A note on virial method for decay estimates

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Abstract

In this note, we show how to prove decay estimates for Schrödinger equations by virial methods. The virial method used in this note are based on the series of work by Kowalczyk, Martel, Munoz and Van Den Bosch [10, 11, 12, 13, 14].

1 Virial method

We start from a well-known formal argument in quantum mechanics. Consider

$$\mathrm{i}\partial_t u = Hu, \tag{1.1}$$

where H is a self-adjoint operator. Let A be another self-adjoint operator and set

$$J(u) = \frac{1}{2}\langle u, Au\rangle .$$

Then, assuming u solves (1.1), by informal computation, we have

$$\begin{aligned}\frac{d}{dt}J(u) &= \frac{1}{2}\langle \dot{u}, Au\rangle + \frac{1}{2}\langle u, A\dot{u}\rangle \\ &= \frac{1}{2}\langle -\mathrm{i}Hu, Au\rangle + \frac{1}{2}\langle u, A(-\mathrm{i}Hu)\rangle \\ &= \frac{1}{2}\langle u, [\mathrm{i}H, A]u\rangle . \end{aligned}\tag{1.2}$$

Here, $\dot{u} = \partial_t u$, $\langle \cdot, \cdot\rangle$ is a real inner-product and $[A, B] = AB - BA$. By (1.2), $J(u)$ is conserved if $[\mathrm{i}H, A] = 0$.

Example 1.1. Let $H = -\partial_x^2$ and $A = \langle \mathrm{i}\partial_x\rangle^{2s}$ for $s \in \mathbb{R}$, where $\langle x\rangle = (1+x^2)^{1/2}$. Then, by $[\mathrm{i}H, A] = 0$, we have $\|u(t)\|_{H^s}^2 = \|u(0)\|_{H^s}^2$ for all $t \in \mathbb{R}$, where $u(t) = e^{\mathrm{i}t\partial_x^2}u(0)$.

If $[\mathrm{i}H, A] \geq 0$ (i.e. $\langle u, [\mathrm{i}H, A]u\rangle \geq 0$ for all u), then, by (1.2), $J(u(t))$ is non-decreasing .

Example 1.2. Let $H = -\partial_x^2$ and $A = \frac{1}{2}\left(x(-\mathrm{i}\partial_x) - \mathrm{i}\partial_x\left(x\cdot\right)\right) = -\mathrm{i}\left(\frac{1}{2} + x\partial_x\right)$. Then,

$$[\mathrm{i}H, A] = [\mathrm{i}(-\partial_x^2), -\mathrm{i}\left(\frac{1}{2} + x\partial_x\right)] = -[\partial_x^2, x\partial_x] = -2\partial_x^2 \geq 0.$$

Thus,

$$\frac{d}{dt}\left(\frac{1}{2}\langle u, Au\rangle\right) = \|\partial_x u\|_{L^2}^2 \geq 0.$$

Remark 1.3. The operator $A = \frac{1}{2}\left(x(-\mathrm{i}\partial_x) - \partial_x\left(x\cdot\right)\right)$ is the generator of scaling, i.e. $\mathrm{i}Au = \frac{d}{d\lambda}\big|_{\lambda=1}\lambda^{1/2}u(\lambda x)$. Notice that A is the quantum counter part of the quantity $\sum_i x_i p_i$ appearing in the proof of virial theorem in classical mechanics, where x_i and p_i are the position and momentum of ith particle.

From a different perspective, suppose we know $\sup_t |J(u(t))| < \infty$ for $u(t) = e^{-\mathrm{i}tH}u_0$. For example, if A is bounded, we have $J(u(t)) \lesssim \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$. Then, integrating (1.2), we have

$$\int_0^\infty \langle u(t), [\mathrm{i}H, A]u(t)\rangle \, dt \leq 2\sup_t |J(u(t))| < \infty. \tag{1.3}$$

One can view (1.3) as a decay estimate, i.e. the nonnegative quantity $\langle u(t), [\mathrm{i}H, A]u(t)\rangle$ converges to 0 subsequently because it is integrable. This will be the estimate which we will be using in this note.

An obvious obstacle for the above estimate are the eigenvalues of H. Suppose H has an eigenvalue $E \in \mathbb{R}$ with the associated eigenfunction ϕ. Then, setting $u_0 = \phi$, we have $u(t) = e^{-\mathrm{i}Et}\phi$, which have no decay. Thus, we have to somehow exclude eigenvalues from the above argument. In this note, based on the idea of Kowalczyk-Martel-Munoz and Van Den Bosch [10, 11, 12, 13, 14] followed by [1, 4, 5, 6, 7, 15], we explain several ideas to use virial estimates when the Schrödinger operator $-\partial_x^2 + V$ posses eigenvalues.

The organization of this note is as follows: In section 2, we explain how to show the decay estimate for Schrödinger equation with repulsive potential, which have no eigenvalues. In section 3, we explain two ways to show the decay estimate for the continuous part of the solution when the potential has eigenvalues (and in particular not repulsive).

For $a \lesssim b$, we mean $a \leq Cb$ for some constant C. By $a \lesssim_\lambda b$, we mean $a \leq C_\lambda b$ for some constant C_λ depending on λ. Also, by $a \sim b$ we mean $a \lesssim b$ and $b \lesssim a$.

