Phase space Feynman path integrals of parabolic type

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Abstract

This paper is a rough survey based on the talk at RIMS about the joint works with Prof. A. S. Vasudeva Murthy and Prof. Keiya Uchida. This survey introduces our results for the phase space path integrals of higher-order parabolic type on $\mathbb{R}^d$ ([10][13][14]) and on the torus $\mathbb{T}^d$ with $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$ ([15]).

1 Introduction to phase space path integral on $\mathbb{R}^d$ ([10])

We begin with an introduction to the phase space path integral on $\mathbb{R}^d$.

Let $T>0$, $m>0$, and $x\in\mathbb{R}^d$. Let $U(T,0)$ be the fundamental solution for the m-th-order parabolic equation on $\mathbb{R}^d$, i.e.,

$$\Big(\partial_T + H(T,x,-i\partial_x)\Big)U(T,0)v(x)=0\,,\quad U(0,0)v(x)=v(x)\,. \tag{1.1}$$

By the Fourier transform with respect to $x_0\in\mathbb{R}^d$ and $\xi_0\in\mathbb{R}^d$, we can write

$$v(x)=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^{2d}} e^{i(x-x_0)\cdot\xi_0}v(x_0)dx_0d\xi_0\,,$$

and the pseudo-differential operator $H(T,x,-i\partial_x)$ on $\mathbb{R}^d$ is defined by

$$H(T,x,-i\partial_x)v(x)=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^{2d}} e^{i(x-x_0)\cdot\xi_0}H(T,x,\xi_0)v(x_0)dx_0d\xi_0 \tag{1.2}$$

with a function $H(T,x,\xi_0)$ (cf. [9]). We consider a function $U(T,0,x,\xi_0)$ such that

$$U(T,0)v(x)=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^{2d}} e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)v(x_0)dx_0d\xi_0\,. \tag{1.3}$$

As an approximation of $U(T,0)v(x)$, we use the operator $I(T,0)$ given by

$$I(T,0)v(x)\equiv\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^{2d}} e^{i(x-x_0)\cdot\xi_0}e^{-\int_0^T H(t,x,\xi_0)dt}v(x_0)dx_0d\xi_0\,. \tag{1.4}$$

Let $\Delta_{T,0}: T=T_{J+1}>T_J>\cdots>T_1>T_0=0$ be an arbitrary division of the interval $[0,T]$ into subintervals. Connecting the solution $U(T,0)v(x)$, we have

$$U(T,0)v(x)=U(T,T_J)U(T_J,T_{J-1})\cdots U(T_2,T_1)U(T_1,0)v(x)\,. \tag{1.5}$$

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Let $t_j = T_j - T_{j-1}$ for $j = 1, 2, \dots, J, J+1$ and $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j$. Set $x_{J+1} = x$. Let $x_j \in \mathbb{R}^d$ and $\xi_j \in \mathbb{R}^d$ for $j = 1, 2, \dots, J$. When $|\Delta_{T,0}| \to 0$, as an approximation of $U(T_j, T_{j-1})v(x_j)$, we use

$$I(T_j, T_{j-1})v(x_j) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^{2d}} e^{i(x_j - x_{j-1})\cdot\xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t, x_j, \xi_{j-1})dt} v(x_{j-1}) dx_{j-1} d\xi_{j-1}.$$

In [10], under a suitable condition, we get

$$\begin{aligned} U(T,0)v(x) &= \lim_{|\Delta_{T,0}|\to 0} I(T, T_J) I(T_J, T_{J-1}) \cdots I(T_2, T_1) I(T_1, 0) v(x) \\ &= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi}\right)^{d(J+1)} \int_{\mathbb{R}^{2d(J+1)}} e^{\sum_{j=1}^{J+1} (i(x_j - x_{j-1})\cdot\xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t, x_j, \xi_{j-1})dt)} v(x_0) \prod_{j=0}^{J} dx_j d\xi_j. \end{aligned} \tag{1.6}$$

By (1.3) and (1.6), we obtain

$$\begin{aligned} & e^{i(x-x_0)\cdot\xi_0} U(T, 0, x, \xi_0) \\ &= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi}\right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\sum_{j=1}^{J+1} (i(x_j - x_{j-1})\cdot\xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t, x_j, \xi_{j-1})dt)} \prod_{j=1}^{J} dx_j d\xi_j \end{aligned} \tag{1.7}$$

(even when $x = x_0$). According Feynman's idea in [5], we introduce a position path $q : [0, T] \to \mathbb{R}^d$ with $q(T_j) = x_j$ and a momentum path $p : [0, T) \to \mathbb{R}^d$ with $p(T_j) = \xi_j$. Then we can formally write (1.7) as

$$e^{i(x-x_0)\cdot\xi_0} U(T, 0, x, \xi_0) = \int_{q(T)=x, p(0)=\xi_0, q(0)=x_0} e^{i\phi(q,p)} \mathcal{D}(q, p)\,, \tag{1.8}$$

where

$$\phi(q, p) = \int_{[0,T)} p(t) \cdot dq(t) + i \int_{[0,T)} H(t, q(t), p(t)) dt \tag{1.9}$$

is the action for the paths (q, p) and the path integral $\int \sim \mathcal{D}(q, p)$ is a sum over all the paths (q, p) with $q(T) = x$, $q(0) = x_0$ and $p(0) = \xi_0$. The expression (1.7) is called the time slicing approximation of the phase space path integral (1.8) on $\mathbb{R}^d$.

2 Introduction to phase space path integral on $\mathbb{T}^d$ ([15])

We would like to give an introduction to the phase space path integral in the torus $\mathbb{T}^d$ because it is similar to the introduction of the phase space path integral on $\mathbb{R}^d$.

