ON THE REGULARITY FOR MULTILINEAR PSEUDO-DIFFERENTIAL OPERATORS

SUNGGEUM HONG

Abstract. In this talk we study a Hörmander type estimates about the multilinear pseudo-differential operators associated with a symbol. The symbol classes can be classified by the derivative conditions concerning both space and frequency variables. We firstly introduce known results about these operators when the symbol is independent of the space variable. We nextly extend the derivative conditions of the symbol to more general ones. Especially, we only assume at most the first time of the differentiability of the symbol with respect to the space variable. Under these weakened conditions, we establish the mapping properties of these multilinear operators on the product Hardy spaces. This is based on the joint work with Yaryong Heo and Chan Woo Yang([1, 2]).

1. Contents

(1) History for multilinear operators
  - The case of multiplier $\mathbf{m} = \mathbf{m}(\vec{\boldsymbol{\xi}})$ on $\vec{\boldsymbol{\xi}} \in (\mathbb{R}^d)^n$
  - The case of multiplier $\mathbf{m} = \mathbf{m}(x, \vec{\boldsymbol{\xi}})$ on $(x, \vec{\boldsymbol{\xi}}) \in \mathbb{R}^d \times (\mathbb{R}^d)^n$

(2) Multilinear pseudo-differential operators on $\mathbb{R}^d \times (\mathbb{R}^d)^n$
  - Littlewood-Paley type decomposition
  - Reduction via limiting arguments
  - Estimates from $H^{p_1}(\mathbb{R}^d) \times \cdots \times H^{p_n}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$

1.1. The inhomogeneous fractional Sobolev spaces.

We firstly define the inhomogeneous fractional Sobolev spaces to describe the known results and state our main theorem.

- For $s \geq 0$ let $(\vec{I} - \vec{\Delta})^{s/2}$ denote the inhomogeneous fractional Laplacian operator acting on functions on $(\mathbb{R}^d)^n$. To be specific,
$$(\vec{I} - \vec{\Delta})^{s/2} F = \Big(\big(1 + 4\pi^2 (|\cdot_1|^2 + \cdots + |\cdot_n|^2)\big)^{s/2} \widehat{F} \Big)^{\vee}$$
for a function F on $(\mathbb{R}^d)^n$, where $\widehat{f}$ denotes the Fourier transform of a Schwartz function f on $\mathbb{R}^d$.
- Now for $s \geq 0$ and $0 < r < \infty$ we define the Sobolev norm
$$\|F\|_{L^r_s((\mathbb{R}^d)^n)} := \big\| (\vec{I} - \vec{\Delta})^{s/2} F \big\|_{L^r((\mathbb{R}^d)^n)}.$$
- For the special case $r = 2$, it can be written that
$$\|F\|_{L^2_s((\mathbb{R}^d)^n)} = \Big(\int_{(\mathbb{R}^d)^n} \big(1 + 4\pi^2(|\xi_1|^2 + \cdots + |\xi_n|^2)\big)^s \big|\widehat{F}(\xi_1, \ldots, \xi_n)\big|^2 d\vec{\boldsymbol{\xi}} \Big)^{1/2}.$$

Date: September 8, 2022.

2010 *Mathematics Subject Classification.* Primary 42B20, Secondary 42B15.

Key words and phrases. Multilinear operators, Hörmander multiplier, regular Calderón-Zygmund kernel, Coifman-Meyer theorem.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology NRF-2022R1A2C1004616.

2. The case $\mathbf{m} = \mathbf{m}(\vec{\boldsymbol{\xi}})$ on $(\mathbb{R}^d)^n$

2.1. The case $\mathbf{m} = \mathbf{m}(\xi)$ on $\mathbb{R}^d$ for $n = 1$.

- Given a bounded function $\mathbf{m}$ on $\mathbb{R}^d$, we define the (linear) multiplier operator $\mathrm{T}_{\mathbf{m}}$ by

$$\mathrm{T}_{\mathbf{m}} f(x) := \int_{\mathbb{R}^d} e^{2\pi i\langle x, \xi\rangle}\, \mathbf{m}(\xi) \widehat{f}(\xi) d\xi$$

