Reliability Evaluation of Linear Consecutive-k-out-of-n:G Systems

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1 Introduction

As most of industrial systems become more complex and multiple-function oriented, such as aircrafts, submarines, military systems, and nuclear systems, it is extremely important to prevent accidents and reduce the causes of failure, which can be dangerous or disastrous [1]. As a result, monitoring and evaluating the performance of the system is essential to ensure the normal operation. Reliability, or the probability of survival, is a critical performance metric of a component or a system, and is defined as the probability that a component or a system will perform its required function under given conditions for a stated time interval [2]. Other measures of performance include failure rate, percentile of system life, mean time to failure, mean time between failures, availability, mean time between repairs, and maintainability.

In realistic, systems are large and complicated, yet they often have characteristic features and structures. In study of these practical systems, we often simplify system models as particular types of coherent systems based on these characteristic features and structures, where a coherent system is one in which every component is relevant for the system and the lifetime is non-decreasing function of components lifetimes. In reliability theory, literature has focused on different types of coherent systems. In 1980, Kontoleon [3] first studied such a system where a cluster of failed components causes system failure, and subsequently, Chiang and Niu [4] formally named it "consecutive-k-out-of-n:F system", where the system consists of n components and it fails if and only if at least k consecutive components fail. On contrast, Tong [5] first introduced the consecutive-k-out-of-n:G system, where the system consists of n components and it works if and only if at least k consecutive components work. Kuo *et al.* [6] well explained the relationship between the consecutive-k-out-of-n:F system and the consecutive-k-out-of-n:G system. The consecutive-k-out-of-n systems also include the series and the parallel systems as special cases, similarly k-out-of-n systems.

For a linear consecutive-k-out-of-n:G system, n components are arranged in a line and it works if and only if at least k consecutive components work. Fig. 1 depicts the linear consecutive-k-out-of-n:G system. To make it easy to understand, we consider an example of a consecutive-2-out-of-4:G system. As shown in Fig. 2, the all system status are listed. Fig. 2 (a) gives the all working status of the consecutive-2-out-of-4:G system, and Fig. 2 (b) gives the all failed status of the consecutive-2-out-of-4:G system. Define the random variable X_j be the state of the jth component during the total n components ($j = 1, 2, \cdots, n$), then

$$X_j = \begin{cases} 0 & \text{if the } j\text{th component failed} \\ 1 & \text{if the } j\text{th component works} \end{cases} \tag{1}$$

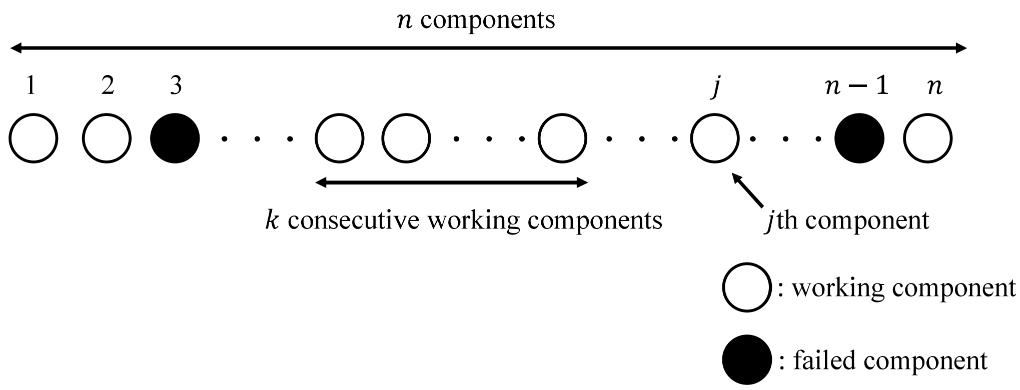

Figure 1： A linear consecutive-k-out-of-n:G system.