Let $T > 0$, $m > 0$, and $x \in \mathbb{T}^d$ with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Let $U(T, 0)$ be the fundamental solution for the m-th-order paraboolic equation on the torus $\mathbb{T}^d$, i.e.,

$$\Big(\partial_T + H(T, x, -i\partial_x)\Big) U(T, 0)v(x) = 0\,, \quad U(0, 0)v(x) = v(x)\,. \tag{2.1}$$

By the Fourier series expansion with respect to $x_0 \in \mathbb{T}^d$ and $\xi_0 \in \mathbb{Z}^d$, we can write

$$v(x) = \left(\frac{1}{2\pi}\right)^d \sum_{\xi_0 \in \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i(x-x_0)\cdot\xi_0} v(x_0) dx_0\,,$$

and the pseudo-differential operator on $\mathbb{T}^d$ is defined by

$$H(T, x, -i\partial_x)v(x) = \left(\frac{1}{2\pi}\right)^d \sum_{\xi_0 \in \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i(x-x_0)\cdot\xi_0} H(T, x, \xi_0) v(x_0) dx_0$$

with a function $H(T,x,\xi_0)$ (cf. [17]). We consider a function $U(T,0,x,\xi_0)$ such that

$$U(T,0)v(x) \equiv \left(\frac{1}{2\pi}\right)^d \sum_{\xi_0\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i(x-x_0)\cdot\xi_0} U(T,0,x,\xi_0)v(x_0)dx_0 \,. \tag{2.2}$$

As an approximation of the solution $U(T,0)v(x)$, we use the operator $I(T,0)$ given by

$$I(T,0)v(x) \equiv \left(\frac{1}{2\pi}\right)^d \sum_{\xi_0\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i(x-x_0)\cdot\xi_0} e^{-\int_0^T H(t,x,\xi_0)dt} v(x_0)dx_0 \,.$$

Let $\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0$ be an arbitrary division of the interval $[0,T]$ into subintervals. Connecting the solution $U(T,0)v(x)$, we have

$$U(T,0)v(x) = U(T,T_J)U(T_J,T_{J-1})\cdots U(T_2,T_1)U(T_1,0)v(x)\,.$$

Let $t_j = T_j - T_{j-1}$ for $j = 1,2,\ldots,J,J+1$ and $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j$. Set $x_{J+1} = x$. Let $x_j \in \mathbb{T}^d$ and $\xi_j \in \mathbb{Z}^d$ for $j = 1,2,\ldots,J$. When $|\Delta_{T,0}| \to 0$, as an approximation of $U(T_j,T_{j-1})v(x_j)$, we use

$$I(T_j,T_{j-1})v(x_j) = \left(\frac{1}{2\pi}\right)^d \sum_{\xi_{j-1}\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt} v(x_{j-1})dx_{j-1}.$$

In [1], under a suitable condition, we get

$$\begin{aligned} U(T,0)v(x) &= \lim_{|\Delta_{T,0}|\to 0} I(T,T_J)I(T_J,T_{J-1})\cdots I(T_2,T_1)I(T_1,0)v(x) \\ &= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi}\right)^{d(J+1)} \sum_{(\xi_J,\cdots,\xi_0)\in\mathbb{Z}^{(d+1)J}} \int_{\mathbb{T}^{(d+1)J}} e^{\sum_{j=1}^{J+1}(i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt)} v(x_0)\prod_{j=0}^{J} dx_j. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain

$$\begin{aligned} &e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) \\ &= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi}\right)^{dJ} \sum_{(\xi_J,\cdots,\xi_1)\in\mathbb{Z}^{dJ}} \int_{\mathbb{T}^{dJ}} e^{\sum_{j=1}^{J+1}(i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt)} \prod_{j=1}^{J} dx_j \end{aligned} \tag{2.4}$$

(even when $x = x_0$). According Feynman's idea in [5], we introduce a position path $q : [0,T] \to \mathbb{T}^d$ with $q(T_j) = x_j$ and a momentum path $p : [0,T) \to \mathbb{Z}^d$ with $p(T_j) = \xi_j$. Then we can formally write (2.4) as

$$e^{i(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0) = \int e^{i\phi(q,p)}\mathcal{D}^{\mathbb{T}}(q,p)\,, \tag{2.5}$$

where

$$\phi(q,p) = \int_{[0,T)} p(t)\cdot dq(t) + i\int_{[0,T)} H(t,q(t),p(t))dt$$

is the action for the paths (q,p), and the path integral $\int \sim \mathcal{D}^{\mathbb{T}}(q,p)$ is a 'sum' over all the paths (q,p) with $q(T) = x$, $q(0) = x_0$ and $p(0) = \xi_0$.

3 Our results [13], [14] for phase space path integrals on $\mathbb{R}^d$

We go back to the phase space path integral on $\mathbb{R}^d$. In [13], [14] , using the time slicing approximation, we proved the existence of the phase space path integrals

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)}F(q,p)\mathcal{D}(q,p) \tag{3.1}$$

of parabolic type with general functional $F(q,p)$ as integrand. We can regard (1.8) as the case of (3.1) with $F(q,p) \equiv 1$. More precisely, we give two general sets $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals such that for any $F(q,p) \in \mathcal{F}_{\mathcal{Q}} \cup \mathcal{F}_{\mathcal{P}}$, the time slicing approximation of (3.1) converges uniformly on compact subsets with respect to the final point x of position paths and to the initial point ξ_0 of momentum paths. Furthermore, we prove some properties of the phase space path integrals similar to some properties of the standard integrals.

Remark 3.1 *In this survey, we treat the phase space path integral of parabolic type. For the phase space path intgral of Schrödinger type, there exist various approaches (cf. [18, §31]). For example, infinite-dimensional oscillatory integrals (cf. [1, §10.5.3], [2], [16, §3.3]), Chernoff formula (cf. [20]), white noise (cf. [3]), coherent states (cf. [4]), and Fourier integral operators (cf. [8], [7]). The approach of [11], [12] for the Schrödinger type is similar to that of [13], [14] for the parabolic type.*

4 Existence of phase space path integrals

Assumption 4.1 *Let* $\langle\xi\rangle \equiv \left(1+|\xi|^2\right)^{1/2}$, $0<T\le \mathbf{T}<\infty$, $0\le\delta<\rho\le 1$ *and* $m>0$. *Let* $H(t,x,\xi)$ *be a complex-valued* C^∞*-function of* $(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$ *satisfying the following:*

(1) *There exist positive constants* c, C, R *such that*

$$0 < c \le \mathrm{Re}\, H(t,x,\xi) \le C\langle\xi\rangle^m \quad \textit{for } |\xi| \ge R\,.$$

Here $\mathrm{Re}\, H(t,x,\xi)$ *is the real part of* $H(t,x,\xi)$.

