R}^2_+ \tag{2.19}$$

*Proof of Lemma 2.7.* Let $\epsilon > 0$ be small enough and let $u_\epsilon \in H^1(\mathbb{R}^2_+)$ be a solution of the problem (2.1)–(2.2) replacing $k$ by $k + i\epsilon$, that is, $u_\epsilon$ satisfies

$$\Delta u_\epsilon + (k + i\epsilon)^2 n u_\epsilon = f \text{ in } \mathbb{R}^2_+, \tag{2.20}$$

$$u_\epsilon = 0 \text{ on } \Gamma_0. \tag{2.21}$$

Let $G_\epsilon(x, y)$ be the Dirichlet Green's function in the upper half plane $\mathbb{R}^2_+$ for $\Delta + (k + i\epsilon)^2$. Let $x \in \mathbb{R}^2_+$ be always fixed such that $x_2 > R$. Let $r > 0$ be large enough such that $x \in B_r(0)$ where $B_r(0) \subset \mathbb{R}^2$ be a open ball with center 0 and radius $r > 0$. By Green's representation theorem in $B_r(0) \cap \mathbb{R}^2_+$ we have

$$\begin{aligned} u_\epsilon(x) &= \int_{\partial B_r(0)\cap\mathbb{R}^2_+} \big[\frac{\partial u_\epsilon}{\partial \nu}(y)G_\epsilon(x, y) - u_\epsilon(y)\frac{\partial G_\epsilon}{\partial \nu}(x, y)\big]ds(y) \\ &- \int_{B_r(0)\cap\mathbb{R}^2_+} \big[\Delta u_\epsilon(y) + (k + i\epsilon)^2 u_\epsilon(y)\big]G_\epsilon(x, y)dy \\ &= \int_{\partial B_r(0)\cap\mathbb{R}^2_+} \big[\frac{\partial u_\epsilon}{\partial \nu}(y)G_\epsilon(x, y) - u_\epsilon(y)\frac{\partial G_\epsilon}{\partial \nu}(x, y)\big]ds(y) \\ &+ (k + i\epsilon)^2 \int_{B_r(0)\cap\mathbb{R}^2_+} (n(y) - 1)u_\epsilon(y)G_\epsilon(x, y)dy \\ &- \int_{B_r(0)\cap\mathbb{R}^2_+} f(y)G_\epsilon(x, y)dy. \end{aligned} \tag{2.22}$$

Since $u_\epsilon \in H^1(\mathbb{R}^2_+)$, the first term of the right hand side converges to zero as $r \to \infty$. Therefore, as $r \to \infty$ we have for $x \in \mathbb{R}^2_+$

$$u_\epsilon(x) = (k+i\epsilon)^2 \int_W (n(y)-1)u_\epsilon(y)G_\epsilon(x,y)dy - \int_W f(y)G_\epsilon(x,y)dy. \tag{2.23}$$

We will show that (2.23) converges as $\epsilon \to 0$ to

$$u(x) = k^2 \int_W (n(y)-1)u(y)G(x,y)dy - \int_W f(y)G(x,y)dy. \tag{2.24}$$

Indeed, by the argument in (3.8) and (3.9) of [2], $G_\epsilon(x,y)$ is of the estimation

$$|G_\epsilon(x,y)| \leq C\frac{x_2 y_2}{1+|x-y|^{3/2}},\ |x-y|>1, \tag{2.25}$$

where above $C$ is independent of $\epsilon > 0$. Then, by Lebesgue dominated convergence theorem we have the second integral in (2.23) converges as $\epsilon \to 0$ to one in (2.24). So, we will consider the convergence of the first integral in (2.23).

By the beginning of the proof of Theorem 6.6 in [8], $u_\epsilon$ can be of the form $u_\epsilon = u_\epsilon^{(1)} + u_\epsilon^{(2)}$ where $u_\epsilon^{(1)}$ converges to $u^{(1)}$ in $H^1(W)$, and $u_\epsilon^{(2)}$ is of the form for $x \in W$

$$u_\epsilon^{(2)}(x) = \sum_{j\in J}\sum_{l=1}^{m_j} y_{l,j} \int_{-1/2}^{1/2} \frac{e^{i\alpha x_1}}{i\epsilon - d_{l,j}\alpha} d\alpha\ \phi_{l,j}(x), \tag{2.26}$$

which converges pointwise to $u^{(2)}(x)$. Here, $y_{l,j} \in \mathbb{C}$ is some constant. From the convergence of $u_\epsilon^{(1)}$ in $H^1(W)$ we obtain that $\int_W (n(y)-1)u_\epsilon^{(1)}(y)G_\epsilon(x,y)dy$ converges $\int_W (n(y)-1)u^{(1)}(y)G(x,y)dy$ as $\epsilon \to 0$.

By the argument of (b) in Lemma 6.1 of [8] we have

$$\begin{aligned}\psi_{l,j,\epsilon}(x_1) &:= \int_{-1/2}^{1/2} \frac{e^{i\alpha x_1}}{i\epsilon - d_{l,j}\alpha} d\alpha \\ &= -\frac{i}{|d_{l,j}|}\int_{-|d_{l,j}|/(2\epsilon)}^{|d_{l,j}|/(2\epsilon)} \frac{\cos(t\epsilon x_1/|d_{l,j}|)}{1+t^2}dt - 2id_{l,j}\int_0^{x_1/2} \frac{t\mathrm{sin}t}{x_1^2\epsilon^2 + d_{l,j}^2 t^2}dt,\end{aligned} \tag{2.27}$$

which implies that for all $x_1 \in \mathbb{R}$

$$\begin{aligned}\bigl|\psi_{l,j,\epsilon}(x_1)\bigr| &\leq C\Bigl(\int_{-\infty}^{\infty}\frac{dt}{1+t^2} + \int_0^{|x_1|/2}\Bigl|\frac{\mathrm{sin}t}{t}\Bigr|dt\Bigr) \\ &\leq C\Bigl(\int_{-\infty}^{\infty}\frac{dt}{1+t^2}dt + \int_0^1\Bigl|\frac{\mathrm{sin}t}{t}\Bigr|dt + \int_1^{|x_1|+1}\frac{1}{t}dt\Bigr) \\ &\leq C\bigl(1+\log(|x_1|+1)\bigr),\end{aligned} \tag{2.28}$$

where above $C$ is independent of $\epsilon > 0$. Then, we have that for $y \in W$

$$\bigl|(n(y)-1)u_\epsilon^{(2)}(y)G_\epsilon(x,y)\bigr| \leq \frac{C\bigl(1+\log(|y_1|+1)\bigr)}{1+|x-y|^{3/2}}, \tag{2.29}$$

where above $C$ is independent of $y$ and $\epsilon$. Then, right hand side of (2.29) is an integrable function in $W$ with respect to $y$. Then, by Lebesgue dominated convergence theorem $\int_W (n(y)-1)u_\epsilon^{(2)}(y)G_\epsilon(x,y)dy$ converges to $\int_W (n(y)-1)u^{(2)}(y)G(x,y)dy$ as $\epsilon \to 0$. Therefore, (2.24) has been shown. □

# 3 Uniqueness of $u^{(2)}$

In Section 3, we will show the uniqueness of $u^{(2)}$ in Theorem 1.2.

**Lemma 3.1.** *Let Assumptions 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3. If $u \in H^1_{loc}(\mathbb{R}^2_+)$ such that*

$$\Delta u + k^2(1+q)nu = 0, \text{ in } \mathbb{R}^2_+, \tag{3.1}$$

$$u = 0 \text{ on } \Gamma_0, \tag{3.2}$$

*and $u$ satisfies the radiation condition in the sense of Definition 2.4, then $u^{(2)} = 0$ in $\mathbb{R}^2_+$.*

**Proof of Lemma 3.1.** By the definition of the radiation condition, $u$ is of the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)}\big|_{\mathbb{R}\times(0,R)} \in H^1(\mathbb{R}\times(0,R))$ for all $R > 0$, and $u^{(2)} \in L^\infty(\mathbb{R}^2_+)$ has the form

$$u^{(2)}(x) = \psi^+(x_1)\sum_{j\in J}\sum_{d_{l,j}>0} a_{l,j}\phi_{l,j}(x) + \psi^-(x_1)\sum_{j\in J}\sum_{d_{l,j}<0} a_{l,j}\phi_{l,j}(x), \tag{3.3}$$

where some $a_{l,j} \in \mathbb{C}$, and $\{d_{l,j}, \phi_{l,j} : l = 1, ..., m_j\}$ are normalized eigenvalues and eigenfunctions of the problem (2.13). Here, by Remark 2.5 the function $\psi^+$ is chosen as a smooth function such that $\psi^+(x_1) = 1$ for $x_1 \geq \eta$ and $\psi^+(x_1) = 0$ for $x_1 \leq -\eta$, and $\psi^- := 1 - \psi^+$ where $\eta > 0$ is some positive number.