Let $\phi_{k|n:G}(X_1, \cdots, X_n)$ denote the structure function of the linear consecutive-k-out-of-n:G system. Then we have

$$\begin{aligned}\phi_{k|n:G}(X_1, \cdots, X_n) &= 1 - \phi_{k|n:F}(1 - X_1, \cdots, 1 - X_n), \\ &= 1 - \prod_{j=1}^{n-k+1} \{1 - \prod_{i=j}^{j+k-1} X_i\}.\end{aligned}$$

Consecutive-k-out-of-n:G systems have been applied to many complex systems. A railroad operation is an application of the consecutive-k-out-of-n:G system [7]. Consider the railroad system with 17 lines numbered from line 1 to line 17. The using density of a line was considered as the probability that the line is not available. When a special train with over-limit loading of some vehicles, the neighbor lines of the line that receives the train must be empty; that is at least three consecutive empty lines is required. The problem of interest is the probability that the special train can enter the station without delay. Kuo and Zuo [8] also gave the example of a photographing of a nuclear accelerator. In analysis of the acceleration activities that occur in a nuclear accelerator, high-speed cameras are used to take pictures of the activities. Because of the high speed of the activities and the high cost involved in implementing such an experiment, the photographing system must be very reliable and accurate. A set of n cameras are installed around the accelerator. If and only if at least k consecutive cameras work properly can the photographing system work properly. The problems of interest include the evaluation of the reliability of the photographing system and the optimal arrangement of the cameras with different reliabilities.

In this paper, we summarize the several proposed methods of system reliability evaluation of the consecutive-k-out-of-n:G system. We first give the result of the expression of system reliability with closed-form, which was proposed in [9]. We furthermore give the other method to calculate system reliability by using system signature. Also, the expected number of failed components at a particular time of system working can also be calculated in this method. On the other hand, in order to grasp the degradation process of the system, the path method to calculate system reliability is also discussed.

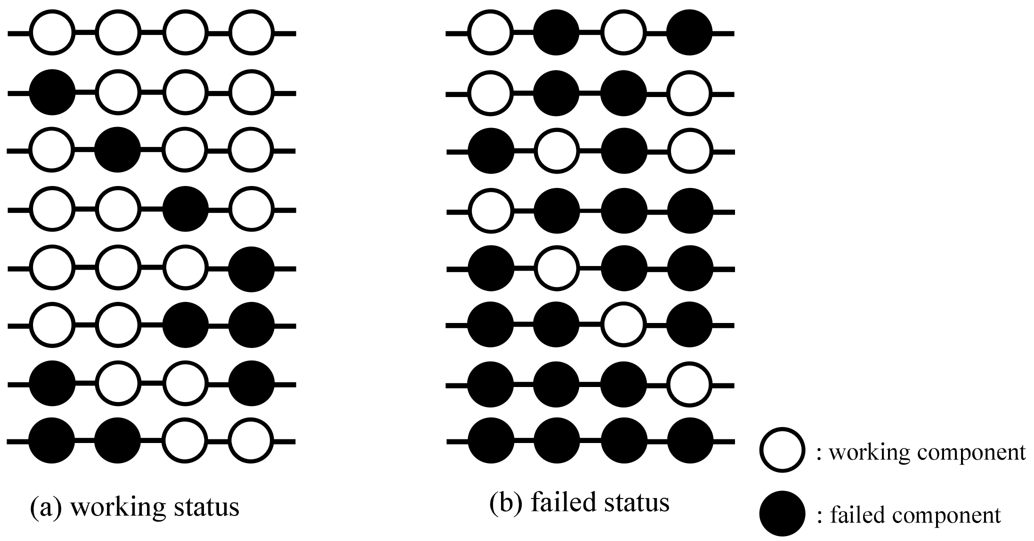

Figure 2 : Status of a consecutive-2-out-of-4:G system.

