WELL-POSEDNESS OF STOCHASTIC NONLINEAR HEAT AND WAVE EQUATIONS DRIVEN BY SUBORDINATE CYLINDRICAL BROWNIAN NOISES ON THE TWO-DIMENSIONAL TORUS

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ABSTRACT. In [12], the author studied stochastic nonlinear heat equations and stochastic nonlinear wave equations driven by subordinate cylindrical Brownian noises on the two-dimensional torus by way of renormalization. More precisely, the author studied the local well-posedness of the renormalized version of the equations. In this note, we make some remarks on the global well-posedness of the stochastic nonlinear wave equation with a cubic nonlinearity after we briefly revisit the results in [12].

1. INTRODUCTION

In [12], the author studied the local well-posedness of stochastic nonlinear heat equations and stochastic nonlinear wave equations driven by subordinate cylindrical Brownian noises on the two-dimensional torus. In this note, we briefly discuss the global well-posedness of the stochastic nonlinear wave equation with a cubic nonlinearity, which we omitted in [12].

First, we recall the setting and the main result of [12]. In that study, the author studied the following Cauchy problems of the stochastic nonlinear heat equation and stochastic nonlinear wave equation on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$:

$$\begin{cases} \partial_t u - \Delta u = \lambda u^k + \partial_t W_L \\ u(0) = u_0 \end{cases} \tag{1.1}$$

$$\begin{cases} \partial_t^2 u - \Delta u = \lambda u^k + \partial_t W_L \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \tag{1.2}$$

where $k \geq 2$ is an integer and $\lambda \in \mathbb{R}$. The stochastic process W_L is called subordinate cylindrical Brownian motion and defined as follows: Let W be a cylindrical Brownian motion on $L^2(\mathbb{T}^2)$ defined on the probability space $(\Omega_1, \mathbb{P}_1)$ written by

$$W(t) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}^2} \beta^l(t) e_l \tag{1.3}$$

where $e_l(x) = e^{\sqrt{-1} l \cdot x}$ is the Fourier basis and $(\beta^l)_{l \in \mathbb{Z}^2}$ is an independent sequence of $\mathbb{C}$-valued Brownian motions conditioned with $\bar{\beta}^l = \beta^{-l}$ and $\mathrm{var}(\beta^l(t)) = t$. We also consider the $\mathbb{R}_+$-valued

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stochastic process L with $L(0) = 0$, non-decreasing and cádlág sample paths defined on another space $(\Omega_2, \mathbb{P}_2)$. Then, we define the process W_L defined on $(\Omega, \mathbb{P}) := (\Omega_1 \otimes \Omega_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ by

$$W_L(t) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}^2} \beta^l(L(t)) e_l. \tag{1.4}$$

It is well-known that when L is a Lévy process with non-decreasing paths, W_L is also a (distribution-valued) Lévy process. See [2, Section 1.3.2] for the example of such subordinators L.

When the noise is given by space time white noise i.e. when $L(t) = t$ in our setting, the equations (1.1) and (1.2) are related to the problem of stochastic quantization and are studied by a lot of researchers, see [5, 8, 9, 4] and references therein. In these equations, some ill-defined products of distributions appear in the equation because of the singularity of the noise, so we need to introduce renormalization in order to give meaning to the products. This kind of equations are called singular SPDEs and have attracted great attentions especially after the invention of the theory of regularity structure [10] and the theory of paracontorolled calculus [7].