**Step1** (Green's theorem in $\Omega_N$): We set $\Omega_N := (-N, N) \times (0, \phi(N))$ where $\psi(N) := N^s$. Later we will choose a appropriate $s \in (0, 1)$. Let $R > h$ be large and always fixed, and let $N$ be large enough such that $\phi(N) > R$. We denote by $I^R_{\pm N} := \{\pm N\} \times (0, R)$, $I^{\phi(N)}_{\pm N} := \{\pm N\} \times (R, \phi(N))$, and $\Gamma_{\phi(N),N} := (-N, N) \times \{\phi(N)\}$. (see the figure below.) We set $I_{\pm N} := I^R_{\pm N} \cup I^{\phi(N)}_{\pm N}$.

By Green's first theorem in $\Omega_N$ and $u = 0$ on $(-N, N) \times \{0\}$, we have

$$\begin{aligned}
\int_{\Omega_N} \{-k^2(1+q)n|u|^2 + |\nabla u|^2\}dx &= \int_{\Omega_N} \{\overline{u}\Delta u + |\nabla u|^2\}dx \\
&= \int_{I_N} \overline{u}\frac{\partial u}{\partial x_1}ds - \int_{I_{-N}} \overline{u}\frac{\partial u}{\partial x_1}ds + \int_{\Gamma_{\phi(N),N}} \overline{u}\frac{\partial u}{\partial x_2}ds \\
&= \int_{I_N} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_1}ds - \int_{I_{-N}} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_1}ds
\end{aligned}$$

$$\begin{aligned}
&+ \int_{I_N} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_1}ds + \int_{I_N} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_1}ds + \int_{I_N} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_1}ds \\
&- \int_{I_{-N}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_1}ds - \int_{I_{-N}} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_1}ds - \int_{I_{-N}} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_1}ds \\
&+ \int_{\Gamma_{\phi(N),N}} \overline{u}\frac{\partial u}{\partial x_2}ds.
\end{aligned} \tag{3.4}$$

By the same argument in Theorem 4.6 of [7] and Lemma 6.3 of [8], we can show that

$$\begin{aligned}
&\int_{I_N} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_1}ds - \int_{I_{-N}} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_1}ds \\
+ &\int_{I_N^R} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_1}ds + \int_{I_N^R} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_1}ds + \int_{I_N^R} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_1}ds \\
- &\int_{I_{-N}^R} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_1}ds - \int_{I_{-N}^R} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_1}ds - \int_{I_{-N}^R} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_1}ds \\
= &\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}>0}\overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}}\overline{\phi_{l,j}}\frac{\partial \phi_{l',j}}{\partial x_1}dx \\
- &\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}<0}\overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}}\overline{\phi_{l,j}}\frac{\partial \phi_{l',j}}{\partial x_1}dx + o(1),
\end{aligned} \tag{3.5}$$

and the first and second term in the right hand side converge as $N \to \infty$ to $\frac{ik}{2\pi}\sum_{j\in J}\sum_{d_{l,j}>0}|a_{l,j}|^2 d_{l,j}$ and $-\frac{ik}{2\pi}\sum_{j\in J}\sum_{d_{l,j}<0}|a_{l,j}|^2 d_{l,j}$ respectively. Therefore, taking an imaginary part in (3.4) yields that

$$\begin{aligned}
0 = &\mathrm{Im}\left[\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}>0}\overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}}\overline{\phi_{l,j}}\frac{\partial \phi_{l',j}}{\partial x_1}dx\right] \\
- &\mathrm{Im}\left[\frac{1}{2\pi}\sum_{j\in J}\sum_{d_{l,j},d_{l',j}<0}\overline{a_{l,j}}a_{l',j}\int_{C_{\phi(N)}}\overline{\phi_{l,j}}\frac{\partial \phi_{l',j}}{\partial x_1}dx\right] \\
+ &\mathrm{Im}\int_{I_N^{\phi(N)}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_1}ds + \mathrm{Im}\int_{I_N^{\phi(N)}} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_1}ds + \mathrm{Im}\int_{I_N^{\phi(N)}} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_1}ds \\
- &\mathrm{Im}\int_{I_{-N}^{\phi(N)}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_1}ds - \mathrm{Im}\int_{I_{-N}^{\phi(N)}} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_1}ds - \mathrm{Im}\int_{I_{-N}^{\phi(N)}} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_1}ds \\
+ &\mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u}\frac{\partial u}{\partial x_2}ds + o(1).
\end{aligned} \tag{3.6}$$

We set

$$J_{\pm}(N) := \pm\mathrm{Im}\int_{I_{\pm N}^{\phi(N)}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_1}ds \pm \mathrm{Im}\int_{I_{\pm N}^{\phi(N)}} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_1}ds \pm \mathrm{Im}\int_{I_{\pm N}^{\phi(N)}} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_1}ds, \tag{3.7}$$

and we will show that $\mathrm{limsup}_{\mathrm{N}\to\infty} J_{\pm}(N) \geq 0$.

**Step2** ($\limsup_{N\to\infty} J_\pm(N) \geq 0$)**:** By the Cauchy Schwarz inequality we have

$$
\begin{aligned}
|J_+(N)| \leq & \Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2\Big)^{1/2} \Big(\int_R^{\phi(N)} \Big|\frac{\partial u^{(1)}}{\partial x_1}(N,x_2)\Big|^2 dx_2\Big)^{1/2} \\
+ & \Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2\Big)^{1/2} \Big(\int_R^{\phi(N)} \Big|\frac{\partial u^{(2)}}{\partial x_1}(N,x_2)\Big|^2 dx_2\Big)^{1/2} \\
+ & \Big(\int_R^{\phi(N)} |u^{(2)}(N,x_2)|^2 dx_2\Big)^{1/2} \Big(\int_R^{\phi(N)} \Big|\frac{\partial u^{(1)}}{\partial x_1}(N,x_2)\Big|^2 dx_2\Big)^{1/2} \\
\leq & \Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2\Big)^{1/2} \Big(\int_R^{\phi(N)} \Big|\frac{\partial u^{(1)}}{\partial x_1}(N,x_2)\Big|^2 dx_2\Big)^{1/2} \\
+ & C(\phi(N)-R)^{1/2}\Big(\int_R^{\phi(N)} |u^{(1)}(N,x_2)|^2 dx_2\Big)^{1/2} \\
+ & C(\phi(N)-R)^{1/2}\Big(\int_R^{\phi(N)} \Big|\frac{\partial u^{(1)}}{\partial x_1}(N,x_2)\Big|^2 dx_2\Big)^{1/2}. \qquad (3.8)
\end{aligned}
$$

In order to estimate $u^{(1)}$, we will show the following lemma.

**Lemma 3.2.** $u^{(1)}$ *has an integral representation of the form*

$$
\begin{aligned}
u^{(1)}(x) &= \int_{y_2>0} \sigma(y)G(x,y)dy + k^2 \int_W \big(n(y)(1+q(y))-1\big)u^{(1)}(y)G(x,y)dy \\
&+ k^2 \int_Q n(y)q(y)u^{(2)}(y)G(x,y)dy, \quad x_2>0, \qquad (3.9)
\end{aligned}
$$

*where* $\sigma := \Delta u^{(2)} + k^2 n u^{(2)}$.