2 System Reliability Evaluation with Closed-Form

Kuo *et al.* [6] focused on the consecutive-k-out-of-n:G system where components are independent and have reliabilities $a_1, \cdots, a_n$. Then the reliability of consecutive-k-out-of-n:G systems is given by

$$R_G(k,n;a_1,\cdots,a_n) = R_G(k,n-1;a_1,\cdots,a_n) + [1-R_G(k,n-k-1;a_1,\cdots,a_n)](1-a_{n-k}) \prod_{i=n-k+1}^{n} a_i. \quad (2)$$

Furthermore, Gera [10] proposed another formulation of recursive equation with i.i.d. components and

$$R_G(k,n;a) = a^k + (1-a)\sum_{l=1}^{k} a^{l-1} R_G(k,n-l;a). \quad (3)$$

Although those recursive algorithms are computationally efficient, they have the usual disadvantage associated with a recursive algorithm of being a black box grinding out only numbers. The dependence of the reliability on the system parameters is hidden in the equations. For the Bernoulli model, reliabilities can be computed by using a combinatorial approach which is more explicit in nature. Fortunately, the closed-form for computing the number of ways of having working consecutive-k-out-of-n:F systems conditional on j failed components was obtained [11].

We propose the reliability of consecutive-k-out-of-n:G systems in closed expression with explicit sums by using the existing results [9]. The relationship between consecutive-k-out-of-n:F system and consecutive-k-out-of-n:G system was proposed by Kuo *et al.* [6]. Then, we can obtain the system reliability of a consecutive-k-out-of-n:G system by using the existing closed expression of the system reliability of a consecutive-k-out-of-n:F system and the relationship between these two systems.

We first give the relationship between these two types of systems.

Lemma 1 *Assume that the components in consecutive-k-out-of-n systems are independent but do not necessarily have the same lifetime distributions. Denote $R_G(k,n;a_1,\cdots,a_n)$ is the reliability of a* *consecutive-k-out-of-n:G system, and $R_F(k, n; b_1, \cdots, b_n)$ is the reliability of a consecutive-k-out-of-n:F system. Then if $a_i = 1 - b_i$ $(i = 1, \cdots, n)$, we have the result that*

$$R_G(k, n; a_1, \cdots, a_n) = 1 - R_F(k, n; b_1, \cdots, b_n).$$

Then we focus on the closed formulation of the reliability for consecutive-k-out-of-n:G systems. Denote that $N_G(j, k, n)$ is the number of combinations to arrange j $(j = k, \cdots, n)$ working components such that at least k consecutive components are working, then by using the duality relationship between consecutive-k-out-of-n:F systems and consecutive-k-out-of-n:G systems in Lemma 1, we have the following result.

Theorem 1 *The reliability of a consecutive-k-out-of-n:G system with i.i.d. components is*

$$R_G(k, n; t) = \sum_{j=k}^{n} N_G(j, k, n)\bar{F}(t)^j F(t)^{n-j}, \tag{4}$$

where

$$N_G(j, k, n) = \binom{n}{j} - \sum_{i=0}^{\lfloor j/k \rfloor} (-1)^i \binom{n-j+1}{i} \binom{n-ik}{n-j} (j = k, \cdots, n). \tag{5}$$

3 System Reliability Evaluation by Using System Signature

In this section, we summarize the results of the system reliability evaluation by using system signature, including the system reliability and the expected number of failed components [9]. Before giving the result, we first explain the definition of system signature and some notations are given as follows.

- $T_1, \cdots, T_n$: the component lifetimes.
- $T = \phi(T_1, \cdots, T_n)$: the lifetime of a coherent system consisting of independent and identical components with lifetime $T_1, \cdots, T_n$.
- $T_{[i]}$: the ith order statistic of n component lifetimes, that is, the time of the ith component failure.
- $\mathbf{s}$: system signature, $\mathbf{s} = (s_1, \cdots, s_n)$.