Similarly to these previous works, it turns out that we need to introduce renormalization to give meaning to the nonlinearity. More precisely, we have to consider the following modified approximation of the equations:

$$\mathcal{L} u_N = \lambda H_k(u_N; C_N) + P_N \partial_t W_L \tag{1.5}$$

where $\mathcal{L} = \partial_t - \Delta$ or $\partial_t^2 - \Delta$, P_N is the approximation operator defined by

$$P_N f = \sum_{l \in \mathbb{Z}^2, |l| \leq N} \hat{f}(l) e_l \quad \text{for } f \in \mathcal{D}'(\mathbb{T}^2), \tag{1.6}$$

H_k is the k-th Hermite polynomial defined via

$$e^{tx - \frac{1}{2}\sigma^2 t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma^2) \tag{1.7}$$

and the renormalization constant C_N is defined by

$$C_N(t) := \begin{cases} \dfrac{1}{2\pi} \displaystyle\sum_{l \in \mathbb{Z}^2, |l| \leq N} \int_0^t e^{2(s-t)|l|^2} dL(s) & \text{when } \mathcal{L} = \partial_t - \Delta \\ \dfrac{1}{2\pi} \displaystyle\sum_{l \in \mathbb{Z}^2, |l| \leq N} \int_0^t \frac{\sin^2((t-s)|l|)}{|l|^2} dL(s) & \text{when } \mathcal{L} = \partial_t^2 - \Delta. \end{cases} \tag{1.8}$$

Note that C_N does not depend on the space variable $x \in \mathbb{T}^2$ but depends on $\omega_2 \in \Omega_2$, so C_N is an L-measurable stochastic process. Then, we decompose the solution u_N as $u_N = v_N + \Phi_N$ where Φ_N solves the linear equation

$$\mathcal{L} \Phi_N = P_N \partial_t W_L \tag{1.9}$$

with initial condition 0, see Section 2 for more precise definition. This decomposition is called Da Prato-Debussche trick in the context of singular SPDEs. Then, v_N solves

$$\begin{aligned} \mathcal{L} v_N &= \lambda H_k(v_N + \Phi_N; C_N) \\ &= \lambda \sum_{l=0}^{k} \binom{k}{l} v_N^{k-l} H_l(\Phi_N; C_N) \end{aligned} \tag{1.10}$$

where we use the algebraic property of the Hermite polynomials

$$H_k(a+b;c) = \sum_{l=0}^{k} \binom{k}{l} a^{k-l} H_l(b;c). \tag{1.11}$$

In the argument of renormalization of singular SPDEs, we usually heavily use the Gaussianity of the noises, so similar kind of argument cannot be made in the case of non-Gaussian noises. In our setting (1.4), although W_L itself is not a Gaussian process in general, it turns out that we can deploy renormalization argument by exploiting the Gaussianity of W: We can show the convergence of the random coefficients $\Phi_N^{\diamond l} \coloneqq H_l(\Phi_N; C_N)$ of the equation (1.10), which are called the Wick powers, see Proposition 2.2. And consequently, we can show the convergence of the solution v_N (and u_N), see Theorems 1.1 and 1.2 below.

As the driving noises of SPDEs, Gaussian noises such as space-time white noise are considered to be the most important examples, but on the other hand, SPDEs driven by Lévy noise (or jump noise) are also studied by a lot of researchers especially after publication of the monograph [14], see [1, 3, 11, 16, 18] and references therein. We also note that our setting of the noise (1.4) includes some intresting type of the noise such as stable noises [6, 15, 17] and compound Poisson noises. Now, we recall the local well-posedness result of the equations (1.1) and (1.2) in [12].

Theorem 1.1. (cf. [12, Theorem 1.1]) *Let $k = 2$ and let v_N be the mild solution of*

$$\begin{cases} \partial_t v_N - \Delta v_N = \lambda \sum_{l=0}^{k} \binom{k}{l} v_N^{k-l} \Phi_N^{\diamond l} \\ v(0) = v_0. \end{cases} \tag{1.12}$$