*Proof of Lemma 3.2.* First, we will consider an integral representation of $u^{(2)}$. Let $N>0$ be large enough. By Green's representation theorem in $(-N,N)\times(0,N^{1/4})$, we have

$$
\begin{aligned}
u^{(2)}(x) &= \int_{(-N,N)\times\{N^{1/4}\}} \big[u^{(2)}(y)\frac{\partial G}{\partial y_2}(x,y) - G(x,y)\frac{\partial u^{(2)}}{\partial y_2}(y)\big]ds(y) \\
&+ \Big(\int_{\{N\}\times(0,N^{1/4})} - \int_{\{-N\}\times(0,N^{1/4})}\Big)\big[u^{(2)}(y)\frac{\partial G}{\partial y_1}(x,y) - G(x,y)\frac{\partial u^{(2)}}{\partial y_1}(y)\big]ds(y) \\
&- \int_{(-N,N)\times(0,N^{1/4})} \big[\sigma(y) + k^2(1-n(y))u^{(2)}(y)\big]G(x,y)dy. \qquad (3.10)
\end{aligned}
$$

By Lemma 3.1 of [2], the Dirichlet Green's function $G(x,y)$ is of the estimation

$$
|G(x,y)|,\ |\nabla_y G(x,y)| \leq C\frac{x_2 y_2}{1+|x-y|^{3/2}},\ |x-y|>1. \qquad (3.11)
$$

By Lemma 2.2 we have that $|u^{(2)}(x)|,\ \big|\frac{\partial u^{(2)}(x)}{\partial x_2}\big| \leq ce^{-\delta|x_2|}$ for all $x\in\mathbb{R}^2_+$, and some $c,\delta>0$. Then, we obtain

$$
\begin{aligned}
&\Big|\int_{(-N,N)\times\{N^{1/4}\}} \big[u^{(2)}(y)\frac{\partial G}{\partial y_2}(x,y) - G(x,y)\frac{\partial u^{(2)}}{\partial y_2}(y)\big]ds(y)\Big| \\
&\leq C\int_{-N}^{N} \frac{x_2 e^{-\delta N^{1/4}}}{|N^{1/4}-x_2|^{3/2}} dy_2 \leq C\frac{x_2 N e^{-\delta N^{1/4}}}{|N^{1/4}-x_2|^{3/2}}. \qquad (3.12)
\end{aligned}
$$

Furthermore,

$$\begin{aligned}&\left|\int_{\{\pm N\}\times(0,N^{1/4})}\big[u^{(2)}(y)\frac{\partial G}{\partial y_1}(x,y)-G(x,y)\frac{\partial u^{(2)}}{\partial y_1}(y)\big]ds(y)\right|\\ &\leq\ C\int_0^{N^{1/4}}\frac{x_2y_2}{|\pm N-x_1|^{3/2}}dy_2\leq C\frac{x_2N^{1/2}}{|\pm N-x_1|^{3/2}}.\end{aligned}\tag{3.13}$$

Therefore, as $N\to\infty$ in (3.10) we get

$$u^{(2)}(x)=-\int_{y_2>0}\sigma(y)G(x,y)dy+k^2\int_W(n(y)-1)u^{(2)}(y)G(x,y)dy.\tag{3.14}$$

By Lemma 2.7, we have (substitute $-k^2qnu$ for $f$ in (2.19))

$$u(x)=k^2\int_W\big(n(y)-1\big)u(y)G(x,y)dy+k^2\int_Q q(y)n(y)u(y)G(x,y)dy.\tag{3.15}$$

Combining (3.14) with (3.15) we have

$$\begin{aligned}u^{(1)}(x)\ &=\ -u^{(2)}(x)+k^2\int_W\big(n(y)-1\big)u(y)G(x,y)dy+k^2\int_Q q(y)n(y)u(y)G(x,y)dy\\ &=\ \int_{y_2>0}\sigma(y)G(x,y)dy-k^2\int_W(n(y)-1)u^{(2)}(y)G(x,y)dy\\ &+\ k^2\int_W\big(n(y)-1\big)u(y)G(x,y)dy+k^2\int_Q q(y)n(y)u(y)G(x,y)dy\\ &=\ \int_{\mathbb{R}^2_+}\sigma(y)G(x,y)dy+k^2\int_W\big(n(y)(1+q(y))-1\big)u^{(1)}(y)G(x,y)dy\\ &+\ k^2\int_Q n(y)q(y)u^{(2)}(y)G(x,y)dy.\end{aligned}\tag{3.16}$$

Therefore, Lemma 3.2 has been shown. □

We set $u^{\pm}(x):=\sum_{j\in J}\sum_{d_{l,j}\lesseqgtr 0}a_{l,j}\phi_{l,j}(x)$. Then, by a simple calculation we can show

$$\sigma(y)=\frac{d^2\psi^+(y_1)}{dy_1^2}u^+(y)+2\frac{d\psi^+(y_1)}{dy_1}\frac{\partial u^+(y)}{\partial y_1}+\frac{d^2\psi^-(y_1)}{dy_1^2}u^-(y)+2\frac{d\psi^-(y_1)}{dy_1}\frac{\partial u^-(y)}{\partial y_1},\tag{3.17}$$

which implies that supp $\sigma\subset(-\eta,\eta)\times(0,\infty)$. By Lemma 3.2 we have for $R<x_2<\phi(N)$

$$\begin{aligned}&|u^{(1)}(N,x_2)|,\ \left|\frac{\partial u^{(1)}}{\partial x_1}(N,x_2)\right|\leq C\int_{(-\eta,\eta)\times(0,\infty)}|\sigma(y)|\frac{\phi(N)y_2}{|N-\eta|^{3/2}}dy\\ &+\ C\int_W|u^{(1)}(y)|\frac{\phi(N)h}{(1+|N-y_1|)^{3/2}}dy+C\int_Q\frac{\phi(N)|u^{(2)}(y)|}{|N-y_1|^{3/2}}dy\\ &\leq\ C\frac{\phi(N)}{N^{3/2}}+C\phi(N)\int_W\frac{|u^{(1)}(y)|}{(1+|N-y_1|)^{3/2}}dy.\end{aligned}\tag{3.18}$$

We have to estimate the second term in right hand side. The following lemma was shown in Lemma 4.12 of [1].

**Lemma 3.3.** *Assume that $\varphi \in L^2_{loc}(\mathbb{R})$ such that*

$$\sup_{A>0}\Big\{(1+A^2)^{-\epsilon}\int_{-A}^{A}|\varphi(t)|^2dt\Big\}<\infty, \tag{3.19}$$

*for some $\epsilon>0$. Then, for every $\alpha\in[0,\frac{1}{2}-\epsilon)$ there exists a constant $C>0$ and a sequence $\{A_m\}_{m\in\mathbb{N}}$ such that $A_m\to\infty$ as $m\to\infty$ and*

$$\int_{K_{A_m}}|\varphi(t)|^2dt\le CA_m^{-\alpha},\ m\in\mathbb{N}, \tag{3.20}$$

*where $K_A:=K_A^+\cup K_A^-$, $K_A^+:=(-A^+,A^+)\setminus(-A,A)$, $K_A^-:=(-A,A)\setminus(-A^-,A^-)$, and $A^\pm:=A\pm A^{1/2}$ for $A\in[1,\infty)$.*

Applying Lemma 3.3 to $\varphi=\big(\int_0^h\big|u^{(1)}(\cdot,y_2)\big|^2dy_2\big)^{1/2}\in L^2(\mathbb{R})$, there exists a sequence $\{N_m\}_{m\in\mathbb{N}}$ such that $N_m\to\infty$ as $m\to\infty$ and

$$\int_{K_{N_m}}\int_0^h|u^{(1)}(y_1,y_2)|^2dy_1dy_2\le CN_m^{-1/4},\ m\in\mathbb{N}. \tag{3.21}$$