2 Decay of solutions of Schrödinger equation with repulsive potential

In the following, we use the norm

$$\|u\|_\Sigma^2 := \|u'\|_{L^2}^2 + \|\langle x\rangle^{-2} u\|_{L^2}^2,$$

and the following elementary lemma.

Lemma 2.1. *Let $W \in L^{1,1} := \{W \in L^1 \mid \langle x\rangle W \in L^1\}$ and $W(x) \geq 0$. Then,*

$$\int W|u|^2 \, dx \lesssim_{\|W\|_{L^{1,1}}} \|u\|_\Sigma^2.$$

Moreover, if $W \neq 0$, then

$$\|u\|_\Sigma^2 \lesssim_{\|W\|_{L^1}, \|W\|_{L^{1,1}}} \|u'\|_{L^2}^2 + \int W|u|^2 \, dx.$$

Proof. It suffices to show the claim for smooth function u. First, let $x_0 \in [-1,1]$ s.t. $|u(x_0)| = \min_{x\in[-1,1]} |u(x)|$. Then, since since $\int_{-1}^1 \langle x\rangle^{-4} \, dx \geq 1/2$ we have

$$|u(x_0)|^2 \leq 2\int_{\mathbb{R}} \langle x\rangle^{-4} |u(x)|^2 \, dx.$$

Thus, by

$$|u(x)| \le |x-x_0|^{1/2}\|u'\|_{L^2} + |u(x_0)|, \tag{2.1}$$

which follows from the fundamental theorem of calculus, we have

$$W(x)|u(x)|^2 \le 4W(x)\langle x\rangle \|u'\|_{L^2}^2 + 4W(x)\int_{\mathbb{R}} \langle x\rangle^{-4} |u(x)|^2\,dx.$$

Therefore, integrating, we have

$$\int W|u|^2\,dx \le 4\int \langle x\rangle W(x)\,dx\|u\|_{\Sigma}^2.$$

We next show the latter claim, which is basically follows by the same argument. Let $R := \frac{2}{\|W\|_{L^1}}\int \langle x\rangle W(x)\,dx > 0$. Let $x_1 \in [-R,R]$ s.t. $|u(x_1)| = \min_{x\in[-R,R]}|u(x)|$. By

$$\int_{|x|\ge R} W(x)\,dx \le \frac{1}{R}\int |x|W(x)\,dx \le \frac{1}{2}\|W\|_{L^1},$$

we see

$$\int_{|x|\le R} W(x)\,dx \ge \frac{1}{2}\|W\|_{L^1}.$$

Then, as before, we have

$$|u(x_1)|^2 \le \frac{2}{\|W\|_{L^1}}\int_{\mathbb{R}} W(x)|u(x)|^2\,dx.$$

So, using (2.1), we have

$$\begin{aligned}\int \langle x\rangle^{-4}|u(x)|^2\,dx &\le 2\int \langle x\rangle^{-4}(|x|+R)\,dx\|u'\|_{L^2}^2 + 4\|W\|_{L^1}^{-2}\int \langle x\rangle^{-4}\,dx\int_{\mathbb{R}} W(x)\,dx\\ &\lesssim_{\|W\|_{L^1},R} \|u'\|_{L^2}^2 + \int W|u|^2\,dx.\end{aligned}$$

Therefore, we have the conclusion. □

We next consider the Schrödinger equation with a potential $V \in \mathcal{S}(\mathbb{R},\mathbb{R})$:

$$\mathrm{i}\partial_t u = (-\partial_x^2 + V)u. \tag{2.2}$$

Computing the commutator, we have

$$[\mathrm{i}(-\partial_x^2+V), -\mathrm{i}\left(\frac{1}{2}+x\partial_x\right)] = -2\partial_x^2\cdot + [V, x\partial_x] = -2\partial_x^2 - xV'.$$

Thus, we say V is *repulsive* if V' is not identically 0 and $-xV'(x) \ge 0$.

Following [14] we set

$$\zeta_A(x) = \exp\left(-\frac{|x|}{A}(1-\chi(x))\right),\ 1_{[-1,1]} \le \chi \le 1_{[-2,2]}, \tag{2.3}$$

and

$$\psi_A(x) = \int_0^x \zeta_A(s)^2\, ds.$$

Notice that if $A \geq 4$, which we always assume, then $e^{-|x|/A} \leq \zeta_A(x) \leq 2e^{-|x|/A}$. For the solution of Schrödinger equation (2.2) with repulsive potential, we have the following decay estimate.

Theorem 2.2. *Let $V \in \mathcal{S}$ be repulsive. Then, there exists $A_0 > 0$, such that for $A \geq A_0$ $u_0 \in H^1$, we have*

$$\int_0^\infty \|\zeta_A e^{-\mathrm{i}(-\partial_x^2+V)} u_0\|_\Sigma^2\, dt \lesssim A\|u_0\|_{H^1}^2, \tag{2.4}$$