Then for any given initial condition $v_0 \in B^{2/\gamma-\delta}_{\infty,\infty}(\mathbb{T}^2)$, there exists $T(\omega) > 0$ such that v_N converges to some v $\mathbb{P}$-almost-surely in

$$L^\gamma([0,T]\,; B^{2/\gamma-\delta}_{\infty,\infty}(\mathbb{T}^2)) \cap C([0,T]; B^{-\delta}_{\infty,\infty}(\mathbb{T}^2))$$

for any $\frac{2}{1-\epsilon} < \gamma < \frac{2}{\epsilon}$, $0 < \delta < \frac{2}{\gamma} - \epsilon$. Moreover the limit process v solves the equation

$$\begin{cases} \partial_t v - \Delta v = \lambda \sum_{l=0}^{k} \binom{k}{l} v^{k-l} \Phi^{\diamond l} \\ v(0) = v_0 \end{cases} \tag{1.13}$$

on the interval $[0, T(\omega)]$ where $\Phi^{\diamond k}$ are the Wick powers of Φ defined in Proposition 2.2.

Theorem 1.2. (cf. [12, Theorem 1.3]) *Let $k \geq 2$ and let v_N be the mild solution of*

$$\begin{cases} \partial_t^2 v_N - \Delta v_N = \lambda \sum_{l=0}^{k} \binom{k}{l} v_N^{k-l} \Psi_N^{\diamond l} \\ (v(0), \partial_t v(0)) = (v_0, v_1). \end{cases} \tag{1.14}$$

Then, for any given initial condition $(v_0, v_1) \in H^{1-\epsilon}(\mathbb{T}^2) \times H^{-\epsilon}(\mathbb{T}^2)$ with sufficiently small $\epsilon > 0$, there exists $T(\omega) > 0$ such that v_N converges to some v $\mathbb{P}$-almost-surely in $C([0,T]; H^{1-\epsilon}(\mathbb{T}^2)) \cap C^1([0,T]; H^{-\epsilon}(\mathbb{T}^2))$. Moreover the limit process v solves the equation

$$\begin{cases} \partial_t^2 v - \Delta v = \lambda \sum_{l=0}^{k} \binom{k}{l} v^{k-l} \Psi^{\diamond l} \\ (v(0), \partial_t v(0)) = (v_0, v_1) \end{cases} \tag{1.15}$$

on the interval $[0, T(\omega)]$, where $\Psi^{\diamond k}$ are the Wick powers of Ψ defined in Proposition 2.2.

Remark 1.3. *When $L(t) = t$, it is known that u_N converges for any positive integer k both heat and wave cases, see* [5, 8]*. On the other hand, we have not been able to obtain the convergence result for $k \geq 3$ in the case of the heat equation for general L. This difference between the heat case and the wave case comes from the difference in the time integrability of the Wick powers, see*

Proposition 2.2. This result (and the argument in the proof of the result) suggests that the heat case behaves worse than the wave case when the singular noise is of jump-type, differently from the case of space-time white noise, see [13].

As we pointed out in [12], when $k = 3$, $\lambda < 0$ and $\mathcal{L} = \partial_t^2 - \Delta$, it is straightforward to extend the locally-in-time convergence of u_N to the globally-in-time convergence by applying the argument with I-method in [9]. In Section 3, we will explain how to adjust their argument to our setting and we will prove the global well-posedness of the stochastic nonlinear wave equation with a cubic nonlinearity.

2. Linear equations

In this section, we give the definition of the mild solution of the stochastic linear heat equation

$$\begin{cases} \partial_t \Phi - \Delta \Phi = \partial_t W_L \\ \Phi(0) = 0, \end{cases} \tag{2.1}$$

and the stochastic linear wave equation

$$\begin{cases} \partial_t^2 \Psi - \Delta \Psi = \partial_t W_L \\ (\Psi(0), \partial_t \Psi(0)) = (0, 0). \end{cases} \tag{2.2}$$

From the Duhamel principle, it is natural to define the solutions of these equations as the following stochastic convolutions:

$$\Phi(t) = \int_0^t e^{(t-s)\Delta} dW_L(s) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}^2} \int_0^t e^{(s-t)|l|^2} d\beta_L^l(s) e_l \tag{2.3}$$

and

$$\Psi(t) = \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} dW_L(s) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}^2} \int_0^t \frac{\sin((t-s)|l|)}{|l|} d\beta_L^l(s) e_l \tag{2.4}$$

where $\beta_L^l(t) = \beta^l(L(t))$. Recall that W_L is defined by (1.4). The integrals $\int_0^t e^{(s-t)|l|^2} d\beta_L^l(s)$ and $\int_0^t \frac{\sin((t-s)|l|)}{|l|} d\beta_L^l(s)$ can be understood as the Young integrals thanks to the following lemma.