(2) *For any multi-indices* α *and* β, $\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)$ *is piecewise continuous with respect to* $t\in[0,T]$ *and there exists a positive constant* $C_{\alpha,\beta}$ *such that*

$$\left|\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)\right| / \mathrm{Re}\, H(t,x,\xi) \le C_{\alpha,\beta}\langle\xi\rangle^{\delta|\alpha|-\rho|\beta|} \quad \textit{for } |\xi|\ge R\,.$$

Let $\Delta_{T,0}: T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0$. Let $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1\le j\le J+1} t_j$. Set $x_{J+1} = x$. Let $x_j\in\mathbb{R}^d$ and $\xi_j\in\mathbb{R}^d$. We define the position path $q_{\Delta_{T,0}} = q_{\Delta_{T,0}}(t, x_{J+1}, x_J, \ldots, x_1, x_0)$ by $q_{\Delta_{T,0}}(0) = x_0$, $q_{\Delta_{T,0}}(t) = x_j$, $T_{j-1} < t \le T_j$ and the momentum path $p_{\Delta_{T,0}} = p_{\Delta_{T,0}}(t,\xi_J,\ldots,\xi_1,\xi_0)$ by $p_{\Delta_{T,0}}(t) = \xi_{j-1}$, $T_{j-1}\le t < T_j$.

Definition 4.2 (Two spaces $\mathcal{Q}$, $\mathcal{P}$ of piecewise constant paths)

(1) *We write* $q\in\mathcal{Q}$ *if* q *is piecewise constant and left-continuous, i.e.,* $q = q_{\Delta_{T,0}}$.

(2) *We write* $p\in\mathcal{P}$ *if* p *is piecewise constant and right-continuous, i.e.,* $p = p_{\Delta_{T,0}}$.

$\phi(q_{\Delta_{T,0}}, p_{\Delta_{T,0}})$ and $F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}})$ are the functions $\phi_{\Delta_{T,0}}$ and $F_{\Delta_{T,0}}$ given by

$$\begin{aligned}
\phi(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) &= \sum_{j=1}^{J+1}\int_{[T_{j-1},T_j)} p_{\Delta_{T,0}}\cdot dq_{\Delta_{T,0}}(t) + i\sum_{j=1}^{J+1}\int_{[T_{j-1},T_j)} H(t, q_{\Delta_{T,0}}, p_{\Delta_{T,0}})dt \\
&= \sum_{j=1}^{J+1}(x_j - x_{j-1})\cdot\xi_{j-1} + i\sum_{j=1}^{J+1}\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt\,, \\
&\equiv \phi_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)\,, \\
F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) &\equiv F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0,x_0)\,.
\end{aligned}$$

Definition 4.3 (Two sets $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals $F(q,p)$) *Let $F(q,p)$ be a functional of $q \in \mathcal{Q}$ and $p \in \mathcal{P}$. For any $\Delta_{T,0}$, we assume*

$$F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) \equiv F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_0, x_0) \in C^\infty(\mathbb{R}^{d(2J+3)}).$$

(1) *We write $F(q,p) \in \mathcal{F}_{\mathcal{Q}}$ if $F(q,p)$ satisfies Assumption 4.4 (1) below.*

(2) *We write $F(q,p) \in \mathcal{F}_{\mathcal{P}}$ if $F(q,p)$ satisfies Assumption 4.4 (2) below.*

Assumption 4.4 *Let $L \geq 0$ and $u_j \geq 0$ with $\sum_{j=1}^{J+1} u_j = U < \infty$ depending $\Delta_{T,0}$.*

(1) *For any non-negative integers ℓ_1, ℓ_2, there exist positive constant A_{ℓ_1,ℓ_2}, B_{ℓ_1,ℓ_2} such that for any $\Delta_{T,0}$ and any multi-indices $|\alpha_j| \leq \ell_1$ and $|\beta_{j-1}| \leq \ell_2$,*

$$\left| \left(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) \right|$$
$$\leq A_{\ell_1,\ell_2}(B_{\ell_1,\ell_2})^{J+1} \left(\sum_{j=1}^{J+1} \langle x_j \rangle + \sum_{j=1}^{J+1} \langle \xi_{j-1} \rangle + \langle x_0 \rangle \right)^L \prod_{j=1}^{J+1} (t_j)^{\min(|\beta_{j-1}|,1)} \prod_{j=1}^{J+1} \langle \xi_{j-1} \rangle^{\delta|\alpha_j| - \rho|\beta_{j-1}|},$$

and for any integer k with $|\alpha_k| > 0$ and $1 \leq k \leq J+1$,

$$\left| \left(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) \right|$$
$$\leq A_{\ell_1,\ell_2}(B_{\ell_1,\ell_2})^{J+1} u_k \left(\sum_{j=1}^{J+1} \langle x_j \rangle + \sum_{j=1}^{J+1} \langle \xi_{j-1} \rangle + \langle x_0 \rangle \right)^L \prod_{j=1, j\neq k}^{J+1} (t_j)^{\min(|\beta_{j-1}|,1)} \prod_{j=1}^{J+1} \langle \xi_{j-1} \rangle^{\delta|\alpha_j| - \rho|\beta_{j-1}|}.$$