Then, by the Cauchy Schwarz inequality we have

$$\begin{aligned}\int_W\frac{|u^{(1)}(y)|}{(1+|N-y_1|)^{3/2}}dy&=\Big(\int_{-N_m^-}^{N_m^-}+\int_{K_{N_m}}+\int_{\mathbb{R}\setminus[-N_m^+,N_m^+]}\Big)\int_0^h\frac{|u^{(1)}(y)|}{(1+|N_m-y_1|)^{3/2}}dy\\ &\le C\Big(\int_{-N_m^-}^{N_m^-}\frac{dy_1}{(1+N_m-|y_1|)^3}\Big)^{1/2}+C\Big(\int_{K_{N_m}}\int_0^h|u^{(1)}(y_1,y_2)|^2dy_1dy_2\Big)^{1/2}\\ &+C\Big(\int_{\mathbb{R}\setminus[-N_m^+,N_m^+]}\frac{dy_1}{(1+|y_1|-N_m)^3}\Big)^{1/2}\\ &\le C\Big(\int_0^{N_m^-}\frac{dy_1}{(1+N_m-y_1)^3}\Big)^{1/2}+CN_m^{-1/8}+C\Big(\int_{N_m^+}^{\infty}\frac{dy_1}{(1+y_1-N_m)^3}\Big)^{1/2}\\ &\le CN_m^{-1/8}.\end{aligned} \tag{3.22}$$

With (3.18) we have for $m\in\mathbb{N}$,

$$|u^{(1)}(N_m,x_2)|,\ \Big|\frac{\partial u^{(1)}}{\partial x_1}(N_m,x_2)\Big|\le C\frac{\phi(N_m)}{N_m^{1/8}}. \tag{3.23}$$

Therefore, by (3.8) we have

$$\begin{aligned}|J_+(N_m)|&\le C(\phi(N_m)-R)\frac{\phi(N_m)^2}{N_m^{1/4}}+C(\phi(N_m)-R)\frac{\phi(N_m)}{N_m^{1/8}}\\ &\le C(\phi(N_m)-R)\frac{\phi(N_m)^2}{N_m^{1/8}}\le C\frac{\phi(N_m)^3}{N_m^{1/8}}.\end{aligned} \tag{3.24}$$

Since $\phi(N)=N^s$, if we choose $s\in(0,1)$ such that $3s<\frac{1}{8}$, that is, $0<s<\frac{1}{24}$ the right hand side in (3.24) converges to zero as $m\to\infty$. Therefore, $\limsup_{N\to\infty}J_+(N)\ge 0$. By the same argument of $J_+$, we can show that $\limsup_{N\to\infty} J_-(N) \geq 0$, which yields Step 2.

Next, we discuss the last term in (3.6). By the same argument in Lemma 3.2 that we apply Green's representation theorem in $x_2 > h$ and use the Dirichlet Green's function $G_h$ of $\mathbb{R}^2_{x_2>h} (:= \mathbb{R}\times(h,\infty))$ instead of $G$, $u^{(1)}$ can also be of another integral representation for $x_2 > h$

$$\begin{aligned} u^{(1)}(x) &= \int_{y_2>h} \sigma(y) G_h(x,y) dy + 2\int_{\Gamma_h} u^{(1)}(y) \frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y) \\ &=: v^1(x) + v^2(x), \end{aligned} \tag{3.25}$$

where $G_h$ is defined by $G_h(x,y) := \Phi_k(x,y) - \Phi_k(x,y_h^*)$ where $y_h^* = (y_1, 2h - y_2)$. We define approximation $u_N^{(1)}$ of $u^{(1)}$ by

$$\begin{aligned} u_N^{(1)}(x) &:= \int_{y_2>0} \chi_{\phi(N)-1}(y_2)\sigma(y) G(x,y) dy + 2\int_{\Gamma_h} \chi_N(y_1) u^{(1)}(y) \frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y) \\ &=: v_N^1(x) + v_N^2(x), \quad x_2 > h, \end{aligned} \tag{3.26}$$

where $\chi_a$ is defined by for $a > 0$,

$$\chi_a(t) := \begin{cases} 1 & \text{for } |t| \leq a \\ 0 & \text{for } |t| > a. \end{cases} \tag{3.27}$$

By Lemma 3.4 of [4] and Lemma 2.1 of [3] we can show that $v_N^1$ and $v_N^2$ satisfy the upward propagating radiation condition, which implies that so does $u_N^{(1)}$. Furthermore, by the definition of $u_N^{(1)}$ we can show that $u_N^{(1)}(\cdot, \phi(N)-1) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, by Lemma 6.1 of [4] we have that

$$\mathrm{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds \geq 0. \tag{3.28}$$

Combining (3.6) with (3.28) we have

$$\begin{aligned} 0 &\geq -\mathrm{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds \\ &= \mathrm{Im}\left[ \frac{1}{2\pi} \sum_{j\in J} \sum_{d_{l,j}, d_{l',j}>0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l,j}}{\partial x_1} dx \right] \\ &- \mathrm{Im}\left[ \frac{1}{2\pi} \sum_{j\in J} \sum_{d_{l,j}, d_{l',j}<0} \overline{a_{l,j}} a_{l',j} \int_{C_{\phi(N)}} \overline{\phi_{l,j}} \frac{\partial \phi_{l,j}}{\partial x_1} dx \right] + J_+(N) + J_-(N) \\ &+ \mathrm{Im} \int_{\Gamma_{\phi(N),N}} \overline{u} \frac{\partial u}{\partial x_2} - \mathrm{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds + o(1). \end{aligned} \tag{3.29}$$

We observe the last term

$$\mathrm{Im} \int_{\Gamma_{\phi(N),N}} \overline{u} \frac{\partial u}{\partial x_2} - \mathrm{Im} \int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}} \frac{\partial u_N^{(1)}}{\partial x_2} ds =: L(N) + M(N), \tag{3.30}$$

where

$$L(N) := \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_2}ds - \mathrm{Im}\int_{\Gamma_{\phi(N)}} \overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds, \tag{3.31}$$

$$M(N) := \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u^{(2)}}{\partial x_2}ds + \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(2)}}\frac{\partial u^{(1)}}{\partial x_2}ds + \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(2)}}\frac{\partial u^{(2)}}{\partial x_2}ds. \tag{3.32}$$

By Lemma 3.2 we can show $|u^{(1)}(x_1,\phi(N))|$, $|\frac{\partial u^{(1)}}{\partial x_2}(x_1,\phi(N))| \leq C\phi(N)$ for $x_1 \in \mathbb{R}$, and by Lemma 2.2 we have $|u^{(2)}(x_1,\phi(N))|$, $|\frac{\partial u^{(2)}}{\partial x_2}(x_1,\phi(N))| \leq Ce^{-\delta\phi(N)}$ for $x_1 \in \mathbb{R}$. Then, we have

$$\begin{aligned}
|M(N)| &\leq \int_{-N}^{N}|u^{(1)}(x_1,\phi(N))|\Big|\frac{\partial u^{(2)}}{\partial x_2}(x_1,\phi(N))\Big|dx_1 \\
&+ \int_{-N}^{N}|u^{(2)}(x_1,\phi(N))|\Big|\frac{\partial u^{(1)}}{\partial x_2}(x_1,\phi(N))\Big|dx_1 \\
&+ \int_{-N}^{N}|u^{(2)}(x_1,\phi(N))|\Big|\frac{\partial u^{(2)}}{\partial x_2}(x_1,\phi(N))\Big|dx_1 \\
&\leq C(N\phi(N)e^{-\delta\phi(N)} + Ne^{-2\delta\phi(N)}) \\
&\leq CN\phi(N)e^{-\delta\phi(N)},
\end{aligned} \tag{3.33}$$

which implies that $M(N) = o(1)$ as $N \to \infty$. Hence, we will show that $\mathrm{limsup}_{N\to\infty}L(N) \geq 0$.