Lemma 2.1. (cf. [12, Lemma 3.1]) *Let $B = B(t)$ be a one-dimensional Brownian motion. Then, the sample paths of $B_L(t) = B(L(t))$ have the finite p-variation on any interval $[0, T]$ for any $p > 2$ almost surely.*

Proof. Let $P[0, T] = \{D = \{0 = t_0 < t_1 < \cdots < t_n = T\}\}$ be the set of all partitions of $[0, T]$. Because the sample paths of B is $\frac{1}{p}$-Hölder continuous almost surely for any $p > 2$, there holds

$$\sup_{D \in P[0,T]} \sum_{i=1}^{N} |B_L(t_i) - B_L(t_{i-1})|^p \lesssim \sup_{D \in P[0,T]} \sum_{i=1}^{N} |L(t_i) - L(t_{i-1})|^{p \times \frac{1}{p}} = L(T) < \infty.$$

□

From this lemma and the theory of the Young integral, we can see that the finite sums

$$\Phi_N(t) := P_N \Phi(t) = \frac{1}{2\pi} \sum_{|l| \le N} \int_0^t e^{(s-t)\langle l \rangle^2} d\beta_L^l(s) e_l \tag{2.5}$$

and

$$\Psi_N(t) := P_N\Psi(t) = \frac{1}{2\pi}\sum_{|l|\le N}\int_0^t \frac{\sin((t-s)\langle l\rangle)}{\langle l\rangle} d\beta_L^l(s)e_l \tag{2.6}$$

are well-defined as stochastic processes. Then, from Proposition 2.2 below, we see that Φ_N and Ψ_N converge as $N\to\infty$.

Now, we recall the convergence result of the Wick powers of Φ_N and Ψ_N defined by

$$\Phi_N^{\diamond k} := H_k(\Phi_N; C_N^H) \tag{2.7}$$

and

$$\Psi_N^{\diamond k} := H_k(\Psi_N; C_N^W). \tag{2.8}$$

which appear in the equation (1.10) where

$$C_N^H := \frac{1}{2\pi}\sum_{l\in\mathbb{Z}^2, |l|\le N}\int_0^t e^{2(s-t)|l|^2} dL(s) \tag{2.9}$$

and

$$C_N^W := \frac{1}{2\pi}\sum_{l\in\mathbb{Z}^2, |l|\le N}\int_0^t \frac{\sin^2((t-s)|l|)}{|l|^2} dL(s). \tag{2.10}$$

Here we state the convergence result in [12].

Proposition 2.2. (cf. [12, Propositions 4.3 and 4.8]) *Let $k\ge 1$ be any positive integer and let $T>0$.*

(i) *$\Phi_N^{\diamond k}$ converges to some $\Phi^{\diamond k}$ $\mathbb{P}$-almost-surely in $L^p([0,T]; W^{-\alpha,\infty}(\mathbb{T}^2))$ for any $\alpha>0$ and $0<p<\frac{2}{k-1}$ with $\frac{2}{p}+\alpha>k$ as $N\to\infty$.*

(ii) *$\Psi_N^{\diamond k}$ converges to some $\Psi^{\diamond k}$ $\mathbb{P}$-almost-surely in $C([0,T]; W^{-\delta,\infty}(\mathbb{T}^2))$ for any $\delta>0$ as $N\to\infty$.*

Remark 2.3. *When $L(t)=t$, it is well-known that both $\Phi_N^{\diamond k}$ and $\Psi_N^{\diamond k}$ converge in the space $C([0,T]; W^{-\delta,\infty}(\mathbb{T}^2))$ for any $\delta>0$ as $N\to\infty$, see* [5, 8] *for example.*