Remark 2.3. If $V \equiv 0$, then (2.4) do not hold. Indeed, let $u_0 \in H^1$ with $\||x|^{2\delta}u\|_{L^2} < \infty$ for some $\delta > 1/2$ and $\mathcal{F}u_0(0) \neq 0$, where $\mathcal{F}$ is the Fourier transform. Then, by $e^{\mathrm{i}t\partial_x^2} = M(t)D(t)\mathcal{F}M(t)$ and

$$\|e^{\mathrm{i}t\partial_x^2}u_0 - M(t)D(t)\mathcal{F}u_0\|_{L^2} \lesssim |t|^{-\delta}\||x|^{2\delta}u_0\|_{L^2},$$

where $M(t)u = e^{\mathrm{i}\frac{|x|^2}{4t}}u$, $D(t)u = (2\mathrm{i}t)^{-1/2}u(x/2t)$, we have

$$\|e^{\mathrm{i}\partial_x^2}u_0\|_{L^2([1,\infty);L^2(|x|\leq R))} \geq \|MD\mathcal{F}u_0\|_{L^2([1,\infty),L^2(|x|\leq R)])} - C\||t|^{-\delta}\||x|^{2\delta}u_0\|_{L^2}\|_{L^2([1,\infty))}. \tag{2.5}$$

Now, since the 2nd term of the r.h.s. of (2.5) is finite, it suffices to show the 1st term of the r.h.s. of (2.5) becomes infinite. By the definition of M and D, we have

$$|MD\mathcal{F}u_0(x)| = (2t)^{-1/2}|\mathcal{F}u_0(x/2t)|.$$

Thus, for $t \geq 1$, we have

$$\int_{|x|\leq R} |MD\mathcal{F}u_0(x)|^2\, dx = \int_{|x|\leq R/2t} |\mathcal{F}u_0(x)|^2\, dx \sim t^{-1}|\hat{u}_0(0)|^2,$$

which imply

$$\|e^{\mathrm{i}\partial_x^2}u_0\|_{L^2([1,T);L^2(|x|\leq R))}^2 \sim |\hat{u}_0(0)|^2 \log T \to \infty \text{ as } T \to \infty.$$

The quantity $\hat{u}_0(0)$ appears because $-\partial_x^2$ has a resonance at 0.

Remark 2.4. Estimate (2.4) is NOT the best estimate we can prove for the *linear* Schrödinger equation. Indeed, the classical Kato smoothness results [9] imply

$$\int_0^\infty \|\langle x\rangle^{-s} e^{-\mathrm{i}t(-\Delta+V)}u_0\|_{L^2}^2 \lesssim_s \|u_0\|_{L^2}, \tag{2.6}$$

for $s > 1$. This inequality can be extended to H^1 setting easily and also holds even if $-\Delta + V$ posses eigenvalues provided u_0 is orthogonal to the eigenfunctions. However, the real strength of virial type inequalities are the ability to handle nonlinearities which is impossible for the linear estimates like (2.6). We will discuss the effect of nonlinearity in section 2.1.

Proof. We set

$$J_A(u) = \frac{1}{2}\left\langle u, -\mathrm{i}\left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)u\right\rangle. \tag{2.7}$$

Remark 2.5. From $-\mathrm{i}\left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)=\frac{1}{2}\left(-\mathrm{i}\partial_x\left(\psi_A\cdot\right)+\psi_A(-\mathrm{i}\partial_x)\right)$, we see $-\mathrm{i}\left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)$ is self-adjoint. This operator is a modification of $-\mathrm{i}\left(\frac{1}{2}+x\partial_x\right)$ and is defined on H^1.

Then, for $u(t)=e^{-\mathrm{i}t\left(-\partial_x^2+V\right)}u_0$ and $w=\zeta_A u$, we have

$$\frac{d}{dt}J_A(u(t))=\|w'\|_{L^2}^2-\frac{1}{2}\int\frac{\psi_A}{\zeta_A^2}V'(x)|w|^2\,dx+\frac{1}{2A}\int\widetilde{V}|w|^2\,dx, \tag{2.8}$$

where

$$\widetilde{V}(x)=2\frac{x}{|x|}\chi'(x)+|x|\chi''(x).$$

We give the proof of (2.8) below.

Notice that $\frac{\psi_A}{\zeta_A^2}V'(x)$ converges to $xV'(x)$ locally uniformly. Thus, taking A sufficiently large, from Lemma 2.1, we have

$$\|w'\|_{L^2}^2-\frac{1}{2}\int\frac{\psi_A}{\zeta_A^2}V'(x)|w|^2\gtrsim\|w\|_{\Sigma}^2,$$

where the implicit constant depends on V but not on A nor u. Again, from Lemma 2.1, we have

$$\frac{1}{2A}\int\widetilde{V}|w|^2\,dx\lesssim\frac{1}{A}\|w\|_{\Sigma}^2,$$

where the implicit constant is independent of A,u. Thus, we have

$$\frac{d}{dt}J_A(u(t))\gtrsim\|w\|_{\Sigma}^2, \tag{2.9}$$

for A sufficiently large. Since $\sup_x|\psi_A(x)|\sim A$, we have $|J_A(u)|\lesssim A\|u\|_{H^1}$. Further, using the energy and mass conservation, we have $\|u(t)\|_{H^1}\sim\|u(0)\|_{H^1}$. Thus, integrating (2.9), we have the conclusion. □

Proof of (2.8). First, we have

$$\frac{d}{dt}J_A(u(t))=\left\langle(-\partial_x^2+V)u,\left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)u\right\rangle.$$

We compute each terms. Recall $w=\zeta_A u$ and $\psi_A'=\zeta_A^2$.