(2) *For any non-negative integers ℓ_1, ℓ_2, there exist positive constant A_{ℓ_1,ℓ_2}, B_{ℓ_1,ℓ_2} such that for any $\Delta_{T,0}$ and any multi-indices $|\alpha_j| \leq \ell_1$ and $|\beta_{j-1}| \leq \ell_2$,*

$$\left| \left(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) \right|$$
$$\leq A_{\ell_1,\ell_2}(B_{\ell_1,\ell_2})^{J+1} \left(\sum_{j=1}^{J+1} \langle x_j \rangle + \sum_{j=1}^{J+1} \langle \xi_{j-1} \rangle + \langle x_0 \rangle \right)^L \prod_{j=1}^{J+1} (t_j)^{\min(|\alpha_j|,1)} \prod_{j=1}^{J+1} \langle \xi_{j-1} \rangle^{\delta|\alpha_j| - \rho|\beta_{j-1}|},$$

and for any integer k with $|\beta_{k-1}| > 0$ and $1 \leq k \leq J+1$,

$$\left| \left(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) \right|$$
$$\leq A_{\ell_1,\ell_2}(B_{\ell_1,\ell_2})^{J+1} u_k \left(\sum_{j=1}^{J+1} \langle x_j \rangle + \sum_{j=1}^{J+1} \langle \xi_{j-1} \rangle + \langle x_0 \rangle \right)^L \prod_{j=1, j\neq k}^{J+1} (t_j)^{\min(|\alpha_j|,1)} \prod_{j=1}^{J+1} \langle \xi_{j-1} \rangle^{\delta|\alpha_j| - \rho|\beta_{j-1}|}.$$

Theorem 1 (Existence of path integrals) *For any $F(q,p) \in \mathcal{F}_{\mathcal{Q}} \cup \mathcal{F}_{\mathcal{P}}$,*

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} F(q,p)\mathcal{D}(q,p)$$
$$\equiv \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi}\right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{i\phi(q_{\Delta_{T,0}},p_{\Delta_{T,0}})} F(q_{\Delta_{T,0}},p_{\Delta_{T,0}}) \prod_{j=1}^{J} d\xi_j dx_j \tag{4.1}$$

*converges uniformly on compact sets of $(x,\xi_0,x_0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$, i.e., the phase space path integral (**??**) is well-defined. Here we treat the multiple integral of (**??**) as an oscillatory integral.*

Remark 4.5 *We explain some hurdles in case we try to treat* (4.1) *mathematically. Even when $F(q,p) \equiv 1$, each integral of the right hand side*

$$\lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi}\right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{\sum_{j=1}^{J+1}\left(i(x_j-x_{j-1})\cdot\xi_{j-1}-\int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt\right)} \prod_{j=1}^{J} dx_j d\xi_j$$

does not converge absolutely. Furthermore, the number J of integrals tends to ∞.

We explain the details of the convergence in Theorem 1.

Theorem 2 *Let $0 < T \le \mathbf{T} < \infty$. For any $F(q,p) \in \mathcal{F}_{\mathcal{Q}} \cup \mathcal{F}_{\mathcal{Q}}$, set*

$$e^{i(x_{J+1}-x_0)\cdot\xi_0} b_{\Delta_{T,0}}(x_{J+1},\xi_0,x_0) \equiv \left(\frac{1}{2\pi}\right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{i\phi(q_{\Delta_{T,0}},p_{\Delta_{T,0}})} F(q_{\Delta_{T,0}},p_{\Delta_{T,0}}) \prod_{j=1}^{J} dx_j d\xi_j\,.$$

Then, for any non-negative integers ℓ_1, ℓ_2, there exist non-negative integers ℓ_1', ℓ_2' and positive constants C_{ℓ_1,ℓ_2}, C'_{ℓ_1,ℓ_2} such that

$$\left|\partial_x^\alpha \partial_{\xi_0}^\beta b_{\Delta_{T,0}}(x,\xi_0,x_0)\right| \le A_{\ell_1',\ell_2'} C_{\ell_1,\ell_2} (\langle x\rangle + \langle \xi_0\rangle + \langle x_0\rangle)^L \langle \xi_0\rangle^{\delta|\alpha|-\rho|\beta|},$$
$$\left|\partial_x^\alpha \partial_{\xi_0}^\beta \left(b_{\Delta_{T,0}}(x,\xi_0,x_0) - b(T,0,x,\xi_0,x_0)\right)\right|$$
$$\le A_{\ell_1',\ell_2'} C'_{\ell_1,\ell_2} |\Delta_{T,0}|(T+U)(\langle x\rangle + \langle \xi_0\rangle + \langle x_0\rangle)^L \langle \xi_0\rangle^{2m+\delta|\alpha|-\rho|\beta|}$$

for any $|\alpha| \le \ell_1$, $|\beta| \le \ell_2$ with $b(T,0,x,\xi_0,x_0) \equiv \lim_{|\Delta_{T,0}|\to 0} b_{\Delta_{T,0}}(x,\xi_0,x_0)$.

5 We can produce many functionals $F(q,p) \in \mathcal{F}_{\mathcal{Q}}$, $F(q,p) \in \mathcal{F}_{\mathcal{P}}$

Theorem 3 (Examaples of $F(q,p) \in \mathcal{F}_{\mathcal{Q}} \cup \mathcal{F}_{\mathcal{P}}$) *Let $L \ge 0$, $0 \le \delta < \rho \le 1$, $0 \le t \le T$ and $0 \le T' \le T'' \le T$.*

(1) *If $|\partial_x^\alpha B(t,x)| \le C_\alpha \langle x\rangle^L$, then $F(q) \equiv B(t,q(t)) \in \mathcal{F}_{\mathcal{Q}}$.*

(2) *If $|\partial_\xi^\beta B(t,\xi)| \le C_\beta \langle \xi\rangle^{L-\rho|\beta|}$, then $F(p) \equiv B(t,p(t)) \in \mathcal{F}_{\mathcal{P}}$.*
In particular, $F(q,p) \equiv 1 \in \mathcal{F}_{\mathcal{Q}} \cap \mathcal{F}_{\mathcal{P}}$.