Remark 2.4. *Because the noise is jump-type, we cannot expect the time-continuity for the heat case Φ. But by working on L^p-space with respect to time variable, we can construct the Wick powers of Φ. Note that when $0<p<1$, L^p-space is not a norm space but a complete metric space. On the other hand, we can show the time-continuity for the wave case Ψ even when subordinator L has jumps.*

3. Global well-posedness of the stochastic nonlinear wave equation with a cubic nonlinearity

In this section, we prove the global well-posedness of the stochastic nonlinear wave equation with a cubic nonlinearity by applying the globalization argument with I-method in [9].

First, we briefly recall their argument. They considered the case when $k=3$, $\lambda<0$, $\mathcal{L}=\partial_t^2-\Delta$ and $L(t)=t$ in our setting. In view of Theorem 1.2, it suffices to consider the following equation:

$$\begin{cases}\partial_t^2 v - \Delta v = \lambda\sum_{l=0}^3\binom{3}{l}v^{3-l}\Psi^{\diamond l}\\ (v(0),\partial_t v(0)) = (v_0,v_1)\end{cases} \tag{3.1}$$

where $\lambda < 0$ is a negative constant. Let $L(t) = t$ for a while. In order to prove the existence of the global solution of this kind of wave equation, one (or only one perhaps) known method is to utilize the formal conservation of the energy

$$E(v)(t) = \frac{1}{2}\int_{\mathbb{T}^2} v(t)^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} (\partial_t v(t))^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} |\nabla v(t)|^2 dx - \frac{\lambda}{4}\int_{\mathbb{T}^2} v(t)^4 dx. \tag{3.2}$$

However, in this setting, the solution $(v, \partial_t v)$ of (3.1) lies in the space $H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$ with $s < 1$ due to the low regularity of the Wick powers $\Psi^{\diamond l}$, so the value of $E(v)$ is expected to be infinite. To avoid this problem, they consider the following mollified energy

$$E^I(v) := E(Iv) = \frac{1}{2}\int_{\mathbb{T}^2} (Iv)^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} (\partial_t Iv)^2 dx + \frac{1}{2}\int_{\mathbb{T}^2} |\nabla Iv|^2 dx - \frac{\lambda}{4}\int_{\mathbb{T}^2} (Iv)^4 dx \tag{3.3}$$

where $I = I_N$ is called I-operator and defined as follows. For fixed $0 < s < 1$ and given $N \geq 1$, we define the operator $I = I_N$ on $\mathcal{D}'(\mathbb{T}^2)$ by

$$I_N f = \frac{1}{2\pi}\sum_{l \in \mathbb{Z}^2} m_N(l)\hat{f}(l)e_l \tag{3.4}$$

where $\hat{f}$ is the Fourier transformation of f and $m_N \in C^\infty(\mathbb{R}_+; [0,1])$ is a rotationally invariant function such that

$$m_N(\xi) = \begin{cases} 1 & \text{when } |\xi| \leq N \\ \left(\dfrac{N}{|\xi|}\right)^{1-s} & \text{when } |\xi| \geq 2N \end{cases} \tag{3.5}$$

and $m_N(\xi)$ is non-decreasing with respect to $|\xi|$. Then, they successfully controlled the growth of mollified energy $E^I(v)$ by the Gronwall-type argument and proved the global well-posedness of the equation (3.1). In that process, we need a more careful observation of the behavior of Ψ than we need in the proof of the local well-posedness. More precisely, the fact that $\Psi^{\diamond k} \in C([0,T]; W^{-\delta,\infty}(\mathbb{T}^2))$ for any $\delta > 0$, which we saw in Proposition 2.2, is not enough to prove the global well-posedness and we need to use the fact that the following quantities are finite for Ψ: For given $\Xi = (\Xi_1, \Xi_2, \Xi_3)$, we define $V = V(\Xi)$ and $R = R(\Xi)$ by