$$\begin{aligned}\left\langle-u'',\frac{\psi_A'}{2}u\right\rangle&=\frac{1}{2}\left\langle(\zeta_A^{-1}w)',(\zeta_A w)'\right\rangle=\frac{1}{2}\left\langle\frac{\zeta_A w'-\zeta_A'w}{\zeta_A^2},\zeta_A'w+\zeta_A w'\right\rangle\\&=\frac{1}{2}\|w'\|_{L^2}^2-\frac{1}{2}\int\left(\frac{\zeta_A'}{\zeta_A}\right)^2w^2\,dx,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\langle-u'',\psi_A u'\rangle&=\frac{1}{2}\int\psi_A'((\zeta_A^{-1}w)')^2\,dx=\frac{1}{2}\int\frac{\zeta_A^2(w')^2-2\zeta_A\zeta_A'ww'+(\zeta_A')^2w^2}{\zeta_A^2}\,dx\\&=\frac{1}{2}\|w'\|_{L^2}^2+\frac{1}{2}\int\left(\frac{\zeta_A'}{\zeta_A}\right)'w^2\,dx+\frac{1}{2}\int\left(\frac{\zeta_A'}{\zeta_A}\right)^2w^2\,dx.\end{aligned} \tag{2.11}$$

Adding, (2.10) and (2.11), we have

$$\left\langle u'', \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle = \|w'\|_{L^2}^2 + \frac{1}{2}\int \left(\frac{\zeta_A'}{\zeta_A}\right)' w^2\, dx.$$

Now, recall (2.3) and notice $\left(\frac{\zeta_A'}{\zeta_A}\right)' = (\log \zeta_A)''$. Since

$$(\log \zeta_A)'' = -\frac{1}{A}(|x|(1-\chi))'' = \frac{1}{A}\left(2\frac{x}{|x|}\chi' + |x|\chi''\right) = \frac{1}{A}\widetilde{V},$$

we have

$$\left\langle u'', \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle = \|w'\|_{L^2}^2 + \frac{1}{2A}\int \widetilde{V} w^2\, dx. \tag{2.12}$$

Finally, for the potential term,

$$\begin{aligned}\left\langle Vu, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle &= \left\langle Vu, \frac{\psi_A'}{2}u \right\rangle + \langle Vu, \psi_A u' \rangle = \left\langle Vu, \frac{\psi_A'}{2}u \right\rangle + \frac{1}{2}\int V\psi_A (|u|^2)'\, dx \\ &= \left\langle Vu, \frac{\psi_A'}{2}u \right\rangle - \frac{1}{2}\int (V\psi_A)'|u|^2\, dx = -\frac{1}{2}\int \psi_A V'|u|^2\, dx \\ &= -\frac{1}{2}\int \frac{\psi_A}{\zeta_A^2} V'(x)|w|^2\, dx. \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), we have (2.8). □

Remark 2.6. It is natural to ask if we can improve the exponential weight $\zeta_A(x) \sim e^{-|x|/A}$ to a polynomial weight like $|x|^{-N}$. At this moment I do not now how to do this (by a computation similar to below, it seems to be possible use polynomial weights if (say) for $x > 1$, $-V'(x) \gtrsim x^{-a}$. However if one wants this tool to be used for asymptotic stability analysis of solitons or kinks, the corresponding potential usually decay exponentially so we will not use such condition), but by a slight generalization of ζ_A, it is possible to make $\zeta_A(x) \sim \exp(-\frac{1}{A}(\log x)^a)$ for any $a > 1$. Notice that if $a = 1$, it is a polynomial order and if $a > 1$, we have

$$e^{-\delta|x|} \ll \exp(-(\log x)^a) \ll |x|^{-\gamma},$$

for any $\delta, \gamma > 0$, where for two positive functions f, g, we mean $f(x) \ll g(x)$ by $\lim_{x\to\infty} f(x)/g(x) = 0$. To show this, we start from a general ζ_A. That is, let

$$\zeta_A(x) = \exp(-\frac{1}{A}(1-\chi(x))f(x)),$$

for some positive increasing function f. Notice that if f is an increasing function, $\zeta_A \to 1$ locally uniformly (and so $\psi_A(x) \to x$ locally uniformly).

The first condition we need is $\psi_A \in L^\infty$ (for each A). This is simply restated as $\zeta_A \in L^2$. Now, suppose $f(x) \gtrsim (\log x)^a$ for some $a > 0$. Then,

$$\begin{aligned}\int_2^\infty \zeta_A(x)^2\, dx &= \int_2^\infty \exp\left(-\frac{1}{A}f(x)\right)\, dx \le \int_2^\infty \exp\left(-\frac{C}{A}(\log x)^a\right)\, dx \\ &= \int_{\log 2}^\infty \exp\left(-\frac{C}{A}y^a + y\right)\, dy.\end{aligned}$$

Thus, we see that if $a > 1$ then $\zeta_A \in L^2$ (of course $\lim_{x\to\infty} \psi_A(x)$ depends on A). On the other hand, if $f(x) \sim \log x$, it is clear from the same computation that for sufficiently large A, $\psi_A(x) \to \infty$.

Now, by replacing ψ_A by the above, $\widetilde{V}$ in (2.8) is replaced by

$$\widetilde{V}_f(x) = A(\log \zeta_A)'' = -(1-\chi)f'' + (2\chi' f' + \chi'' f)\,.$$

The contribution of $(2\chi' f' + \chi'' f)$ in the virial computation can by bounded by $A^{-1}\|w\|_\Sigma^2$ just as the proof of Theorem 2.2. On the other hand, if $f''(x) \le 0$ for $|x| \ge 1$, then the contribution of $-(1-\chi)f''$ in the virial estimate will have a good sign. In the case, $f(x) = (\log x)^a$, we have

$$f''(x) = -ax^{-2}(\log x)^{a-2}\,(\log(x) - a + 1)\,.$$

So, slightly modifying f such as $f(x) = \left(\log(e^{a-1}x)\right)^a$, we have $f''(x) \le 0$ for $x > 1$.