**Step3 (**$\mathrm{limsup}_{N\to\infty}L(N) \geq 0$**):** First, we observe that

$$\begin{aligned}
|L(N)| &\leq \Big|\mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u^{(1)}}{\partial x_2}ds - \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds\Big| \\
&+ \Big|\mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds - \mathrm{Im}\int_{\Gamma_{\phi(N),N}} \overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds\Big| \\
&+ \Big|\mathrm{Im}\int_{\Gamma_{\phi(N)}\setminus\Gamma_{\phi(N),N}} \overline{u_N^{(1)}}\frac{\partial u_N^{(1)}}{\partial x_2}ds\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{-N}^{N}|u^{(1)}(x_1,\phi(N))|\Big|\frac{\partial u^{(1)}}{\partial x_2}(x_1,\phi(N)) - \frac{\partial u_N^{(1)}}{\partial x_2}(x_1,\phi(N))\Big|ds \\
&+ \int_{-N}^{N}|u^{(1)}(x_1,\phi(N)) - u_N^{(1)}(x_1,\phi(N))|\Big|\frac{\partial u_N^{(1)}}{\partial x_2}(x_1,\phi(N))\Big|ds \\
&+ \int_{\mathbb{R}\setminus(-N,N)}|u_N^{(1)}(x_1,\phi(N))|\Big|\frac{\partial u_N^{(1)}}{\partial x_2}(x_1,\phi(N))\Big|ds.
\end{aligned} \tag{3.34}$$

By Lemma 2.2 $\sigma$ has a exponential decay in $y_2$. Then, we have for $x_1 \in \mathbb{R}$,

$$\begin{aligned}
&|v^1(x_1,\phi(N))|,\ \Big|\frac{\partial v^1}{\partial x_2}(x_1,\phi(N))\Big|,\ |v_N^1(x_1,\phi(N))|,\ \Big|\frac{\partial v_N^1}{\partial x_2}(x_1,\phi(N))\Big| \\
&\leq C\int_{(-\eta,\eta)\times(0,\infty)}\frac{e^{-\delta y_2}\phi(N)y_2}{(1+|x_1-y_1|)^{3/2}}dy \leq C\frac{\phi(N)}{(1+|x_1|)^{3/2}},
\end{aligned} \tag{3.35}$$

and

$$
\begin{aligned}
& |v^1(x_1,\phi(N)) - v^1_N(x_1,\phi(N))|, \ \left|\frac{\partial v^1}{\partial x_2}(x_1,\phi(N)) - \frac{\partial v^1_N}{\partial x_2}(x_1,\phi(N))\right| \\
\leq & \ C\int_{(-\eta,\eta)\times(\phi(N)-1,\infty)} \frac{e^{-\delta y_2}\phi(N)y_2}{(1+|x_1-y_1|)^{3/2}}dy \\
\leq & \ C\Big(\int_{\phi(N)}^{\infty} e^{-\delta y_2} y_2 dy_2\Big)\frac{\phi(N)}{(1+|x_1|)^{3/2}}dy \leq \frac{e^{-\delta\phi(N)}\phi(N)}{(1+|x_1|)^{3/2}}.
\end{aligned} \tag{3.36}
$$

Since the fundamental solution to Helmholtz equation $\Phi(x,y)$ is of the following estimation (see e.g., [2]) for $|x-y|\geq 1$

$$
\left|\frac{\partial \Phi}{\partial y_2}(x,y)\right| \leq C\frac{|x_2-y_2|}{1+|x-y|^{3/2}}, \quad \left|\frac{\partial^2 \Phi}{\partial x_2\partial y_2}(x,y)\right| \leq C\frac{|x_2-y_2|^2}{1+|x-y|^{3/2}}, \tag{3.37}
$$

we can show that for $x_1\in\mathbb{R}$

$$
|v^2(x_1,\phi(N))| \leq C\phi(N)W_\infty(x_1), \quad |v^2_N(x_1,\phi(N))| \leq C\phi(N)W_N(x_1), \tag{3.38}
$$

and

$$
\left|\frac{\partial v^2}{\partial x_2}(x_1,\phi(N))\right| \leq C\phi(N)^2 W_\infty(x_1), \quad \left|\frac{\partial v^2_N}{\partial x_2}(x_1,\phi(N))\right| \leq C\phi(N)^2 W_N(x_1), \tag{3.39}
$$

and

$$
|v^2(x_1,\phi(N)) - v^2_N(x_1,\phi(N))| \leq C\phi(N)\big(W_\infty(x_1) - W_N(x_1)\big), \tag{3.40}
$$

and

$$
\left|\frac{\partial v^2}{\partial x_2}(x_1,\phi(N)) - \frac{\partial v^2_N}{\partial x_2}(x_1,\phi(N))\right| \leq C\phi(N)^2\big(W_\infty(x_1) - W_N(x_1)\big), \tag{3.41}
$$

where $W_N$ is defined by for $N\in(0,\infty]$

$$
W_N(x_1) := \int_{-N}^{N} \frac{|u^{(1)}(y_1,h)|}{(1+|x_1-y_1|)^{3/2}}dy_1, \quad x_1\in\mathbb{R}. \tag{3.42}
$$

Using (3.35)–(3.41), we continue to estimate (3.34). By the Cauchy Schwarz inequality we have

$$
\begin{aligned}
|L(N)| \leq C\int_{-N}^{N} & \Big\{\frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N)W_\infty(x_1)\Big\} \\
& \times\Big\{\frac{\phi(N)e^{-\sigma\phi(N)}}{(1+|x_1|)^{3/2}} + \phi(N)^2\big(W_\infty(x_1)-W_N(x_1)\big)\Big\}dx_1 \\
+ \ \int_{-N}^{N} & \Big\{\frac{\phi(N)e^{-\sigma\phi(N)}}{(1+|x_1|)^{3/2}} + \phi(N)\big(W_\infty(x_1)-W_N(x_1)\big)\Big\} \\
& \times\Big\{\frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N)^2 W_N(x_1)\Big\}dx_1 \\
+ \ \int_{\mathbb{R}\setminus(-N,N)} & \Big\{\frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N)W_N(x_1)\Big\}\Big\{\frac{\phi(N)}{(1+|x_1|)^{3/2}} + \phi(N)^2 W_N(x_1)\Big\}dx_1
\end{aligned}
$$

$$
\begin{aligned}
&\le C\phi(N)^3\int_{-N}^{N} W_\infty(x_1)\big(W_\infty(x_1)-W_N(x_1)\big)dx_1\\
&+ C\phi(N)^3\int_{-N}^{N}\frac{1}{(1+|x_1|)^{3/2}}\big(W_\infty(x_1)-W_N(x_1)\big)dx_1\\
&+ C\phi(N)^2\int_{\mathbb{R}\setminus(-N,N)}\frac{1}{(1+|x_1|)^3}dx_1 + C\phi(N)^2\int_{\mathbb{R}\setminus(-N,N)}\frac{1}{(1+|x_1|)^{3/2}}W_N(x_1)dx_1\\
&+ C\phi(N)^3\int_{\mathbb{R}\setminus(-N,N)}|W_N(x_1)|^2dx_1+o(1)\\
&\le C\phi(N)^3\Big\{\Big(\int_{-N}^{N}\big(W_\infty(x_1)-W_N(x_1)\big)^2dx_1\Big)^{1/2}+\Big(\int_{\mathbb{R}\setminus(-N,N)}W_N(x_1)^2dx_1\Big)^{1/2}\Big\}\\
&\qquad +o(1).
\end{aligned}
\tag{3.43}
$$

Finally, we will estimate $\big(W_\infty(x_1)-W_N(x_1)\big)$ and $W_N(x_1)$. Since $u^{(1)}(\cdot,h)\in L^2(\mathbb{R})$, by Lemma 3.3 there exists a sequence $\{N_m\}_{m\in\mathbb{N}}$ such that $N_m\to\infty$ as $m\to\infty$ and

$$
\int_{K_{N_m}}|u^{(1)}(y_1,h)|^2dy_1\le CN_m^{-\frac{1}{4}},\ m\in\mathbb{N}, \tag{3.44}
$$

where $K_A := K_A^+\cup K_A^-$, $K_A^+ := (-A^+,A^+)\setminus(-A,A)$, $K_A^- := (-A,A)\setminus(-A^-,A^-)$, and $A^\pm := A\pm A^{1/2}$ for $A\in[1,\infty)$.