$$e^{V^{\frac{1}{3}}} := \sum_{j=0}^{\infty} e^{-\theta j} e^{V_j(\Xi)^{\frac{1}{3}}} \tag{3.6}$$

and

$$R := \sum_{N=1}^{\infty}\sum_{j=1}^{\infty} e^{-\theta j \log N} \int_0^j \int_{\mathbb{T}^2} e^{|I_N \Xi_1(t,x)|} dx dt \tag{3.7}$$

where $V_j(\Xi) := \max_{1 \leq k \leq 3} \|\Xi_k\|_{L^\infty([j,j+1]; W^{-\delta,\infty})}$ and $\delta, \theta > 0$. Note that the definition of V above is slightly different from the one in [9] but this does not cause any problem in the argument below. Then, it turns out that $V(\Psi, \Psi^{\diamond 2}, \Psi^{\diamond 3})$ and $R(\Psi, \Psi^{\diamond 2}, \Psi^{\diamond 3})$ are finite almost surely. In [9], they essentially proved the following theorem which is purely deterministic result and by applying this to the equation (3.1) pathwisely, they obtained the desired result.

Theorem 3.1. (cf. [9]) *Let $s > \frac{4}{5}$. Assume that $\Xi = (\Xi_1, \Xi_2, \Xi_3)$ satisfies $V(\Xi) < \infty$ and $R(\Xi) < \infty$ for some $\theta > 0$ and sufficiently small $\delta > 0$. Then, the equation*

$$\begin{cases} \partial_t^2 v - \Delta v = \lambda v^3 + \lambda \sum_{l=1}^{3} \binom{3}{l} v^{3-l}\Xi_l \\ (v(0), \partial_t v(0)) = (v_0, v_1) \end{cases} \tag{3.8}$$

has a unique mild solution in the space

$$C([0,T];H^s(\mathbb{T}^2))\cap C^1([0,T];H^{s-1}(\mathbb{T}^2))$$

for any given $(v_0,v_1)\in H^s(\mathbb{T}^2))\times H^{s-1}(\mathbb{T}^2))$ *and* $T>0$.

Now, we go back to the original setting i.e. let L be any cádlág and non-decreasing stochastic process with $L(0)=0$. Then, unfortunately, $(\Psi,\Psi^{\diamond 2},\Psi^{\diamond 3})$ does not satisfy the assumption of Theorem 3.1. However, it turns out that we can easily overcome this problem: Fix $T>0$. We define $\Psi_T^{\diamond k}(t)=\Psi^{\diamond k}(t\wedge T)$. Then, we can prove that $(\Psi_T,\Psi_T^{\diamond 2},\Psi_T^{\diamond 3})$ satisfies the assumption of Theorem 3.1:

Lemma 3.2. *For any* $T>0$, $\delta>0$ *and* $\omega_2\in\Omega_2$, *there exists some* $\theta=\theta(\omega_2,T)>0$ *such that the values of* $V(\Psi_T,\Psi_T^{\diamond 2},\Psi_T^{\diamond 3})$ *and* $R(\Psi_T,\Psi_T^{\diamond 2},\Psi_T^{\diamond 3})$ *are finite* $\mathbb{P}_1$*-almost surely.*

Before the proof we recall the following lemma.

Lemma 3.3. (cf. [12, Lemma 4.1]) *Let* f *and* g *be smooth deterministic functions on* $[0,T]$. *Then, for any* $l,l'\in\mathbb{Z}^2$,

$$\mathbb{E}^{\mathbb{P}_1}\left[\int_0^T f(t)d\beta_L^l(t)\int_0^T g(t)d\beta_L^{l'}(t)\right]=\delta_{l,-l'}\int_0^T f(t)g(t)dL(t). \tag{3.9}$$