2.1 Handling nonlinearity

As noted in Remark 2.4, the real benefit of the virial inequality comes from considering nonlinearities. Let $G \in C^1(\mathbb{R},\mathbb{R})$ with $G(0) = 0$ and

$$|G^{(n)}(s)| \lesssim |s|^{p+1-n},\ \ n = 0, 1, \tag{2.14}$$

for some $p > 0$. We set $g = G'$ and consider the following nonlinear Schrödinger equation:

$$\mathrm{i}\partial_t u = -\partial_x^2 u + Vu + g(|u|^2)u. \tag{2.15}$$

As the linear case, we try to obtain a decay estimate for the solution of (2.15) by investigating the time derivative of J_A given by (2.7). For u satisfying (2.15), we have

$$\frac{d}{dt}J_A(u) = \left\langle (-\partial_x^2 + V)u, \left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)u\right\rangle + \left\langle g(|u|^2)u, \left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)u\right\rangle. \tag{2.16}$$

Since the contribution of the first term in the r.h.s. of (2.16) is given by the r.h.s. of (2.8), it remains to study the second term. By the relation $G' = g$ and $\psi_A' = \zeta_A^2$,

$$\langle g(|u|^2)u, \psi_A u'\rangle = \frac{1}{2}\int \psi_A (G(|u|^2))'\,dx = -\frac{1}{2}\int \zeta_A^2 G(|u|^2)\,dx.$$

Thus,

$$\left\langle g(|u|^2)u, \left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)u\right\rangle = \frac{1}{2}\int \left(g(|u|^2)|u|^2 - G(|u|^2)\right)\zeta_A^2\,dx. \tag{2.17}$$

We say the nonlinearity g is *repulsive* if for all $s \ge 0$, $g(s)s - G(s) \ge 0$. Since for the repulsive nonlinearity, the contribution of the nonlinear term has a good sign, we can immediately have the following theorem.

Theorem 2.7. *Let V and g be repulsive. Then, there exists $A_0 > 0$ s.t. for $A \ge A_0$ and $u_0 \in H^1$, we have*

$$\int_0^\infty \|\zeta_A u(t)\|_\Sigma^2, dt \lesssim A\|u_0\|_{H^1}^2,$$

where u is the solution of (2.15) *with $u(0) = u_0$.*

When the nonlinearity is not repulsive, Theorem 2.7 do not hold in general because there can be a nonlinear bound state, i.e. solution of the form $e^{\mathrm{i}\omega t}\phi(x)$ for some $\omega \in \mathbb{R}$. However, restricting ourselves to small (in H^1) solutions we can recover the above theorem for general nonlinearity satisfying (2.14).

Theorem 2.8. *Let V be repulsive. Then, there exists $A_0 > 0$ s.t. for $A \geq A_0$, there exists $\delta_A > 0$ s.t. for $u_0 \in H^1$ with $\|u_0\|_{H^1} < \delta_A$, we have*

$$\int_0^\infty \|\zeta_A u(t)\|_\Sigma^2, dt \lesssim A\|u_0\|_{H^1}^2,$$

where u is the solution of (2.15) with $u(0) = u_0$.

For the proof of Theorem 2.1, we need the following lemma (taken from [1, 4, 11]):

Lemma 2.9. *For any $\varepsilon_0 > 0$. There exists $A_0 > 0$ s.t. for $A \geq A_0$, there exists $\delta_A > 0$ s.t. for $\|u\|_{H^1} \leq \delta_A$ we have*

$$\left|\left\langle g(|u|^2)u, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u\right\rangle\right| \leq \varepsilon_0 \|w'\|_{L^2}^2 \text{ for } w = \zeta_A u \ .$$

Proof of Theorem 2.8. Let $\widetilde{A}_0$ be the constant given given in Lemma 2.9 for $\varepsilon_0 = 1/2$. Take $A > \widetilde{A}_0$ and let $\widetilde{\delta}_A$ be the constant given in Lemma 2.9. By standard argument using energy and mass conservation, it is easy to show that there exists $\delta_A > 0$ s.t. we have $\|u(t)\|_{H^1} \leq \widetilde{\delta}_A$ if $\|u_0\|_{H^1} \leq \delta_A$. Then, by Lemma 2.9, taking A_0 larger if necessary, we have

$$\frac{d}{dt} J_A(u(t)) \geq \frac{1}{2}\|w'\|_{L^2} - \frac{1}{2}\int \frac{\psi_A}{\zeta_A^2} V'(x)|w|^2\, dx + \frac{1}{2A}\int \widetilde{V}|w|^2\, dx \gtrsim \|w\|_\Sigma^2.$$

Therefore, we have the conclusion. ∎

It remains to prove Lemma 2.9.