(3) *If $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta} (\langle x\rangle + \langle \xi\rangle)^L \langle \xi\rangle^{\delta|\alpha|-\rho|\beta|}$, then $F(q,p) \equiv \int_{[T',T'')} B(t,q(t),p(t))dt \in \mathcal{F}_{\mathcal{Q}} \cap \mathcal{F}_{\mathcal{P}}$.*

(4) *If $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta} \langle \xi\rangle^{\delta|\alpha|-\rho|\beta|}$, then $F(q,p) = e^{\int_{[T',T'')} B(t,q(t),p(t))dt} \in \mathcal{F}_{\mathcal{Q}} \cap \mathcal{F}_{\mathcal{P}}$.*

Remark 5.1 *To avoid the uncertain principle, we do not treat $q(t)$ and $p(t)$ at the same time t. We have $q(t) \in \mathcal{F}_\mathcal{Q}$, $p(t) \notin \mathcal{F}_\mathcal{Q}$ and $q(t) \notin \mathcal{F}_\mathcal{P}$, $p(t) \in \mathcal{F}_\mathcal{P}$. If $t \neq s$, we can treat $q(t) \cdot p(s)$ as the product of the two operators.*

Definition 5.2 (Functional derivatives) *For any q, $q' \in \mathcal{Q}$ and any p, $p' \in \mathcal{P}$, we define the functional derivatives along (q', p') by*

$$D_{(q',p')}F(q,p) = \lim_{\theta \to 0} \frac{F(q+\theta q', p+\theta p') - F(q,p)}{\theta} \,.$$

Remark 5.3 *Let $\Delta_{T,0}$ contain all times when q, q', p or p' breaks. Set $q(T_j) = x_j$, $q'(T_j) = x'_j$, $p(T_{j-1}) = \xi_{j-1}$ and $p'(T_{j-1}) = \xi'_{j-1}$. By $(q+\theta q')(t) = x_j + \theta x'_j$ on $(T_{j-1}, T_j]$, $(p+\theta p')(t) = \xi_{j-1} + \theta \xi'_{j-1}$ on $[T_{j-1}, T_j)$, we have*

$$F(q+\theta q', p+\theta p') = F_{\Delta_{T,0}}(x_{J+1} + \theta x_{J+1}, \xi_J + \theta \xi'_J, \ldots, \xi_0 + \theta \xi'_0, x_0 + \theta x'_0) \,.$$

Therefore, we can treat $D_{(q',p')}F(q,p)$ as a function as follows.

$$\begin{aligned} D_{(q',p')}F(q,p) &= \sum_{j=0}^{J+1} (\partial_{x_j} F_{\Delta_{T,0}})(x_{J+1}, \xi_J, x_J, \ldots, \xi_0, x_0) \cdot x'_j \\ &\quad + \sum_{j=0}^{J} (\partial_{\xi_j} F_{\Delta_{T,0}})(x_{J+1}, \xi_J, x_J, \ldots, \xi_0, x_0) \cdot \xi'_j \,. \end{aligned}$$

Thus, we have the linearity of functional derivatives

$$D_{c'(q',p')+c''(q'',p'')}F(q,p) = c' D_{(q',p')}F(q,p) + c'' D_{(q'',p'')}F(q,p)$$

for q', $q'' \in \mathcal{Q}$, p', $p'' \in \mathcal{P}$ and c', $c'' \in \mathbb{R}$. By $F_{\Delta_{T,0}} \in C^\infty(\mathbb{R}^{d(2J+3)})$, we can treat the functional derivatives of higher order.

Theorem 4 (Algebra on $\mathcal{F}_\mathcal{Q}$, $\mathcal{F}_\mathcal{P}$)

(1) *For any $F(q,p)$, $G(q,p) \in \mathcal{F}_\mathcal{Q}$, any $q' \in \mathcal{Q}$, any $p' \in \mathcal{P}$, any $d \times d$ real matrix V and any $d \times d$ real-regular matrix W, we have*

$$F(q,p) + G(q,p) \in \mathcal{F}_\mathcal{Q} \,, \ F(q,p)G(q,p) \in \mathcal{F}_\mathcal{Q} \,,$$

$$F(q+q', p+p') \in \mathcal{F}_\mathcal{Q} \,, \ F(Vq, Wp) \in \mathcal{F}_\mathcal{Q}, \ D_{(p',q')}F(q,p) \in \mathcal{F}_\mathcal{Q}.$$

(2) *For any $F(q,p)$, $G(q,p) \in \mathcal{F}_\mathcal{P}$, any $q' \in \mathcal{Q}$, any $p' \in \mathcal{P}$, any $d \times d$ real matrix V and any $d \times d$ real-regular matrix W, we have*

$$F(q,p) + G(q,p) \in \mathcal{F}_\mathcal{P} \,, \ F(q,p)G(q,p) \in \mathcal{F}_\mathcal{P} \,,$$

$$F(q+q', p+p') \in \mathcal{F}_\mathcal{P} \,, \ F(Vq, Wp) \in \mathcal{F}_\mathcal{P} \,, \ D_{(p',q')}F(q,p) \in \mathcal{F}_\mathcal{P} \,.$$

Remark 5.4 *The two sets $\mathcal{F}_\mathcal{Q}$, $\mathcal{F}_\mathcal{P}$ are closed under addition, multiplication, translation, real linear transformation and functional differentiation. Applying Theorem 4 to Theorem 3, we can produce more functionals $F(q,p) \in \mathcal{F}_\mathcal{Q} \cup \mathcal{F}_\mathcal{P}$. However, the part $\int_{[0,T)} p(t) \cdot dq(t)$ of $e^{i\phi(q,p)}$ does not always have good properties under these operations. Therefore, we must pay attention to which properties are valid in the phase space path integrals.*

6 Properties of phase space path integrals

We can interchange the order of the path integration and some integrations as follows.