  for a Schwartz function f on $\mathbb{R}^d$.
- By Plancherel's identity, $\|\mathrm{T}_{\mathbf{m}}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = \|\mathbf{m}\|_{L^\infty(\mathbb{R}^d)}$.
- According to the classical Mihlin multiplier theorem (1956), $\mathrm{T}_{\mathbf{m}}$ admits an L^p-bounded extension for $1 < p < \infty$ whenever

$$\big|\partial_\xi^\alpha \mathbf{m}(\xi)\big| \le C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0 \tag{2.1}$$

  for all multi-indices α with $|\alpha| \le [d/2] + 1$.
- This result was refined by Hörmander (1960) who replaced (2.1) by the weaker condition

$$\sup_{j\in\mathbb{Z}} \big\|\mathbf{m}(2^j\cdot)\widehat{\psi}(\cdot)\big\|_{L^2_s(\mathbb{R}^d)} < \infty \quad \text{for } s > d/2, \tag{2.2}$$

  where $L^2_s(\mathbb{R}^d)$ stands for the fractional Sobolev space on $\mathbb{R}^d$ and ψ is a Schwartz function on $\mathbb{R}^d$ whose Fourier transform $\widehat{\psi}$ is supported in the annulus $\{\xi \in \mathbb{R}^d : 1/2 < |\xi| < 2\}$ and satisfies $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$.
- Calderón and Torchinsky (1977): extended it to the (real) Hardy space $H^p(\mathbb{R}^d)$ for $p \le 1$ assuming (2.2) for $s > d/p - d/2$. They proved that there exists a $C > 0$ such that

$$\big\|\mathrm{T}_{\mathbf{m}}\big\|_{H^p(\mathbb{R}^d)\to H^p(\mathbb{R}^d)} \le C \sup_{j\in\mathbb{Z}} \big\|\mathbf{m}(2^j\cdot)\widehat{\psi}(\cdot)\big\|_{L^2_s(\mathbb{R}^d)}.$$

- The Hardy space $H^p(\mathbb{R}^d)$ is actually defined for all $0 < p \le \infty$ and coincides with $L^p(\mathbb{R}^d)$ for $1 < p \le \infty$.
- Recently, their result has been reformulated by Grafakos, He, Honzík, and Nguyen (2014): Let $1 < p < \infty$ and $1 < r < \infty$. Suppose that $s > d/r$ and $s > |d/p - d/2|$. Then there exists a $C > 0$ such that

$$\big\|\mathrm{T}_{\mathbf{m}}\big\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} \le C \sup_{j\in\mathbb{Z}} \big\|\mathbf{m}(2^j\cdot)\widehat{\psi}\big\|_{L^r_s(\mathbb{R}^d)}. \tag{2.3}$$

- We remark that the two conditions $s > d/r$ and $s > |d/p - d/2|$ are sharp in the sense that if one of them does not hold, then there exists a bounded function $\mathbf{m}$ for which (2.3) does not hold (Grafakos, Park (2022); Slavíková (2020)).

2.2. The case $\mathbf{m} = \mathbf{m}(\vec{\boldsymbol{\xi}})$ on $(\mathbb{R}^d)^n$ for $n > 1$.

- For a bounded function $\mathbf{m}$ on $(\mathbb{R}^d)^n$, the corresponding n-linear Fourier multiplier operator $\mathrm{T}_{\mathbf{m}}$ is defined by

$$\mathrm{T}_{\mathbf{m}}\big(f_1, \cdots, f_n\big)(x) := \int_{(\mathbb{R}^d)^n} e^{2\pi i\langle x, \xi_1+\cdots+\xi_n\rangle}\, \mathbf{m}(\vec{\boldsymbol{\xi}})\, \widehat{f_1}(\xi_1)\cdots\widehat{f_n}(\xi_n)\, d\vec{\boldsymbol{\xi}}$$

  for Schwartz functions $f_1, \cdots, f_n$ on $\mathbb{R}^d$, where $\vec{\boldsymbol{\xi}} := (\xi_1, \dots, \xi_n) \in (\mathbb{R}^d)^n$ and $d\vec{\boldsymbol{\xi}} := d\xi_1 \cdots d\xi_n$.