Proof of Lemma 2.9. From (2.17) and (2.14), we have

$$\left|\left\langle g(|u|^2)u, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u\right\rangle\right| = \frac{1}{2}\int \left(|g(|u|^2)||u|^2 + |G(|u|^2)|\right) \zeta_A^2\, dx \lesssim \int_{\mathbb{R}} \zeta_A^2 |u|^{2(p+1)}\, dx.$$

Let $q = \frac{2p}{3} > 0$. Then, by the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \zeta_A^2 |u|^{2(p+1)}\, dx \lesssim \|u\|_{H^1}^q \int_{\mathbb{R}} \zeta_A^2 |u|^{2(p+1)-q} dx.$$

Therefore, it suffices to prove

$$\int_{\mathbb{R}} \zeta_A^2 |u|^{2(p+1)-q}\, dx \lesssim \|w'\|_{L^2}^2. \tag{2.18}$$

Recall $\zeta_A \sim e^{-\frac{|x|}{A}}$. Since $2(2(p-q)+1) = 2(p+1)-q$,

$$\begin{aligned}
\int_{\mathbb{R}} \zeta_A^2 |u|^{2(p+1)-q} dx &= \int_{\mathbb{R}} \zeta_A^{-(2p-q)} |w|^{2(p+1)-q} dx \\
&\lesssim \int_0^\infty e^{\frac{2p-q}{A}x} |w|^{2(p+1)-q} dx + \int_{-\infty}^0 e^{-\frac{2p-q}{A}x} |w|^{2(p+1)-q} dx \\
&\le -\frac{2A}{2p-q} |w(0)|^{2(p+1)-q} + \frac{A}{2p-q} \int_{\mathbb{R}} e^{\frac{2p-q}{A}|x|} \left| (|w|^{2(p+1)-q})' \right| dx \\
&\lesssim A \int_{\mathbb{R}} \zeta_A^{-2p+q} |w|^{2p-q+1} |w'| \, dx = A \int_{\mathbb{R}} \zeta_A |u|^{2p-q+1} |w'| \, dx \\
&\lesssim A \|u\|_{H^1}^q \int_{\mathbb{R}} \zeta_A |u|^{2(p-q)+1} |w'| \, dx \\
&\le \frac{A}{p} \|u\|_{H^1}^q \left(\int_{\mathbb{R}} \zeta_A^2 |u|^{2(p+1)-q} \, dx \right)^{\frac{1}{2}} \|w'\|_{L^2} \\
&\le \|w'\|_{L^2}^2 + \frac{1}{4p^2} A^2 \|u\|_{H^1}^{2q} \int_{\mathbb{R}} \zeta_A^2 |u|^{2(p+1)-q} \, dx.
\end{aligned}$$

Here, notice that in the third line, we have used

$$\left[\frac{2A}{2p-q} e^{\frac{2p-q}{A}x} |w(x)|^{2(p+1)-q} \right]_{x=0}^{\infty} = -\frac{2A}{2p-q} |w(0)|^{2(p+1)-q},$$

which follows from the fact $w = \zeta_A(x) u(x)$, $u \in H^1 \hookrightarrow L^\infty$ and

$$\lim_{x\to\infty} e^{\frac{2p-q}{A}x} |w(x)|^{2(p+1)-q} = \lim_{x\to\infty} e^{-\frac{2}{A}x} |u(x)|^{2(p+1)-q} = 0.$$

Thus, taking $\frac{1}{4p^2} A^2 \delta_A^{2q} \ll 1$, we have (2.18). □

Remark 2.10. It is not clear how to show Lemma 2.9 for subexponential weight like the one considered in Remark 2.6.

3 How to handle the eigenvalues?

For the case that the Schrödinger operator $-\partial_x^2 + V$ has an eigenvalue it seems difficult to obtain decay estimate by virial method. This is because the spectral information of the Schrödinger operator is somewhat abstract but virial method is based on a concrete calculation. Indeed, we got the inequalities mainly from integration by parts. On the other hand, decay estimate will only hold for the continuous spectral part of the solution. However, there are several ways to obtain decay estimates for the continuous part as we will see below. Below, we only consider linear Schrödinger equations, but both method we introduce can be extended for the nonlinear case.

3.1 Delta potential

We start from a "solvable" potential [2], which is the delta potential. We set

$$H_a := -\partial_x^2 - a\delta_0,$$

where δ_0 is the Dirac delta function.

Remark 3.1. For the rigorous definition, see [2] or [4].

We know the eigenvalue and eigenfunction of H_a explicitly. Indeed, $\sigma_{\mathrm{d}}(H_a) = \{-a^2/4\}$ and $\ker(H_a + a^2/4) = \mathrm{span}\{e^{-\frac{a}{2}|x|}\}$ for $a > 0$. Therefore, the continuous spectral part of the solution is given explicitly taking the projection to the orthogonal complement of the eigenfunction. Restricting the initial data to the continuous space $\mathcal{H}_c^1 := \{e^{-\frac{1}{2}|x|}\}^{\perp} \cap H^1$, we have the desired decay.

Theorem 3.2. *There exists $A_0 > 0$ s.t. for $A \geq A_0$ and $u_0 \in H_c^1$, (2.4) holds with $-\partial_x^2 + V$ replaced by H_1.*

Remark 3.3. One can also show similar estimate in the nonlinear case, see [4].

Proof. First, we note

$$\langle H_{-1}u, u\rangle \gtrsim \|u\|_{\Sigma}^2, \tag{3.1}$$

see (2.11) of [4].