Kochar *et al.* [12] has given the survival function of any coherent system with i.i.d. components and

$$\Pr\{T > t\} = \sum_{i=1}^{n} s_i \cdot \Pr\{T_{[i]} > t\}, \tag{6}$$

where s_i is the probability that the system failed upon the occurrence of the ith component failure, i.e., $s_i = \Pr\{T = T_{[i]}\}$. Such the vector $\mathbf{s} = (s_1, \cdots, s_n)$ is called the system signature [13]. System signature has been found to be widely used in the evaluation of the system reliability and comparison of the performance among different systems [12, 14–16]. Furthermore, Boland [17] studied the system signature for any coherent system and gave the calculation formula by considering the number of path sets of any system with j working components, where

$$s_{n-j} = a_{j+1} - a_j, \tag{7}$$

in which

$$a_j = \frac{\#\text{ path sets of size } j \text{ for the system}}{\binom{n}{j}}.$$

We then give the signature-based expression of the reliability for the consecutive-k-out-of-n:G system. Denote that $F(t)$ is the lifetime distribution of each component and $\bar{F}(t)$ is the survival function of each component, we have

$$\begin{aligned}\Pr\{T > t\} &= \sum_{i=1}^{n-k+1} s_i \cdot \Pr\{T_{[i]} > t\}, \\ &= \sum_{i=1}^{n-k+1} s_i \cdot \sum_{j=n-i+1}^{n} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j}. \end{aligned} \tag{8}$$

To compute system signature of the consecutive-k-out-of-n:G system, note that N_{n-i} is the number of path sets with $(n-i)$ working components of the consecutive-k-out-of-n:G system where

$$N_{n-i} = \binom{n}{i} - \sum_{q=0}^{\lfloor (n-i)/k \rfloor} (-1)^q \binom{i+1}{q} \binom{n-qk}{i}, \tag{9}$$

and using Eq. (7), we can derive the expression of the system signature of the consecutive-k-out-of-n:G system, where

$$s_i = \frac{N_{n-i+1}}{\binom{n}{i-1}} - \frac{N_{n-i}}{\binom{n}{i}}. \tag{10}$$

Clearly, it is not difficult to proof that Eq. (4) and Eq. (8) are the same.

In addition, the number of failed components at the time of system failure gives the information that how many spare components should be available to replace failed components. Furthermore, the number of failed/working components when the system is working at a particular time also gives useful information to understand the behavior of the system. If the number of failed components when system is working is near the maximum number of failures that causes system failure, then we could consider to take maintenance and estimate that how many spare components should be prepared. Knowing the signature of a system is equivalent to knowing the distribution of the number of failed components at the moment when system failure occurs. Denote that X_n is the number of failed components in a failed system, and we give the following result which has been proposed in [18].

Proposition 1 *The expected number of failed components at the moment when system failure occurs for a consecutive-k-out-of-n:G system is*

$$\begin{aligned} E[X_n] &= \sum_{i=1}^{n-k+1} i \cdot s_i, \\ &= \sum_{i=1}^{n-k+1} i \cdot \left(\frac{N_{n-i+1}}{\binom{n}{i-1}} - \frac{N_{n-i}}{\binom{n}{i}} \right), \end{aligned} \tag{11}$$

where N_{n-i} is obtained in Eq. (9).

On the other hand, we also summarize the expected number of failed components at a particular time t [18]. As shown in Fig. 3, two cases are existed, where case 1 is that system failure occurs before time t, and case 2 is that system is working at time t.

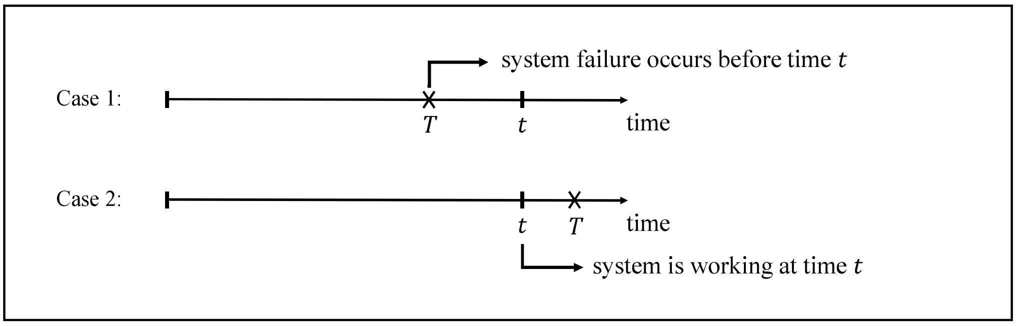

Figure 3： Case analysis for the expected number of failed components.