2.3. A multilinear extension of Mihlin's result.

- Coifman and Meyer (1978) proved that if L is sufficiently large and $\mathbf{m}$ satisfies

$$\big|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n}\mathbf{m}(\xi_1,\dots,\xi_n)\big| \le C_{\alpha_1,\dots,\alpha_n}\big(|\xi_1|+\cdots+|\xi_n|\big)^{-(|\alpha_1|+\cdots+|\alpha_n|)} \tag{2.4}$$

  for multi-indices $\alpha_1,\dots,\alpha_n$ with $|\alpha_1|+\cdots+|\alpha_n|\le L$, then $\mathrm{T}_{\mathbf{m}}$ is bounded from $L^{p_1}(\mathbb{R}^d)\times\cdots\times L^{p_n}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for all $1<p_1,\dots,p_n\le\infty$ and $1<p<\infty$ with $1/p_1+\cdots+1/p_n=1/p$.
- The result was extended to $p\le 1$ by Kenig and Stein (1999) and Grafakos and Torres (2002).
- Later, the research naturally proceeded toward improving the condition (2.4) to obtain multilinear analogues of the classical Hörmander multiplier theorem.
- A multilinear extension of Hörmander's result was first established by Tomita (2010).

2.4. A multilinear extension of Hörmander's result.

Let Ψ be the n-linear counterpart of ψ. That is, Ψ is a Schwartz function on $(\mathbb{R}^d)^n$ having the properties that

$$\mathrm{supp}(\widehat{\Psi})\subset\big\{\vec{\boldsymbol{\xi}}\in(\mathbb{R}^d)^n : 1/2\le|\vec{\boldsymbol{\xi}}|\le 2\big\},\ \sum_{j\in\mathbb{Z}}\widehat{\Psi}(2^{-j}\vec{\boldsymbol{\xi}})=1,\ \vec{\boldsymbol{\xi}}\neq\vec{\mathbf{0}}.$$

Theorem 2.1 (Tomita (2010)). *Let $\mathbf{m}=\mathbf{m}(\vec{\boldsymbol{\xi}})$. Let $1<p,p_1,\dots,p_n<\infty$ be such that $1/p=1/p_1+\cdots+1/p_n$. Suppose that*

$$\sup_{j\in\mathbb{Z}}\big\|\mathbf{m}(2^j\,\vec{\cdot}\,)\widehat{\Psi}(\,\vec{\cdot}\,)\big\|_{L^2_s((\mathbb{R}^d)^n)}<\infty \tag{2.5}$$

for $s>nd/2$. Then we have

$$\big\|\mathrm{T}_{\mathbf{m}}\big\|_{L^{p_1}\times\cdots\times L^{p_n}\to L^p}\le C\sup_{j\in\mathbb{Z}}\big\|\mathbf{m}(2^j\,\vec{\cdot}\,)\widehat{\Psi}(\,\vec{\cdot}\,)\big\|_{L^2_s((\mathbb{R}^d)^n)}. \tag{2.6}$$

This result was later extended by Grafakos and Si (2012) to the range $p\le 1$ by using L^r-based Sobolev space condition $L^r_s((\mathbb{R}^d)^n)$ for $1<r\le 2$.

2.5. The necessary condition for multilinear operators.

We state the necessary condition for multilinear operators associated with $\mathbf{m}=\mathbf{m}(\vec{\boldsymbol{\xi}})$.

Theorem 2.2 (Grafakos, He, Honzík (2018)). *Let $\mathbf{m}=\mathbf{m}(\vec{\boldsymbol{\xi}})$. Let $0<p,p_1,\dots,p_n<\infty$ be such that $1/p=1/p_1+\cdots+1/p_n$. Let $0<r,s<\infty$. Suppose that*

$$\|\mathrm{T}_{\mathbf{m}}\|_{L^{p_1}\times\cdots\times L^{p_n}\to L^p}\le C\sup_{j\in\mathbb{Z}}\big\|\mathbf{m}(2^j\,\vec{\cdot}\,)\widehat{\Psi}(\,\vec{\cdot}\,)\big\|_{L^r_s((\mathbb{R}^d)^n)} \tag{2.7}$$

for all bounded functions $\mathbf{m}$ for which

$$\sup_{j\in\mathbb{Z}}\big\|\mathbf{m}(2^j\,\vec{\cdot}\,)\widehat{\Psi}(\,\vec{\cdot}\,)\big\|_{L^r_s((\mathbb{R}^d)^n)}<\infty.$$