By the Cauchy Schwarz inequality we have for $|x_1|>N_m$,

$$
\begin{aligned}
\int_{-N_m^-}^{N_m^-}\frac{|u^{(1)}(y_1,h)|}{(1+|x_1-y_1|)^{3/2}}dy_1 &\le \Big(\int_{-N_m^-}^{N_m^-}|u^{(1)}(y_1,h)|^2dy_1\Big)^{1/2}\Big(\int_{-N_m^-}^{N_m^-}\frac{dy_1}{(1+|x_1|-y_1)^3}\Big)^{1/2}\\
&\le \frac{C}{1-|x_1|-N_m^-},
\end{aligned}
\tag{3.45}
$$

and

$$
\begin{aligned}
\int_{K_{N_m^-}}\frac{|u^{(1)}(y_1,h)|}{(1+|x_1-y_1|)^{3/2}}dy_1 &\le \Big(\int_{K_{N_m}}|u^{(1)}(y_1,h)|^2dy_1\Big)^{1/2}\Big(\int_{K_{N_m}^-}\frac{dy_1}{(1+|x_1|-y_1)^3}\Big)^{1/2}\\
&\le \frac{C}{N_m^{1/8}(1+|x_1|-N_m)}.
\end{aligned}
\tag{3.46}
$$

Therefore, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus(-N_m,N_m)}W_N(x_1)^2dx_1\\
&\quad\le C\int_{N_m}^{\infty}\frac{dx_1}{(1-|x_1|-N_m^-)^2}+\frac{C}{N_m^{1/4}}\int_{N_m}^{\infty}\frac{dx_1}{(1-|x_1|-N_m)^2}\\
&\quad\le \frac{C}{1+N_m^{1/2}}+\frac{C}{N_m^{1/4}}\ \le\ \frac{C}{N_m^{1/4}}.
\end{aligned}
\tag{3.47}
$$

By the Cauchy Schwarz inequality we have for $|x_1| < N_m$,

$$\begin{aligned}
&\int_{\mathbb{R}\setminus(-N_m^+,N_m^+)} \frac{|u^{(1)}(y_1,h)|}{(1+|x_1-y_1|)^{3/2}}dy_1 \\
&\le \Big(\int_{\mathbb{R}\setminus(-N_m^+,N_m^+)} |u^{(1)}(y_1,h)|^2 dy_1\Big)^{1/2}\Big(\int_{\mathbb{R}\setminus(-N_m^+,N_m^+)} \frac{dy_1}{(1+y_1-|x_1|)^3}\Big)^{1/2} \\
&\le \frac{C}{1+N_m^+-|x_1|}, \qquad (3.48)
\end{aligned}$$

and

$$\begin{aligned}
\int_{K_{N_m^+}} \frac{|u^{(1)}(y_1,h)|}{(1+|x_1-y_1|)^{3/2}}dy_1 &\le \Big(\int_{K_{N_m}} |u^{(1)}(y_1,h)|^2 dy_1\Big)^{1/2}\Big(\int_{K_{N_m}^+} \frac{dy_1}{(1+y_1-|x_1|)^3}\Big)^{1/2} \\
&\le \frac{C}{N_m^{1/8}(1+N_m-|x_1|)}. \qquad (3.49)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{-N_m}^{N_m} \big(W_\infty(x_1)-W_N(x_1)\big)^2 dx_1 \\
&\le C\int_{-N_m}^{N_m} \frac{dx_1}{(1+N_m^+-|x_1|)^2} + \frac{C}{N_m^{1/4}}\int_{-N_m}^{N_m}\frac{dx_1}{(1+N_m-|x_1|)^2} \\
&\le \frac{C}{1+N_m^{1/2}} + \frac{C}{N_m^{1/4}} \le \frac{C}{N_m^{1/4}}. \qquad (3.50)
\end{aligned}$$

Therefore, Collecting (3.43), (3.47), and (3.50) we conclude that $|L(N_m)| \le C\frac{\phi(N_m)^3}{N_m^{1/8}}$. Since $\phi(N) = N^s$, if we choose $s \in (0,1)$ such that $3s < \frac{1}{8}$, that is, $0 < s < \frac{1}{24}$, the term $\frac{\phi(N_m)^3}{N_m^{1/8}}$ converges to zero as $m \to \infty$. Therefore, $\mathrm{limsup}_{N\to\infty} L(N) \ge 0$, which yields Step 3.

By taking $\mathrm{limsup}_{N\to\infty}$ in (3.29) we have that

$$\begin{aligned}
0 \ &\ge \ \frac{k}{2\pi}\sum_{j\in J}\Big[\sum_{d_{l,j}>0}|a_{l,j}|^2 d_{l,j} - \sum_{d_{l,j}<0}|a_{l,j}|^2 d_{l,j}\Big] \\
&+ \ \mathrm{limsup}_{N\to\infty}\Big(J_+(N)+J_-(N)+L(N)\Big). \qquad (3.51)
\end{aligned}$$

By Steps 2 and 3 and choosing $0 < s < \frac{1}{24}$ the right hand side is non-negative. Therefore, $a_{l,j} = 0$ for all $l, j$, which yields $u^{(2)} = 0$. Lemma 3.1 has been shown, and in next section we will show the uniqueness of $u^{(1)}$. □

## 4 Uniqueness of $u^{(1)}$

In Section 4, we will show the following lemma.

**Lemma 4.1.** *If $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies*

**(i)** *$u \in H^1(\mathbb{R} \times (0, R))$ for all $R > 0$,*

**(ii)** $\Delta u + k^2(1 + q)nu = 0$ in $\mathbb{R}^2_+$,

**(iii)** *$u$ vanishes for $x_2 = 0$,*

**(iv)** *There exists $\phi \in L^\infty(\Gamma_h) \cap H^{1/2}(\Gamma_h)$ with $u(x) = 2 \int_{\Gamma_h} \phi(y) \frac{\partial \Phi_k(x,y)}{\partial y_2} ds(y)$ for $x_2 > h$,*

*then, $u \in H^1_0(\mathbb{R}^2_+)$.*

If we can use Lemma 4.1, we have the uniqueness of the solution in Theorem 1.2.

**Theorem 4.2.** *Let Assumptions 1.1 and 2.1 hold and let $k > 0$ be regular in the sense of Definition 2.3. If $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfies (3.1), (3.2), and the radiation condition in the sense of Definition 2.4, then $u$ vanishes for $x_2 > 0$.*

**Proof of Theorem 4.2.** Let $u \in H^1_{loc}(\mathbb{R}^2_+)$ satisfy (3.1), (3.2), and the radiation condition in the sense of Definition 2.4. By Lemma 3.1, $u^{(2)} = 0$ for $x_2 > 0$. Then, $u^{(1)}$ satisfies the assumptions (i)–(iv) of Lemma 4.1, which implies that $u^{(1)} \in H^1_0(\mathbb{R}^2_+)$. By Assumption 1.1, $u^{(1)}$ vanishes for $x_2 > 0$, which yields the uniqueness. □

Finally in this section we will show Theorem 4.2.