Case 1: System failure occurs before time t

Let $\{X(T) = i|T \le t\}$ denotes the number of failed components under the condition that system failure occurs before time t, then for a consecutive-k-out-of-n:G system with signature $\mathbf{s} = (s_1, \cdots, s_n)$, the conditional distribution of the number of failed components at the time when the system fails before time t is

$$\Pr\{X(T) = i|T \le t\} = \frac{s_i \cdot \sum_{j=0}^{n-i} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j}}{\Pr\{T \le t\}}. \tag{12}$$

As a result, the expected number of failed components under this condition is derived as

$$E[X(T)|T \le t] = \frac{\sum_{i=1}^{n-k+1} i s_i \sum_{j=0}^{n-i} \binom{n}{j} \bar{F}(t)^j F(t)^{n-j}}{\Pr\{T \le t\}}. \tag{13}$$

Case 2: System is working at time t

Denote that $X(t)$ is the number of failed components until time t, then for a consecutive-k-out-of-n:G system with signature $\mathbf{s} = (s_1, \cdots, s_n)$, the conditional distribution of the number of failed components at a particular time t when system is working is

$$\Pr\{X(t) = i|T > t\} = \frac{\binom{n}{i} \bar{F}(t)^{n-i} F(t)^i}{\Pr\{T > t\}} \sum_{j=i+1}^{n-k+1} s_j. \tag{14}$$

Then, the expected number of failed components under this condition is

$$E[X(t)|T > t] = \frac{\sum_{i=0}^{n-k} i \left(\sum_{j=i+1}^{n-k+1} s_j \right) \binom{n}{i} \bar{F}(t)^{n-i} F(t)^i}{\Pr\{T > t\}}. \tag{15}$$

These results of the expected number of failed components have been well applied in age-based preventive maintenance policies [18].

4 System Reliability Evaluation by Using Path Method

In section 2 and 3, we gave the system reliability expressions in two ways, which are the closed-form evaluation and the way by using system signature. These methods can perform reliability calculations efficiently, but cannot effectively predict the working state of the internal parts of the system. Therefore, limitations in the application of these calculation methods in maintenance problems exists. As a result, we proposed a path method to calculate the reliability of the linear consecutive-k-out-of-n:G system.

We consider a binary state: 1 for working state and 0 for failed state. At the beginning, all n identical components are working so that the system state vector is $(1, 1, \cdots, 1)$. The component fails one by one and the component failure sequences are constructed. These sequences from the beginning state (all working component states) to the system failure state, which consists of no k consecutive working components, are called the system failure paths. We use the path method proposed by Endharta *et al.* [19] to estimate the system failure distribution. Obviously, there are at most $n!$ paths to the system failure for a consecutive-k-out-of-n:G system. Suppose the number of steps (the number of failed components) until the system failure in path j ($1 \leq j \leq n!$) is denoted as N_j, the sum of failure rates of working components after ith failure ($1 \leq i \leq N_j - 1$) in path j is denoted as α_{ji}, and the failure rate of a failed component which will be failed after ith failure in path j is denoted as β_{ji}.

In order to obtain the system failure time distribution, a lemma is firstly introduced [20].

Lemma 2 *Let $Y_1, Y_2, \cdots, Y_m$ be exponentially distributed random variables with failure rates $h_1, \cdots, h_m$, and let $Z = \min\Gamma(Y_1, Y_2, \cdots, Y_m)$. Then Z is also exponentially distributed with failure rate $\sum h_i$ and $\Pr\{Z = Y_i\} = h_i / \sum h_i$.*