Then we must necessarily have

1. $s\ge\max\big\{\frac{(n-1)d}{2},\frac{nd}{r}\big\}$,
2. $\frac{1}{p}-\frac{1}{2}\le\frac{s}{d}+\sum_{i\in I}\Big(\frac{1}{p_i}-\frac{1}{2}\Big)$ *where I is an arbitrary subset of $J_n=\{1,2,\dots,n\}$ which may also be empty (in which case the sum is supposed to be zero).*

2.6. **The Hörmander multiplier theorem for multilinear operators.** Most recently, Lee et al (2021) in [1] consider the case $r = 2$ in (2.7), and proved that the necessary conditions in the above are also "almost" sufficient for the $H^{p_1} \times \cdots \times H^{p_n} \to L^p$ boundedness for $\mathrm{T}_{\mathbf{m}}$.

Theorem 2.3. ([1]) Let $\mathbf{m} = \mathbf{m}(\vec{\boldsymbol{\xi}})$. Let $0 < p_1, \cdots, p_n \le \infty$ and $0 < p < \infty$ satisfy $1/p = 1/p_1 + \cdots + 1/p_n$. Suppose that

(1) $s > \frac{nd}{2}$,

(2) $\frac{1}{p} - \frac{1}{2} < \frac{s}{d} + \sum_{i \in I} \left(\frac{1}{p_i} - \frac{1}{2}\right)$

where I is an arbitrary subset of $J_n = \{1, 2, \ldots, n\}$ which may also be empty (in which case the sum is supposed to be zero). Then we have

$$\left\|\mathrm{T}_{\mathbf{m}}(f_1, \ldots, f_n)\right\|_{L^p(\mathbb{R}^d)} \le C \sup_{j \in \mathbb{Z}} \left\|\mathbf{m}(2^j\,\vec{\cdot}\,)\widehat{\Psi}(\,\vec{\cdot}\,)\right\|_{L^2_s((\mathbb{R}^d)^n)} \prod_{i=1}^{n} \|f_i\|_{H^{p_i}(\mathbb{R}^d)}, \tag{2.8}$$

for Schwartz functions $f_1, \ldots, f_n$ on $\mathbb{R}^d$.

- This is called the Hörmander multiplier theorem for multilinear operators.
- The conditions (1) and (2) in Theorem (Lee et al [1]) are "almost" sharp in view of Theorem(Grafakos, He, Honzík(2018)).
- However, it does not give us any information in the critical case
$$s = \frac{nd}{2} \qquad \text{or} \qquad \frac{1}{p} - \frac{1}{2} = \frac{s}{d} + \sum_{i \in I}\left(\frac{1}{p_i} - \frac{1}{2}\right) \quad \text{for some } \; I \subset J_n.$$
- Also two conditions $s > nd/2$ and $1/p - 1/2 < s/d$ are necessary for the $H^{p_1} \times \cdots \times H^{p_n} \to L^p$ boundedness of $\mathrm{T}_{\mathbf{m}}$ in (2.8) with the Hörmander multiplier condition
$$\sup_{j \in \mathbb{Z}} \left\|\mathbf{m}(2^j\,\vec{\cdot}\,)\widehat{\Psi}(\,\vec{\cdot}\,)\right\|_{L^2_s((\mathbb{R}^d)^n)} < \infty.$$

3. Multilinear pseudo-differential operators on $\mathbb{R}^d \times (\mathbb{R}^d)^n$

- Let n be a positive integer greater than 1.
- Compared to the previous case, properties of the multilinear operators $\mathrm{T}_{\mathbf{m}}$ associate with multipliers depending on the variable x
$$\mathrm{T}_{\mathbf{m}}(f_1, \cdots, f_n)(x) := \int_{(\mathbb{R}^d)^n} e^{2\pi i \langle x, \xi_1 + \cdots + \xi_n \rangle}\, \mathbf{m}(x, \vec{\boldsymbol{\xi}}) \widehat{f_1}(\xi_1) \cdots \widehat{f_n}(\xi_n)\, d\vec{\boldsymbol{\xi}}$$
has not been well understood.
- Most results for $\mathrm{T}_{\mathbf{m}}$ were obtained by assuming $\mathbf{m}$ belongs to some symbol class n-$\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$, $0 \le \delta \le \rho \le 1$, $0 \le \delta < 1$ for some $m \le 0$. That is,
$$\left|\partial_x^\alpha \partial_{\vec{\boldsymbol{\xi}}}^\beta \mathbf{m}(x, \vec{\boldsymbol{\xi}})\right| \le C_{\alpha,\beta}(1 + |\vec{\boldsymbol{\xi}}|)^{m + \delta|\alpha| - \rho|\beta|} \tag{3.1}$$
for all multi-indices α and β. Here, the number m is called the order of $\mathbf{m}$.