Theorem 5 (Interchange of the order of path integrals and integrals) *Let $L \geq 0$. Let $0 \leq \delta < \rho \leq 1$ and $0 \leq T' \leq T'' \leq T$.*

(1) *Assume that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous on $[0,T] \times \mathbb{R}^d$ and $|\partial_x^\alpha B(t,x)| \leq C_\alpha \langle x \rangle^L$. Then, for any $F(q,p) \in \mathcal{F}_{\mathcal{Q}}$ including $F(q,p) \equiv 1$, we have*

$$\begin{aligned}&\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} \int_{[T',T'')} B(t,q(t))F(q,p)dt\mathcal{D}(q,p) \\ &\quad = \int_{[T',T'')} \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} B(t,q(t))F(q,p)\mathcal{D}(q,p)dt\,. \end{aligned} \tag{6.1}$$

(2) *Assume that for any multi-index β, $\partial_\xi^\beta B(t,\xi)$ is continuous on $[0,T]\times\mathbb{R}^d$ and $|\partial_\xi^\beta B(t,\xi)| \leq C_\beta \langle \xi \rangle^{L-\rho|\beta|}$. Then, for any $F(q,p) \in \mathcal{F}_{\mathcal{P}}$ including $F(q,p) \equiv 1$, we have*

$$\begin{aligned}&\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} \int_{[T',T'')} B(t,p(t))F(q,p)dt\mathcal{D}(q,p) \\ &\quad = \int_{[T',T'')} \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} B(t,p(t))F(q,p)\mathcal{D}(q,p)dt\,. \end{aligned} \tag{6.2}$$

Remark 6.1 *On the left-hand sides of* (6.1) *and* (6.2)*, we perform the path integration after the integration with respect to time t. On the right-hand side of* (6.1) *and* (6.2)*, we perform the path integration before the integration with respect to time t.*

Remark 6.2 *To avoid the uncertain principle, we do not treat $q(t)$ and $p(t)$ at the same time t.*

We can interchange the order of the path integration and some limit operations, and have the perturbative expansion as follows.

Theorem 6 (Perturbative expansion) *Assume that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous on $[0,T] \times \mathbb{R}^d$ and $|\partial_x^\alpha B(t,x)| \leq C_\alpha$. Then we have*

$$\begin{aligned}&\int_{q(T)=x,\,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)+\int_{[T',T'')} B(t,q(t))dt}\mathcal{D}(q,p) \\ &\quad = \sum_{n=0}^{\infty} \int_{[T',T'')} d\tau_n \cdots \int_{[T',\tau_3)} d\tau_2 \int_{[T',\tau_2)} d\tau_1 \\ &\qquad \times \int_{q(T)=x,\,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)} B(\tau_n,q(\tau_n)) \cdots B(\tau_2,q(\tau_2))B(\tau_1,q(\tau_1))\mathcal{D}(q,p)\,. \end{aligned}$$

Theorem 7 (Orthogonal transformation w.r.t. paths) *For any $F(q,p) \in \mathcal{F}_{\mathcal{Q}} \cup \mathcal{F}_{\mathcal{P}}$ and any $d\times d$ orthogonal matrix Q, we have*

$$\begin{aligned}&\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(Qq,Qp)} F(Qq,Qp)\mathcal{D}(q,p) \\ &\quad = \int_{q(T)=Qx,p(0)=Q\xi_0,q(0)=Qx_0} e^{i\phi(q,p)} F(q,p)\mathcal{D}(q,p)\,. \end{aligned} \tag{6.3}$$

Remark 6.3 *On the left-hand side of* (6.3)*, we perform the orthogonal transformation of all paths. On the right-hand side of* (6.3)*, we perform the orthogonal transformation of the endpoints.*

Theorem 8 (Translation w.r.t. momentum paths) *For any $p' \in \mathcal{P}$, we have*

$$G(q) \equiv e^{i\int_{[0,T)} p'(t)dq(t)} \in \mathcal{F}_{\mathcal{Q}}\,.$$

Furthermore, for any $F(q,p) \in \mathcal{F}_{\mathcal{Q}}$, we have

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p+p')}F(q,p+p')\mathcal{D}(q,p) = \int_{q(T)=x,p(0)=\xi_0+p'(0),q(0)=x_0} e^{i\phi(q,p)}F(q,p)\mathcal{D}(q,p). \tag{6.4}$$

Remark 6.4 *On the left-hand side of* (6.4)*, we perform the translation of all paths with respect to the momentum path. On the right-hand side of* (6.4)*, we perform the translation of the endpoint with respect to the momentum path.*

Remark 6.5 *The author has not proved that $G(p) \equiv e^{i\int_{[0,T)} p(t)dq'(t)} \in \mathcal{F}_{\mathcal{P}}$ for any $q' \in \mathcal{Q}$. Therefore, the author has not proved that for any $F(q,p) \in \mathcal{F}_{\mathcal{P}}$,*

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q+q',p)}F(q+q',p)\mathcal{D}(q,p) = \int_{q(T)=x+q'(T),p(0)=\xi_0,q(0)=x_0+q'(0)} e^{i\phi(q,p)}F(q,p)\mathcal{D}(q,p)\,,$$

but which was given in the Schrödinger case [11, Theorem 6 (2)*].*

Theorem 9 (Integration by parts formula w.r.t. momentum paths) *Let $0(t) \equiv 0$. Then, for any $p' \in \mathcal{P}$, we have $D_{(0,p')}\phi(q,p) \in \mathcal{F}_{\mathcal{Q}}$. Furthermore, for any $F(q,p) \in \mathcal{F}_{\mathcal{Q}}$ and any $p' \in \mathcal{P}$ with $p'(0) = 0$, we have*

$$\begin{aligned}&\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)}(D_{(0,p')}F)(q,p)\mathcal{D}(q,p)\\ &\quad = -i\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{i\phi(q,p)}(D_{(0,p')}\phi)(q,p)F(q,p)\mathcal{D}(q,p).\end{aligned} \tag{6.5}$$

Remark 6.6 *On the left-hand side of* (6.5)*, we differentiate $F(q,p)$ with respect to momentum paths. On the right hand side of* (6.5)*, we differentiate $e^{i\phi(q,p)}$ with respect to momentum paths.*

7 Outline for Proof of Theorem 1

We explain the outline for the proof of Theorem 1.