**Proof of Lemma 4.1.** Let $R > h$ be fixed. We set $\Omega_{N,R} := (-N, N) \times (0, R)$ where $N > 0$ is large enough. We denote by $I^R_{\pm N} := \{\pm N\} \times (0, R)$, $\Gamma_{R,N} := (-N, N) \times \{R\}$, and $\Gamma_R := (-\infty, \infty) \times \{R\}$. By Green's first theorem in $\Omega_{N,R}$ and assumptions **(ii)**, **(iii)** we have

$$\int_{\Omega_{N,R}} \{-k^2(1 + q)n|u|^2 + |\nabla u|^2\} dx = \int_{\Omega_{N,R}} \{\overline{u} \Delta u + |\nabla u|^2\} dx$$
$$= \int_{I^R_N} \overline{u} \frac{\partial u}{\partial x_1} ds - \int_{I^R_{-N}} \overline{u} \frac{\partial u}{\partial x_1} ds + \int_{\Gamma_{R,N}} \overline{u} \frac{\partial u}{\partial x_2} ds. \tag{4.1}$$

By the assumption **(i)**, the first and second term in the right hands side of (4.1) go to zero as $N \to \infty$. Then, by taking an imaginary part and as $N \to \infty$ in (4.1) we have

$$\mathrm{Im} \int_{\Gamma_R} \overline{u} \frac{\partial u}{\partial x_2} ds = 0. \tag{4.2}$$

By considering the Floquet Bloch transform with respect to $x_1$ (see the notation of (2.5)), we can show that

$$\int_{\Gamma_R} \overline{u} \frac{\partial u}{\partial x_2} ds = \int_{-1/2}^{1/2} \int_0^{2\pi} \overline{\tilde{u}_\alpha}(x_1, R) \frac{\partial \tilde{u}_\alpha(x_1, R)}{\partial x_2} dx_1 d\alpha. \tag{4.3}$$

Since the upward propagating radiation condition is equivalent to the Rayleigh expansion by the Floquet Bloch transform (see the proof of Theorem 6.8 in [8]), we can show that

$$\tilde{u}_\alpha(x) = \sum_{n \in \mathbb{Z}} u_n(\alpha) e^{inx_1 + i\sqrt{k^2 - (n+\alpha)^2}(x_2 - h)}, \ x_2 > h, \tag{4.4}$$

where $u_n(\alpha) := (2\pi)^{-1}\int_0^{2\pi} u_\alpha(x_1,h)e^{-inx_1}dx_1$. From (4.2)–(4.4) we obtain that

$$\begin{aligned} 0 &= \mathrm{Im}\int_{-1/2}^{1/2}\int_0^{2\pi}\overline{\tilde{u}_\alpha}(x_1,R)\frac{\partial\tilde{u}_\alpha(x_1,R)}{\partial x_2}dx_1d\alpha \\ &= \mathrm{Im}\sum_{n\in\mathbb{Z}}\int_{-1/2}^{1/2}2\pi|u_n(\alpha)|^2 i\sqrt{k^2-(n+\alpha)^2}, \end{aligned} \tag{4.5}$$

Here, we denote by $k = n_0 + r$ where $n_0 \in \mathbb{N}_0$ and $r \in [-1/2, 1/2)$. Then by (4.5) we have

$$u_n(\alpha) = 0 \text{ for } |n| < n_0, \text{ a.e. } \alpha \in (-1/2, 1/2),$$

$$\begin{aligned} u_{n_0}(\alpha) &= 0 \text{ for } \alpha \in (-1/2, r), \\ u_{-n_0}(\alpha) &= 0 \text{ for } \alpha \in (-r, 1/2). \end{aligned} \tag{4.6}$$

By (4.6) we have

$$\begin{aligned} &\int_{-1/2}^{1/2}\int_0^{2\pi}\int_R^\infty|\tilde{u}_\alpha(x)|^2dx_2dx_1d\alpha \\ =\;& 2\pi\int_{-1/2}^{1/2}\sum_{|n|>n_0}|u_n(\alpha)|^2\int_R^\infty e^{-\sqrt{(n+\alpha)^2-k^2}(x_2-h)}dx_2d\alpha \\ +\;& 2\pi\int_r^{1/2}|u_{n_0}(\alpha)|^2\int_R^\infty e^{-\sqrt{(n_0+\alpha)^2-k^2}(x_2-h)}dx_2d\alpha \\ +\;& 2\pi\int_{-1/2}^{-r}|u_{-n_0}(\alpha)|^2\int_R^\infty e^{-\sqrt{(-n_0+\alpha)^2-k^2}(x_2-h)}dx_2d\alpha \end{aligned}$$

$$\begin{aligned} \le\;& 2\pi\sum_{|n|>n_0}\int_{-1/2}^{1/2}\frac{|u_n(\alpha)|^2e^{-\sqrt{(n+\alpha)^2-k^2}(R-h)}}{\sqrt{(n+\alpha)^2-k^2}}d\alpha \\ +\;& 2\pi\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2e^{-\sqrt{(n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(n_0+\alpha)^2-k^2}}d\alpha \\ +\;& 2\pi\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2e^{-\sqrt{(-n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(-n_0+\alpha)^2-k^2}}d\alpha \\ \le\;& C\sum_{|n|>n_0}\int_{-1/2}^{1/2}|u_n(\alpha)|^2d\alpha \\ +\;& C\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}}d\alpha + C\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}}d\alpha, \end{aligned} \tag{4.7}$$

and

$$
\begin{aligned}
&\int_{-1/2}^{1/2}\int_0^{2\pi}\int_R^\infty |\partial_{x_1}\tilde{u}_\alpha(x)|^2 dx_2dx_1d\alpha \\
=&\ 2\pi\sum_{|n|>n_0}\int_{-1/2}^{1/2}\frac{|u_n(\alpha)|^2n^2e^{-\sqrt{(n+\alpha)^2-k^2}(R-h)}}{\sqrt{(n+\alpha)^2-k^2}}d\alpha \\
+&\ 2\pi\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2n_0^2e^{-\sqrt{(n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(n_0+\alpha)^2-k^2}}d\alpha \\
+&\ 2\pi\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2n_0^2e^{-\sqrt{(-n_0+\alpha)^2-k^2}(R-h)}}{\sqrt{(-n_0+\alpha)^2-k^2}}d\alpha \\
\le&\ C\sum_{|n|>n_0}\int_{-1/2}^{1/2}|u_n(\alpha)|^2d\alpha \\
+&\ C\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}}d\alpha+C\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}}d\alpha. \qquad (4.8)
\end{aligned}
$$

By the same argument in (4.8) we have

$$
\begin{aligned}
\int_{-1/2}^{1/2}\int_0^{2\pi}\int_R^\infty |\partial_{x_2}\tilde{u}_\alpha(x)|^2 dx_2dx_1d\alpha \le C\sum_{|n|>n_0}\int_{-1/2}^{1/2}|u_n(\alpha)|^2d\alpha \\
+C\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}}d\alpha+C\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}}d\alpha. \qquad (4.9)
\end{aligned}
$$

It is well known that the Floquet Bloch Transform is an isomorphism between $H^1(\mathbb{R}^2_+)$ and $L^2\big((-1/2,1/2)_\alpha;H^1((0,2\pi)\times\mathbb{R})_x\big)$ (e.g., see Theorem 4 in [9]). Therefore, we obtain from (4.7)–(4.9)

$$
\begin{aligned}
\|u\|^2_{H^1(\mathbb{R}\times(R,\infty))} \le&\ C\int_{-1/2}^{1/2}\int_0^{2\pi}\int_R^\infty |\tilde{u}_\alpha(x)|^2+|\partial_{x_1}\tilde{u}_\alpha(x)|^2+|\partial_{x_2}\tilde{u}_\alpha(x)|^2dx_2dx_1d\alpha \\
\le&\ C\sum_{|n|>n_0}\int_{-1/2}^{1/2}|u_n(\alpha)|^2d\alpha \\
+&\ C\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}}d\alpha+C\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}}d\alpha. \\
\le&\ C\int_{-1/2}^{1/2}\int_0^{2\pi}|\tilde{u}_\alpha(x_1,h)|^2dx_1d\alpha \\
+&\ C\int_r^{1/2}\frac{|u_{n_0}(\alpha)|^2}{\sqrt{\alpha-r}}d\alpha+C\int_{-1/2}^{-r}\frac{|u_{-n_0}(\alpha)|^2}{\sqrt{-\alpha-r}}d\alpha. \qquad (4.10)
\end{aligned}
$$

If we can show that

$$\exists\delta>0 \ \text{ and } \ \exists C>0 \ \text{ s.t. } \ |u_{\pm n_0}(\alpha)|\le C \ \text{ for all } \ \alpha\in(-\delta\pm r,\delta\pm r), \qquad (4.11)$$

then the right hands side of (4.10) is finite, which yields Lemma 4.1.