4. The case $\mathbf{m} = \mathbf{m}(x, \vec{\boldsymbol{\xi}})$ on $\mathbb{R}^d \times (\mathbb{R}^d)^n$

4.1. **The case of $\mathbf{m} \in 1$-$\mathcal{S}^0_{\rho,\delta}(\mathbb{R}^d)$ for $n = 1$.**

- L. Hörmander (1967) $\mathrm{T}_{\mathbf{m}} : L^2 \to L^2$ for $0 < \delta < \rho < 1$.
- Calderón-Vaillancourt (1972) $\mathrm{T}_{\mathbf{m}} : L^2 \to L^2$ for $\mathbf{m} \in 1$-$\mathcal{S}^0_{0,0}(\mathbb{R}^d)$.
- Coifman and Meyer (1978) $\mathrm{T}_{\mathbf{m}} : L^p \to L^p$, $1 < p < \infty$ for $\mathbf{m} \in 1$-$\mathcal{S}^0_{1,0}(\mathbb{R}^d)$.
- E.M. Stein (1993): $\mathrm{T}_{\mathbf{m}}$ is unbounded for $\mathbf{m} \in 1$-$\mathcal{S}^0_{1,1}(\mathbb{R}^d)$.

4.2. The case of $\mathbf{m} \in n\text{-}\mathcal{S}^0_{\rho,\delta}(\mathbb{R}^d)$ for $n > 1$.

- Calderón-Vaillancourt (1972) : $\mathrm{T}_{\mathbf{m}}$ is unbounded for $\mathbf{m} \in n\text{-}\mathcal{S}^0_{0,0}(\mathbb{R}^d)$.
- Coifman and Meyer (1978) : $\mathrm{T}_{\mathbf{m}}$ is bounded from $L^{p_1} \times \cdots \times L^{p_n}$ to L^p, $1 < p_1, \cdots, p_n < \infty$, $1 < p < \infty$ and $1/p = \sum_{i=1}^n 1/p_i$ for $\mathbf{m} \in n\text{-}\mathcal{S}^0_{1,0}(\mathbb{R}^d)$.
- Grafakos-Torres (2002) extended Coifman and Meyer (1978)'s result up to optimal range for $p > 1/n$.
- Bényi-Torres (2004) : $\mathrm{T}_{\mathbf{m}}$ is unbounded for $\mathbf{m} \in 2\text{-}\mathcal{S}^0_{\rho,\delta}(\mathbb{R}^d)$ and $0 \le \rho < 1, 0 \le \delta \le 1$.
- Bényi et al (2010) : $\mathrm{T}_{\mathbf{m}} : L^p \times L^q \to L^r$, $1 < p, q < \infty$, $1/p+1/q = 1/r$ for $\mathbf{m} \in 2\text{-}\mathcal{S}^0_{1,\delta}(\mathbb{R}^d)$.

4.3. The case of $\mathbf{m} \in 2\text{-}\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ and $m < 0$.

- Miyachi-Tomita (2013): $\mathrm{T}_{\mathbf{m}} : L^p \times L^q \to L^r$, $1 < p, q < \infty$, $1/p + 1/q = 1/r$ for $\mathbf{m} \in 2\text{-}\mathcal{S}^m_{0,0}(\mathbb{R}^d)$ and $m = m_0(p, q) = -n\big(\max\{1/2, 1/p, 1/q, 1 - 1/r, 1/r - 1/2\}\big)$.
- Miyachi-Tomita (2020). $\mathrm{T}_{\mathbf{m}} : L^p \times L^q \to L^r$, $1 < p, q \le \infty$, $1/p + 1/q = 1/r$ for $\mathbf{m} \in 2\text{-}\mathcal{S}^m_{\rho,\rho}(\mathbb{R}^d)$, $0 \le \rho < 1$ and $m = (1 - \rho)m_0(p, q)$.