Lemma 2 gives the probability of selecting the component which will fail in step i ($1 \leq i \leq N_j - 1$) among the working components. Define X_i as the time between $(i-1)$th failure and ith failure in the system, X_{ji} as the time between $(i-1)$th failure and ith failure in path j, and π_j as the probability that the system failure follows path j, then Endharta *et al.* [19] gave the following results:

$$\begin{aligned} \pi_j &= \Pr\{X_1 = X_{j1}, X_2 = X_{j2}, \cdots, X_{N_j} = X_{jN_j}\}, \\ &= \Pr\{X_1 = X_{j1}\} \cdot \prod_{i=2}^{N_j} \Pr\{X_i = X_{ji} | X_1 = X_{j1}, \cdots, X_{i-1} = X_{j(i-1)}\}, \end{aligned} \tag{16}$$

where

$$\Pr\{X_1 = X_{j1}\} = \frac{\beta_{j0}}{\alpha_{j0}},$$

and

$$\Pr\{X_i = X_{ji} | X_1 = X_{j1}, X_2 = X_{j2}, \cdots, X_{i-1} = X_{j(i-1)}\} = \frac{\beta_{j(i-1)}}{\alpha_{j(i-1)}}.$$

As a result, the probability that system states follows path j becomes

$$\pi_j = \prod_{i=0}^{N_j - 1} \frac{\beta_{ji}}{\alpha_{ji}}. \tag{17}$$

Denote s_j^{op} as the step when only one set with i ($k \leq i \leq 2k-1$) working components exists in path j. Then it is easy to understand that until step s_j^{op}, system will never fail. That is, system failure distribution probability in path j can be calculated from step s_j^{op}. Then based on the results in [19], system failure probability in path j for the consecutive-k-out-of-n:G system can be calculated by

$$F_j(t) = 1 - \sum_{i=0}^{N_j - 1} A_{ji} \mathrm{e}^{-\alpha_{ji} t}, \tag{18}$$

where $A_{ji} = \prod_{m=0,m\neq i}^{N_j-1} \frac{\alpha_{jm}}{\alpha_{jm}-\alpha_{ji}}$. As a result, denote that P is the number of total paths to system failure, then the system failure probability can be estimated as

$$F(t) = 1 - \sum_{j=1}^{P} \pi_j \sum_{i=0}^{N_j-1} A_{ji} \mathrm{e}^{-\alpha_{ji} t}, \tag{19}$$

where π_j is given in Eq. (17).

By using the path method, we discuss the expected number of failed components again. Same as section 3, the expected number of failed components includes two cases. In Case 1, we define N^{PM} as the number of failed components at preventive maintenance time, and N_j^{PM} as the number of failed components at preventive maintenance time in path j. Then we have

$$E[N^{PM}] = \sum_{j=1}^{P} \pi_j E[N_j^{PM}], \tag{20}$$

where

$$E[N_j^{PM}] = \sum_{i=s_j^{op}}^{N_j-1} i \sum_{m=s_j^{op}}^{i-1} \frac{B_{jm}\alpha_{jm}}{\alpha_{ji}-\alpha_{jm}} (e^{-\alpha_{jm} t_{pm}} - e^{-\alpha_{ji} t_{pm}}),$$

in which

$$B_{jm} = \prod_{l=s_j^{op}, l\neq m}^{i-1} \frac{\alpha_{jl}}{\alpha_{jl}-\alpha_{jm}}.$$

Furthermore, in Case 2, define N^{SF} as the number of failed components when system fails, then we can easily obtain that

$$E[N^{SF}] = \sum_{j=1}^{P} \pi_j N_j F_j(t_{pm}), \tag{21}$$

where $F_j(t)$ is proposed in Eq. (18).

5 Conclusion

In this paper, we summarized several proposed methods to calculate reliability of the linear consecutive-k-out-of-n:G system. For the method of closed-form, it can efficiently obtain the reliability of the system. For the method by using system signature, it can not only obtain the reliability efficiently, but also can investigate the expected number of failed components whether system is working or failed. It gives more information of the system and can well improve the preventive maintenance for the system. Finally, we expain the path method for calculating system reliability. Although this method is not as efficient as the two methods before, but it focuses on the each component state in the system and give the possibility for condition maintenance policy or other efficient maintenance policies.

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