Finally, we will show (4.11). By the same argument in section 3 of [8] we have

$$(I - K_\alpha)\tilde{u}_\alpha = f_\alpha \text{ in } H^1_{0,per}(C_h), \tag{4.12}$$

where the operator $K_\alpha$ is defined by (2.12) and $f_\alpha := -(T_{per}k^2 nqu)(\cdot, \alpha)$. Since the function $k^2nqu$ has a compact support, $\|f_\alpha\|^2_{H^1(C_h)}$ is bounded with respect to $\alpha$. By Assumption 2.1 and the operator $K_\alpha$ is compact, $(I-K_\alpha)$ is invertible if $\alpha \in A_k$. Since $\pm r \in A_k$, $(I-K_\pm)$ is invertible. Since the exceptional values are finitely many (see Lemma 2.2), $(I-K_\alpha)$ is also invertible if $\alpha$ is close to $\pm r$. Therefore, there exists $\delta > 0$ such that $(I-K_\alpha)$ is invertible for all $\alpha \in (-\delta+r, \delta+r) \cup (-\delta-r, \delta-r)$.

The operator $(I - K_\alpha)$ is of the form

$$(I - K_\alpha) = (I - K_{\pm r})\Big(I - (I - K_{\pm r})^{-1}[I - K_{\pm r} - (I - K_\alpha)]\Big) = (I - K_{\pm r})(I - M_\alpha), \tag{4.13}$$

where $M_\alpha := (I - K_{\pm r})^{-1}(K_\alpha - K_{\pm r})$. Next, we will estimate $(K_\alpha - K_{\pm r})$. By the definition of $K_\alpha$ we have for all $v, w \in H^1_{0,per}(C_h)$,

$$\begin{aligned}\langle (K_\alpha - K_{\pm r})v, w\rangle_* &= -\int_{C_h}\left[i(\alpha \mp r)\left(v\frac{\partial \overline{w}}{\partial x_1} - \overline{v}\frac{\partial \overline{w}}{\partial x_1}\right) + (\alpha^2 - r^2)v\overline{w}\right]dx \\ &+ 2\pi i \sum_{|n|\neq n_0} v_n\overline{w_n}\big(\sqrt{k^2-(n+\alpha)^2} - \sqrt{k^2-(n\pm r)^2}\big) \\ &+ 2\pi i \sum_{|n|= n_0} v_n\overline{w_n}\big(\sqrt{k^2-(n+\alpha)^2} - \sqrt{k^2-(n\pm r)^2}\big).\end{aligned} \tag{4.14}$$

Since

$$\begin{aligned}|\sqrt{k^2-(n+\alpha)^2} - \sqrt{k^2-(n\pm r)^2}| &= \left|\frac{\pm 2nr + r^2 - 2n\alpha - \alpha^2}{\sqrt{k^2-(n+\alpha)^2} + \sqrt{k^2-(n\pm r)^2}}\right| \\ &\leq \begin{cases} \frac{|n||\alpha\pm r| + |r^2-\alpha^2|}{\sqrt{|k^2-(n\pm r)^2|}} & \text{for } |n|\neq n_0 \\ \frac{|n||\alpha\pm r| + |r^2-\alpha^2|}{\sqrt{|r+\alpha||r-\alpha|}} & \text{for } |n| = n_0, \end{cases}\end{aligned} \tag{4.15}$$

we have for all $\alpha \in (-\delta + r, \delta + r) \cup (-\delta - r, \delta - r)$

$$\begin{aligned}|\langle (K_\alpha - K_{\pm r})v, w\rangle_*| &\leq C|\alpha \mp r|\,\|v\|_{H^1(C_h)}\|w\|_{H^1(C_h)} \\ &+ C\sum_{|n|\neq n_0}|v_n||w_n|\frac{|n||\alpha\mp r|}{\sqrt{|k^2-(n\pm r)^2|}} \\ &+ C\sum_{|n|=n_0}|v_n||w_n|n_0\sqrt{|\alpha\mp r|} \\ &\leq C\sqrt{|\alpha\mp r|}\,\|v\|_{H^1(C_h)}\,\|w\|_{H^1(C_h)}.\end{aligned} \tag{4.16}$$

(we retake very small $\delta > 0$ if needed.) This implies that there is a constant number $C > 0$ which is independent of $\alpha$ such that $\|K_\alpha - K_{\pm r}\| \leq C\sqrt{|\alpha \mp r|}$. Therefore, by the property of Neumann series, there is a small $\delta > 0$ such that for all $\alpha \in (-\delta + r, \delta + r) \cup (-\delta - r, \delta - r)$

$$(I - M_\alpha)^{-1} = \sum_{n=0}^{\infty} M_\alpha^n \quad \text{and} \quad \|M_\alpha\| \leq 1/2. \tag{4.17}$$

By the Cauchy Schwarz inequality, the boundedness of trace operator, and (4.17) we have

$$\begin{aligned} |u_{\pm n_0}(\alpha)| &\leq \int_0^{2\pi} |\tilde{u}_\alpha(x_1,h)|dx_1 \leq C\left\|\tilde{u}_\alpha\right\|_{H^1(C_h)} \\ &= C\left\|(I-M_\alpha)^{-1}(I-K_{\pm r})^{-1}f_\alpha\right\|_{H^1(C_h)} \\ &\leq C\left\|(I-M_\alpha)^{-1}\right\|\left\|(I-K_{\pm r})^{-1}f_\alpha\right\| \\ &\leq C\sum_{n=0}^{\infty}\|M_\alpha\|^n < C\sum_{n=0}^{\infty}(1/2)^j < \infty, \end{aligned} \tag{4.18}$$

where constant number $C>0$ is independent of $\alpha$. Therefore, we have shown (4.11). □

# 5 Existence

In previous sections we discussed the uniqueness of Theorem 1.2. In Section 5, we will show the existence. Let Assumptions 1.1 and 2.1 hold and let $k>0$ be regular in the sense of Definition 2.3. Let $f \in L^2(\mathbb{R}^2_+)$ such that $\mathrm{supp} f = Q$. We define the solution operator $S : L^2(Q) \to L^2(Q)$ by $Sg := v\big|_Q$ where $v$ satisfies the radiation condition and

$$\Delta v + k^2 n v = g, \ \text{ in } \mathbb{R}^2_+, \tag{5.1}$$

$$v = 0 \text{ on } \Gamma_0. \tag{5.2}$$

Remark that by Theorem 2.6 we can define such a operator $S$, and $S$ is a compact operator since the restriction to $Q$ of the solution $v$ is in $H^1(Q)$. We define the multiplication operator $M : L^2(Q) \to L^2(Q)$ by $Mh := k^2nqh$. We will show the following lemma.

**Lemma 5.1.** *$I_{L^2(Q)} + SM$ is invertible.*

**Proof of Lemma 5.1.** By the definition of operators $S$ and $M$ we have $SMg = v\big|_Q$ where $v$ is a radiating solution of (5.1)–(5.2) replacing $g$ by $k^2nqg$. If we assume that $(I_{L^2(Q)} + SM)g = 0$, then $g = -v\big|_Q$, which implies that $v$ satisfies $\Delta v + k^2n(1+q)v = 0$ in $\mathbb{R}^2_+$. By the uniqueness we have $v = 0$ in $\mathbb{R}^2_+$, which implies that $I_{L^2(Q)} + SM$ is injective. Since the operator $SM$ is compact, by Fredholm theory we conclude that $I_{L^2(Q)} + SM$ is invertible. □

We define $u$ as the solution of

$$\Delta u + k^2 n u = f - M(I_{L^2(Q)} + SM)^{-1}Sf, \ \text{ in } \mathbb{R}^2_+. \tag{5.3}$$

satisfying the radiation condition and $u = 0$ on $\Gamma_0$. Since

$$\begin{aligned} u\big|_Q &= S(f - M(I_{L^2(Q)} + SM)^{-1}Sf) \\ &= (I_{L^2(Q)} + SM)(I_{L^2(Q)} + SM)^{-1}Sf - SM(I_{L^2(Q)} + SM)^{-1}Sf \\ &= (I_{L^2(Q)} + SM)^{-1}Sf, \end{aligned} \tag{5.4}$$

we have that

$$\Delta u + k^2 n u = f - k^2 nqu, \ \text{ in } \mathbb{R}^2_+, \tag{5.5}$$

and $u$ is a radiating solution of (1.8)–(1.9). Therefore, Theorem 1.2 has been shown.

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