4.4. The case of $\mathbf{m} \in n\text{-}\mathcal{S}^0_{\rho,\delta}(\mathbb{R}^d)$ for $n > 2$.

- Calderón-Vaillancourt (1972): $\mathrm{T}_{\mathbf{m}}$ is unbounded for $\mathbf{m} \in n\text{-}\mathcal{S}^0_{0,0}(\mathbb{R}^d)$.
- Coifman-Meyer (1978): $\mathrm{T}_{\mathbf{m}} : L^{p_1} \times \cdots \times L^{p_n} \to L^p$, $1 < p_1, \cdots, p_n < \infty$ and $1 \le p < \infty$ for $\mathbf{m} \in n\text{-}\mathcal{S}^0_{1,0}(\mathbb{R}^d)$.
- Grafakos-Torres (2002) extended Coifman-Meyer (1978)'s result up to optimal range for $p > 1/n$.

4.5. The case of $\mathbf{m} \in n\text{-}\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$ and $m < 0$.

- Kato-Miyachi-Tomita (2022) : local Hardy space estimates $\mathrm{T}_{\mathbf{m}} : h^{p_1} \times \cdots \times h^{p_n} \to h^p$, $0 < p, p_1, \cdots, p_n \le \infty$ and $1/p \le \sum_{i=1}^n 1/p_i$ for $\mathbf{m} \in n\text{-}\mathcal{S}^m_{0,0}(\mathbb{R}^d)$.

5. The goal of this presentation

(1) We generalize the condition of a symbol class $\mathbf{m} \in n\text{-}\mathcal{S}^m_{\rho,\delta}(\mathbb{R}^d)$, $0 \le \delta \le \rho \le 1$, $0 \le \delta < 1$ for some $m \le 0$ in (3.1) to obtain multilinear analogues of the classical Hörmander multiplier theorem associated with $\mathbf{m}(x, \vec{\boldsymbol{\xi}})$.

(2) Especially, we only use at most the first time of the differentiability of the symbol concerning the space variable $x \in \mathbb{R}^d$.

(3) Under these weakened conditions, we establish that the operator $\mathrm{T}_{\mathbf{m}}$ is bounded from $H^{p_1}(\mathbb{R}^d) \times \cdots \times H^{p_n}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for all $0 < p_1, \ldots, p_n \le \infty$ and $0 < p < \infty$ with $1/p_1 + \cdots + 1/p_n = 1/p$.

(4) The condition (3.1) is weakened as the Hörmander type condition as below:

$$\|\mathbf{m}\|_{\mathcal{L}^2_{s,\delta}} := \sup_{x\in\mathbb{R}^d} \left(\sum_{|\alpha|\le 1} \big\| \partial_x^\alpha \mathbf{m}(x, \vec{\cdot}\,)\widehat{\Phi}(\vec{\cdot}\,) \big\|_{L^2_s((\mathbb{R}^d)^n)} \right) + \sup_{j\ge 0} \sup_{x\in\mathbb{R}^d} \left(\sum_{|\alpha|\le 1} 2^{-j\delta|\alpha|} \big\| \partial_x^\alpha \mathbf{m}(x, 2^j\,\vec{\cdot}\,)\widehat{\Psi}(\vec{\cdot}\,) \big\|_{L^2_s((\mathbb{R}^d)^n)} \right).$$

- Here Φ is a Schwartz function on $(\mathbb{R}^d)^n$ whose Fourier transform $\widehat{\Phi}$ is supported in $|\vec{\boldsymbol{\xi}}| < 1$ and $\widehat{\Phi}(\vec{\boldsymbol{\xi}}) = 1$ for $|\vec{\boldsymbol{\xi}}| \le 1/2$.
- With this Φ, we define another function Ψ by $\widehat{\Psi}(\vec{\boldsymbol{\xi}}) = \widehat{\Phi}(\vec{\boldsymbol{\xi}}) - \widehat{\Phi}(2\vec{\boldsymbol{\xi}})$.