To prove the convergence of the multiple (oscillatory) integral

$$\lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi}\right)^{dJ}\int_{\mathbb{R}^{2dJ}} e^{i\phi(q_{\Delta_{T,0}},p_{\Delta_{T,0}})}F(q_{\Delta_{T,0}},p_{\Delta_{T,0}})\prod_{j=1}^{J} d\xi_j dx_j \tag{7.1}$$

we have only to add many assumptions to $F(q_{\Delta_{T,0}},p_{\Delta_{T,0}}) = F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_0,x_0)$ so that (7.1) converges, and to define two sets $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ by these assumptions. Do not consider other things. Then $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ will be larger as sets and contain at least one example $F(q,p) \equiv 1$. Add assumptions closed under addtion and multiplication. Then $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ will be closed under addition and multiplication. Our proof consists of the following 3 steps:

1. Control (7.1) by C^J as $J \to \infty$. Here we use H. Kumano-go-Taniguchi-Tsutsumi's idea ([19], [9]).

2. Control (7.1) by C independent of $J \to \infty$ ($|\Delta_{T,0}| \to 0$). Here we use Fujiwara's idea ([6]).

3. Add assumptions so that (7.1) converges as $|\Delta_{T,0}| \to 0$.

For the properties of the path integrals, we have only to prove the properties that we can prove.

7.1 Do the same thing over and over again

The oscillatory integral is defined by the integration by parts. For the multiple oscillatory integrals in [19], [9], H. Kumano-go-Taniguchi-Tsutsumi did the integration by parts over and over again and obtained a sharper result. We use this idea.

Let

$$\psi = \sum_{j=1}^{J+1} (x_j - x_{j-1}) \cdot \xi_{j-1}, \quad a = e^{-\sum_{j=1}^{J+1} \int_{[T_{j-1},T_j)} H(t,x_j,\xi_{j-1})dt} F_{\Delta_{T,0}} .$$

Using the differential operators $M_j = \dfrac{1 - i\psi \cdot \partial_{\xi_j}}{1 + |\psi|^2}$, $N_j = \dfrac{1 - i\psi \cdot \partial_{x_j}}{1 + |\psi|^2}$ with $M_j e^{i\psi} = e^{i\psi}$, $N_j e^{i\psi} = e^{i\psi}$, we repeat the integration by parts.

$$\left(\frac{1}{2\pi}\right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{i\psi} a \prod_{j=1}^{J} dx_j d\xi_j = \left(\frac{1}{2\pi}\right)^{dJ} \int_{\mathbb{R}^{2dJ}} e^{i\psi} a^{\spadesuit} \prod_{j=1}^{J} dx_j d\xi_j \, ,$$

where $a^{\spadesuit} = (N_J^*)^\ell \cdots (N_2^*)^\ell (N_1^*)^\ell (M_J^*)^{\ell'} \cdots (M_2^*)^{\ell'} (M_1^*)^{\ell'} a$ with the adjoint operators M_j^*, N_j^* of M_j, N_j. Generally speaking, we can not control the multiple oscillatory integral of (7.1) by C^J as $J \to \infty$. However, if we assume the following, we can control the multiple integral of (7.1) by C^J as $J \to \infty$.

Assumption 7.1 *For any non-negative integers ℓ_1, ℓ_2, there exist positive constants A_{ℓ_1,ℓ_2}, B_{ℓ_1,ℓ_2} such that for any multi-indices $|\alpha_j| \le \ell_1$ and $|\beta_{j-1}| \le \ell_2$,*

$$\begin{aligned} &\left| \left(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, \dots, x_1, \xi_0, x_0) \right| \\ &\le A_{\ell_1,\ell_2} (B_{\ell_1,\ell_2})^{J+1} \left(\sum_{j=1}^{J+1} \langle x_j \rangle + \sum_{j=1}^{J+1} \langle \xi_{j-1} \rangle + \langle x_0 \rangle \right)^L \prod_{j=1}^{J+1} \langle \xi_{j-1} \rangle^{\delta|\alpha_j| - \rho|\beta_{j-1}|}. \end{aligned} \tag{7.2}$$

Example 7.2 *Let $F(q,p) \equiv e^{\int_{[0,T)} B(t,q(t),p(t))dt}$ with $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{\delta|\alpha| - \rho|\beta|}$. Then $F_{\Delta_{T,0}} = e^{\sum_{j=1}^{J+1} \int_{[T_{j-1},T_j)} B(t,x_j,\xi_{j-1})dt} = \prod_{j=1}^{J+1} e^{\int_{[T_{j-1},T_j)} B(t,x_j,\xi_{j-1})dt}$ satisfies the inequalities of (7.2), (7.3), (7.4) with $L = 0$ and $u_j = t_j$.*

7.2 Do simple things

For the multiple integrals in [6], Fujiwara did only simple integrals that appear as main terms in the calculations of the multiple integral. He skipped many complicated integrals that appear as remainder terms. Carrying out this rule until the end, he forced all the many complicated integrals of remainder terms on H. Kumano-go-Taniguchi-Tsutsumi's estimate. We use this idea.

Let $2m-(\rho-\delta)N \le 0$. The asymptotic expansion of pseudo-differential operator with double symbol $a(x_2,\xi_1,x_1,\xi_0)$ is written by

$$\sum_{|\alpha_1|<N} \frac{(-i)^{|\alpha_1|}}{\alpha_1!}(\partial_{\xi_1}^{\alpha_1}\partial_{x_1}^{\alpha_1}a)(x_2,\xi_0,x_2,\xi_0)+r_N(x_2,\xi_0)\,,\quad |r_N(x_2,\xi_0)|\le Ct_2.$$

The main term $a(x_2,\xi_0,x_2,\xi_0)$ are obtained by setting $\xi_1=\xi_0$ and $x_1=x_2$ in $a(x_2,\xi_1,x_1,\xi_0)$. We also note the key lemma below.

Lemma 7.3 *If $\xi_1=\xi_0$ and $x_1=x_2$, we have*

$$q_{T_2,T_1,0}(x_2,x_2,x_0)=q_{T_2,0}(x_2,x_0),\quad p_{T_2,T_1,0}(\xi_0,\xi_0)=p_{T_2,0}(\xi_0),$$

$$F_{T_2,T_1,0}(x_2,\xi_0,x_2,\xi_0,x_0)=F(q_{T_2,T_1,0},p_{T_2,T_1,0})=F(q_{T_2,0},p_{T_2,0})=F_{T_2,0}(x_2,\xi_0,x_0)\,.$$

If we assume the following for any $\Delta_{T,0}$ as the inequality of Assumption 4.4 (1) for $F(q,p)\in\mathcal{F}_{\mathcal{Q}}$ and use $e^{\sum_{j=1}^{J+1}t_j}=e^T<\infty$, then we can control the multiple integral of (7.1) by C indepent of $J\to\infty$.