Next, we compute $\frac{d}{dt}J_A(u)$. Then, we have

$$\frac{d}{dt}J_A(u(t)) = \left\langle H_{1/2}w, w\right\rangle + \frac{1}{2A}\int \widetilde{V}|w|^2\,dx, \tag{3.2}$$

see Lemma 2.3 of [4]. Of course, the r.h.s. of (3.2) is not positive (recall $H_{1/2}$ has negative delta potential, also if it is positive, it means that the bound state decays, which is an absurd). However, the crucial observation is that $H_{1/2}$ is less negative compared to H_1.

By decomposing $H_{1/2} = \frac{1}{4}H_{-1} + \frac{3}{4}H_1$ and using (3.1), we have

$$\frac{d}{dt}J_A(u(t)) \gtrsim (1 - A^{-1})\|w\|_{\Sigma}^2 + \langle H_1 w, w\rangle\,.$$

Now, we use the condition $u \perp e^{-|x|/2}$. Notice that since $w = \zeta_A u \to u$ locally uniformly as $A \to \infty$, w is almost orthogonal to $e^{-|x|/2}$. In particular, we have

$$\liminf_{A\to\infty}\langle H_1 w, w\rangle \geq 0.$$

Therefore, taking A sufficiently large, we have

$$\frac{d}{dt}J_A(u(t)) \gtrsim \|w\|_{\Sigma}^2.$$

For more details, see [4]. □

3.2 Darboux transform

Another way to handle the eigenvalues is to use Darboux transform.

Remark 3.4. Darboux transform has a long history, which can date back at least to Darboux in 18th century. In the context of virial inequality, Darboux transform was first used by Kowalczyk, Martel, Munoz and Van Den Bosch [14]. We also refer [3] and [8].

Consider $H = -\partial_x^2 + V$ with $V \in \mathcal{S}(\mathbb{R},\mathbb{R})$ and $\sigma_{\mathrm{d}}(V) \neq \emptyset$. Let $\min\sigma(H) = \omega < 0$ and ϕ be the ground state (eigenfunction of H w.r.t. ω. Recall that we can take ϕ to be positive). Set

$$A_V := \phi\partial_x\left(\phi^{-1}\cdot\right) = \partial_x - \frac{\phi'}{\phi}.$$

Then,

$$A_V^* := -\phi^{-1}\partial_x(\phi\cdot) = -\partial_x + \frac{\phi'}{\phi}.$$

Proposition 3.5. *We have*

$$A_V^* A_V = H - \omega, \tag{3.3}$$

and

$$A_V A_V^* = H_1 - \omega = -\partial_x^2 + V_1 - \omega,$$

where $V_1 = V - 2(\log\phi)'' \in \mathcal{S}$. *Further, we have* $\sigma_{\mathrm{d}}(H_1) = \sigma_{\mathrm{d}}(H) \setminus \{\omega\}$.

Proof. We start from the formal computations. First,

$$\begin{aligned} A_V^* A_V u &= -\phi^{-1}\partial_x\left(\phi^2\partial_x(\phi^{-1}u)\right) = -\phi^{-1}\partial_x(u'\phi - u\phi') \\ &= -\phi^{-1}(u''\phi - u\phi'') = -u'' + \phi^{-1}\phi''. \end{aligned}$$

Thus, using $\phi'' = V\phi - \omega\phi$, we have (3.3). Next,

$$\begin{aligned} A_V A_V^* u &= -\phi\partial_x\left(\phi^{-2}\partial_x(\phi u)\right) = -\phi\partial_x\left(\frac{\phi' u + u'\phi}{\phi^2}\right) \\ &= -\phi\frac{(\phi'' u + 2\phi' u' + \phi u'')\phi^2 - 2(\phi' u + u'\phi)\phi\phi'}{\phi^4} \\ &= -\phi^{-1}\phi'' u - u'' + 2\phi^{-2}(\phi')^2 u \\ &= -u'' + \frac{2(\phi')^2 - \phi\phi''}{\phi^2}u = -u'' + \frac{\phi\phi'' - 2(\phi\phi'' - (\phi')^2)}{\phi^2}u \\ &= -u'' + (V-\omega)u - 2(\log\phi)'' u. \end{aligned}$$

From the definition of V_1, we have the conclusion. □

Having in mind of Proposition 3.5, we define another notion of repulsivity.

Definition 3.6. We say V is repulsive in the sense of Darboux if $V - 2(\log\phi)''$ is repulsive.

Remark 3.7. The above definition is for the case $-\partial_x^2 + V$ has exactly one negative eigenvalue. It is clear that one can define more general notation of repulsivity by iterating Darboux transform, see [5].

We use the norm $\|\cdot\|_{\Sigma_A}$ defined by

$$\|u\|_{\Sigma_A}^2 := \|e^{-\frac{|x|}{A}}u'\|_{L^2}^2 + A^{-2}\|e^{-\frac{|x|}{A}}u\|_{L^2}^2.$$

Theorem 3.8. *Assume* $V \in \mathcal{S}(\mathbb{R},\mathbb{R})$ *with* $\max_{j=0,1,2}\sup_{x\in\mathbb{R}} e^{a_1|x|}|V^{(j)}(x)| < \infty$ *for some* $a_1 > 0$. *Assume* $-\partial_x^2 + V$ *has exactly one negative eigenvalue with the eigenpair* (ω,ϕ) *and* V *is repulsive in the sense of Darboux. Then, there exists* $A_0 > 0$ *s.t. for* $A \geq A_0$ *and* $u_0 \in H^1$ *satisfying* $u_0 \perp \phi$, *we have*

$$\int_0^\infty \|e^{-\mathrm{i}t(-\partial_x^2+V)}u_0\|_{\Sigma_A}^2, dt \lesssim A\|u_0\|_{H^1}^2.$$

Remark 3.9. For nonlinear case, see [5].