Proof of Lemma 3.2. Because

$$V_j(\Psi_T,\Psi_T^{\diamond 2},\Psi_T^{\diamond 3})=\max_{1\le k\le 3}\|\Psi_T^{\diamond k}\|_{L^\infty([j,j+1];W^{-\delta,\infty})}\le\max_{1\le k\le 3}\|\Psi_T^{\diamond k}\|_{L^\infty([0,T];W^{-\delta,\infty})}<\infty$$

uniformly in $j\in\mathbb{N}$ from Proposition 2.2, we can see that

$$e^{V^{\frac13}}=\sum_{j=0}^\infty e^{-\theta j}e^{V_j(\Psi_T,\Psi_T^{\diamond 2},\Psi_T^{\diamond 3})^{\frac13}}<\infty$$

for any $\theta>0$. Next, we check that $R(\Psi_T)$ is finite. We write $\Psi_T=P_N\Psi_T+P_N^\perp\Psi_T$. From Lemma 3.3,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_1}\left[|I_NP_N\Psi_T(x,t)|^2\right]&=\mathbb{E}^{\mathbb{P}_1}\left[|P_N\Psi_T(x,t)|^2\right]\\&=\frac{1}{2\pi}\sum_{l\in\mathbb{Z}^2,|l|\le N}\int_0^{t\wedge T}\frac{\sin^2((t-s)|l|)}{|l|^2}dL(s)\\&\lesssim L(T)\log N\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_1}\left[\left|I_NP_N^\perp\Psi_T(x,t)\right|^2\right]&=\frac{1}{2\pi}\sum_{l\in\mathbb{Z}^2,|l|>N}m_N(l)^2\int_0^{t\wedge T}\frac{\sin^2((t-s)|l|)}{|l|^2}dL(s)\\&\lesssim\sum_{l\in\mathbb{Z}^2,|l|>N}\frac{N^{2-2s}}{|l|^{2-2s}}\times\frac{L(T)}{|l|^2}\lesssim L(T).\end{aligned}$$

Therefore, for any fixed $\omega_2\in\Omega_2$, $I_N\Psi_T(x,t)$ is a centered Gaussian random variable on $(\Omega_1,\mathbb{P}_1)$ with

$$\mathbb{E}^{\mathbb{P}_1}\left[|I_N\Psi_T(x,t)|^2\right]\le C_0L(T)\log N$$

where $C_0 > 0$ is a positive constant which is independent of $t \in \mathbb{R}_+$ and $x \in \mathbb{T}^2$. Thus, we can calculate as

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}_1}\left[R(\Psi_T)\right] &= \mathbb{E}^{\mathbb{P}_1}\left[\sum_{N=1}^{\infty}\sum_{j=1}^{\infty} e^{-\theta j \log N} \int_0^j \int_{\mathbb{T}^2} e^{|I_N \Psi_T(t,x)|} dxdt\right] \\
&\leq \sum_{N=1}^{\infty}\sum_{j=1}^{\infty} e^{-\theta j \log N} \int_0^j \int_{\mathbb{T}^2} e^{C_0 L(T) \log N} dxdt \\
&\lesssim \sum_{N=1}^{\infty}\sum_{j=1}^{\infty} e^{-\theta j \log N} e^{C_0 L(T) \log N} \times j \\
&= \sum_{N=1}^{\infty}\sum_{j=1}^{\infty} N^{-\theta j} N^{C_0 L(T)} \times j < \infty
\end{aligned}$$

if θ is sufficiently larger than $L(T)$. □

From Theorem 3.1 and Lemma 3.2, we can see that the global well-posedness of the following equation holds.

$$\begin{cases} \partial_t^2 v - \Delta v = \lambda v^3 + \lambda \sum_{l=1}^{3} \binom{3}{l} v^{3-l} \Psi_T^{\diamond l} \\ (v(0), \partial_t v(0)) = (v_0, v_1). \end{cases} \tag{3.10}$$

Here, the solutions of (3.1) and (3.10) coincide on the interval $[0, T]$ if the same initial value (v_0, v_1) is given. Therefore, we obtain the global well-posedness of the original equation (3.1).

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