Assumption 7.4 *For any non-negative integers ℓ_1, ℓ_2, there exist positive constants A_{ℓ_1,ℓ_2}, B_{ℓ_1,ℓ_2} such that for any multi-indices $|\alpha_j|\le\ell_1$ and $|\beta_{j-1}|\le\ell_2$,*

$$\left|\left(\prod_{j=1}^{J+1}\partial_{x_j}^{\alpha_j}\partial_{\xi_{j-1}}^{\beta_{j-1}}\right)F_{\Delta_{T,0}}(x_{J+1},\xi_J,\dots,x_1,\xi_0,x_0)\right|$$
$$\le A_{\ell_1,\ell_2}(B_{\ell_1,\ell_2})^{J+1}\left(\sum_{j=1}^{J+1}\langle x_j\rangle+\sum_{j=1}^{J+1}\langle\xi_{j-1}\rangle+\langle x_0\rangle\right)^L\prod_{j=1}^{J+1}(t_j)^{\min(|\beta_{j-1}|,1)}\prod_{j=1}^{J+1}\langle\xi_{j-1}\rangle^{\delta|\alpha_j|-\rho|\beta_{j-1}|}\,. \quad (7.3)$$

Remark 7.5

1. *If $J=0$, $F_{T,0}(x_1,\xi_0,x_0)$ is controlled by $(B_{\ell_1,\ell_2})^1$.*
2. *If $J=1$, $F_{T,T_1,0}(x_2,\xi_1,x_1,\xi_0,x_0)$ is controlled by $(B_{\ell_1,\ell_2})^2$.*
3. *If $J=2$, $F_{T,T_2,T_1,0}(x_3,\xi_2,x_2,\xi_1,x_1,\xi_0,x_0)$ is controlled by $(B_{\ell_1,\ell_2})^3$.*

7.3 Consider multiple integrals by paths.

To prove the convergence, we have only to make a Cauchy sequence. To make a Cauchy sequence, we compare two multiple integrals:

$$\int\cdots\int\cdots\int\cdots\int \qquad \prod_{j=1}^{J}d\xi_j dx_j\,.$$

and

$$\int\cdots\int \qquad \int\cdots\int \quad \prod_{j=N+1}^{J}d\xi_j dx_j\prod_{j=1}^{n-1}d\xi_j dx_j\,.$$

Two integrands of two multiple integrals are different in the numbers of variables. However we can compare the two integrand using the lemma below.

Lemma 7.6 *Set $x_j^\circ = x_{N+1}$, $\xi_j^\circ = \xi_{n-1}$, $j = n, \cdots, N$. Then we have*

$$
\begin{aligned}
&F_{\Delta_{T,0}}(x_{J+1}, \xi_J, \ldots, x_{N+1}, \xi_N^\circ, x_N^\circ, \ldots, \xi_n^\circ, x_n^\circ, \xi_{n-1}, \ldots, x_1, \xi_0, x_0) \\
&\quad = F(q_{\Delta_{T,0}}, p_{\Delta_{T,0}}) = F(q_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}, p_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}) \\
&\quad = F_{(\Delta_{T,T_{N+1}}, \Delta_{T_{n-1},0})}(x_{J+1}, \xi_J, \ldots, x_{N+1}, \xi_{n-1}, \ldots, x_1, \xi_0, x_0)\,.
\end{aligned}
$$

Add the parameter $u_j \geq 0$ with $\sum_{j=1}^{J+1} u_j = U < \infty$ depending $\Delta_{T,0}$ and for any $|\alpha_k| > 0$ $(1 \leq k \leq J)$ and assume the following as the second inequality of Assumption 4.4 (1) for $F(q,p) \in \mathcal{F}$. Then the multiple integral of (7.1) converges as $|\Delta_{T,0}| \to 0$.

Assumption 7.7 *Let $L \geq 0$. Let $u_j \geq 0$ with $\sum_{j=1}^{J+1} u_j = U < \infty$ depending $\Delta_{T,0}$. For any non-negative integers ℓ_1, ℓ_2, there exist positive constant A_{ℓ_1,ℓ_2}, B_{ℓ_1,ℓ_2} such that for any $\Delta_{T,0}$ and any multi-indices $|\alpha_j| \leq \ell_1$, $|\beta_{j-1}| \leq \ell_2$, and $|\alpha_k| > 0$ $(1 \leq k \leq J)$,*

$$
\begin{aligned}
&\left| \left(\prod_{j=1}^{J+1} \partial_{x_j}^{\alpha_j} \partial_{\xi_{j-1}}^{\beta_{j-1}} \right) F_{\Delta_{T,0}}(x_{J+1}, \xi_J, \ldots, x_1, \xi_0, x_0) \right| \\
&\quad \leq A_{\ell_1,\ell_2} (B_{\ell_1,\ell_2})^{J+1} u_k \left(\sum_{j=1}^{J+1} \langle x_j \rangle + \sum_{j=1}^{J+1} \langle \xi_{j-1} \rangle + \langle x_0 \rangle \right)^L \\
&\quad\quad \times \prod_{j=1, j\neq k}^{J+1} (t_j)^{\min(|\beta_{j-1}|,1)} \prod_{j=1}^{J+1} \langle \xi_{j-1} \rangle^{\delta|\alpha_j| - \rho|\beta_{j-1}|}\,.
\end{aligned} \tag{7.4}
$$

Remark 7.8 *Measure theory considers the base. However, integration theory considers the area, i.e., the product of the base and the height. This assumption implies that, if the difference of two paths is small, the difference $\partial_{x_k} F_{\Delta_{T,0}}$ of the two heights is controlled by u_k with $\sum_{j=1}^{J+1} u_j = U < \infty$.*

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