Sketch of the proof of Theorem 3.8. We set $\kappa = \frac{1}{100}\min(a_1, \sqrt{-\omega})$. As Theorem 2.2, we start from computing $\frac{d}{dt}J_A(u(t))$, where $u(t) = e^{-\mathrm{i}t(-\partial_x^2+V)}u_0$. Then, we have (2.8). However, this time we do not have $\psi_A V' \geq 0$ so we only have the estimate

$$\| (\zeta_A u)' \|_{L^2} \lesssim \frac{d}{dt}J_A(u(t)) + \|e^{-\kappa|x|}u\|_{L^2}.$$

At this point we have not used the fact that $u_0 \perp \phi$. Thus, it is natural to have some term in the right hand side. By the estimate (see (19) of [10] or (6.5) of [7]),

$$\|u\|_{\Sigma_A}^2 \lesssim \|(\zeta_A u)'\|_{L^2}^2 + A^{-1}\|e^{-\kappa|x|}u\|_{L^2}^2,$$

we have

$$\|u\|_{\Sigma_A}^2 \lesssim \frac{d}{dt}J_A(u(t)) + \|e^{-\kappa|x|}u\|_{L^2}. \tag{3.4}$$

We now use Darboux transformation and set $v = \langle \mathrm{i}\epsilon\partial_x\rangle^{-1} A_V u$ for $\epsilon > 0$ to be chosen. By the estimate (see Lemma 7.4 of [7]),

$$\|e^{-\kappa|x|}u\|_{L^2} \lesssim \|e^{-\frac{\kappa}{2}|x|}v\|_{L^2}, \tag{3.5}$$

our task will be to estimate $\|e^{-\frac{\kappa}{2}|x|}v\|_{L^2}$. The transformed function v satisfies

$$\mathrm{i}\partial_t v = (-\partial_x^2 + V_1)v + [\langle \mathrm{i}\varepsilon\partial_x\rangle^{-1}, V_1]A_V u.$$

Thus, for $B > 1$ to be chosen, we have

$$\begin{aligned}
\frac{d}{dt}J_B(v) &= \left\langle (-\partial_x^2 + V_1)v, \left(\frac{\psi_B'}{2} + \psi_B\partial_x\right)v\right\rangle + \left\langle [\langle \mathrm{i}\varepsilon\partial_x\rangle^{-1}, V_1]A_V u, \left(\frac{\psi_{A,B}'}{2} + \psi_{A,B}\partial_x\right)v\right\rangle \\
&= \| (\zeta_B v)' \|_{L^2}^2 - \frac{1}{2}\int \frac{\psi_B}{\zeta_B^2}V'(x)|\zeta_B v|^2\,dx + \frac{1}{2B}\int \widetilde{V}|\zeta_B v|^2\,dx \\
&\quad + \left\langle [\langle \mathrm{i}\varepsilon\partial_x\rangle^{-1}, V_1]A_V u, \left(\frac{\psi_B'}{2} + \psi_B\partial_x\right)v\right\rangle.
\end{aligned}$$

Now, for $1 \ll B \ll A$, we have (see (8.5) of [7]),

$$\begin{aligned}
\|e^{-\frac{\kappa}{2}|x|}v'\|_{L^2}^2 + \|e^{-\frac{\kappa}{2}|x|}v\|_{L^2}^2 \lesssim& \| (\zeta_B v)' \|_{L^2}^2 - \frac{1}{2}\int \frac{\psi_B}{\zeta_B^2}V'(x)|\zeta_B v|^2\,dx \\
&+ \frac{1}{2B}\int \widetilde{V}|\zeta_B v|^2\,dx + A^{-1}\|u\|_{\Sigma_A}^2.
\end{aligned}$$

Further, by the estimate (see Lemma 7.6 of [7]),

$$\|e^{-\kappa|x|}[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-1}, V_1]A_V u\|_{L^2} \lesssim \epsilon\|e^{-\kappa|x|}v\|_{L^2},$$

and Schwartz inequality, we have

$$\left|\left\langle [\langle \mathrm{i}\varepsilon\partial_x\rangle^{-1}, V_1]A_V u, \left(\frac{\psi_B'}{2} + \psi_B\partial_x\right)v\right\rangle\right| \lesssim \epsilon\left(\|e^{-\frac{\kappa}{2}|x|}v'\|_{L^2}^2 + \|e^{-\frac{\kappa}{2}|x|}v\|_{L^2}^2\right).$$

Thus, we arrive to

$$\|e^{-\frac{\kappa}{2}|x|}v'\|_{L^2}^2 + \|e^{-\frac{\kappa}{2}|x|}v\|_{L^2}^2 \lesssim \frac{d}{dt}J_B(v) + A^{-1}\|u\|_{\Sigma_A}^2. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we have

$$\|u\|_{\Sigma_A}^2 \lesssim \frac{d}{dt}\left(J_A(u) + J_B(v)\right).$$

By $|J_A(u)| \lesssim A\|u\|_{H^1}^2$ and $|J_B(v)| \lesssim B\epsilon^{-1}\|u\|_{H^1}^2$, we have the conclusion by taking $1 \ll \epsilon^{-1} \ll B \ll A$. □

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