- Then we have the following "partition of unity" of the $\vec{\boldsymbol{\xi}}$-space:

$$1 = \widehat{\Phi}(\vec{\boldsymbol{\xi}}) + \sum_{j=0}^{\infty} \widehat{\Psi}(2^{-j}\vec{\boldsymbol{\xi}}), \quad \text{for all } \vec{\boldsymbol{\xi}}, \tag{5.1}$$

where $\widehat{\Psi}$ is supported in the annulus $\{\vec{\boldsymbol{\xi}} : 1/2 < |\vec{\boldsymbol{\xi}}| < 2\}$.

6. The sufficient condition for multilinear pseudo-differential operators

We first observe that the sufficient condition of Theorem 2.3(Lee et al) in [1]

$$\frac{1}{p} - \frac{1}{2} < \frac{s}{d} + \sum_{i\in I}\Big(\frac{1}{p_i} - \frac{1}{2}\Big)$$

is equivalent to

$$\sum_{i\in I^c} \frac{1}{p_i} < \frac{s}{d} + \frac{1-\sharp I}{2} = (\frac{s}{d} + \frac{1}{2}) + \frac{\sharp I^c - n}{2}. \tag{6.1}$$

where I is an arbitrary subset of $J_n = \{1, 2, \dots, n\}$. Then we may rewrite this condition as

$$\sum_{i\in I^c} (\frac{1}{p_i} - \frac{1}{2}) + \frac{n}{2} < (\frac{s}{d} + \frac{1}{2}).$$

So by replacing the set I^c in (6.1) by I.

Lemma 6.1. *The set of all collections of n-tuples $(1/p_1, \cdots, 1/p_n) \in (0,\infty)^n$ that satisfies*

$$\frac{1}{p} - \frac{1}{2} < \frac{s}{d} + \sum_{i\in I}\Big(\frac{1}{p_i} - \frac{1}{2}\Big)$$

in Theorem 2.3([1]) is equivalent to the set $\mathrm{B_n}(\frac{s}{d} + \frac{1}{2})$ where

$$\mathrm{B_n}(\alpha) := \bigcap_{I\subset J_n} \Big\{(x_1, \cdots, x_n) \in (0,\infty)^n : \sum_{i\in I}(x_i - \frac{1}{2}) + \frac{n}{2} < \alpha\Big\}.$$

Moreover, for $\alpha > 0$ if we denote

$$\mathrm{A_n}(\alpha) = \Big\{(x_1, \cdots, x_n) \in (0,\infty)^n : \sum_{i=l}^{n} \max(x_i, \frac{1}{2}) < \alpha\Big\},$$

then we have $\mathrm{B_n}(\alpha) = \mathrm{A_n}(\alpha)$.

7. Main Theorem

Now we state our main results. For a bounded function $\mathbf{m}(x, \vec{\boldsymbol{\xi}})$ on $\mathbb{R}^d \times (\mathbb{R}^d)^n$, the corresponding n-linear pseudo-differential operator $\mathrm{T}_{\mathbf{m}}$ is defined by

$$\mathrm{T}_{\mathbf{m}}(f_1, \cdots, f_n)(x) := \int_{(\mathbb{R}^d)^n} e^{2\pi \mathrm{i}\langle x, \xi_1 + \cdots + \xi_n\rangle}\, \mathbf{m}(x, \vec{\boldsymbol{\xi}}) \widehat{f_1}(\xi_1) \cdots \widehat{f_n}(\xi_n)\, d\vec{\boldsymbol{\xi}}$$

for Schwartz functions $f_1, \cdots, f_n$ on $\mathbb{R}^d$, where $\vec{\boldsymbol{\xi}} := (\xi_1, \dots, \xi_n) \in (\mathbb{R}^d)^n$ and $d\vec{\boldsymbol{\xi}} := d\xi_1 \cdots d\xi_n$.

Theorem 7.1. ([2]) Let $\mathbf{m} = \mathbf{m}(x, \vec{\boldsymbol{\xi}})$. Let $0 < p_1, \cdots, p_n < \infty$ and $0 < p < \infty$ satisfy $1/p = 1/p_1 + \cdots + 1/p_n$. Suppose that

(1) $s > \frac{nd}{2}$, and
(2) $\mathrm{B_n}(\frac{s}{d}) = \Big\{(x_1, \cdots, x_n) \in (0,\infty)^n : \sum_{i=l}^{n} \max(x_i, \frac{1}{2}) < \frac{s}{d}\Big\}$.

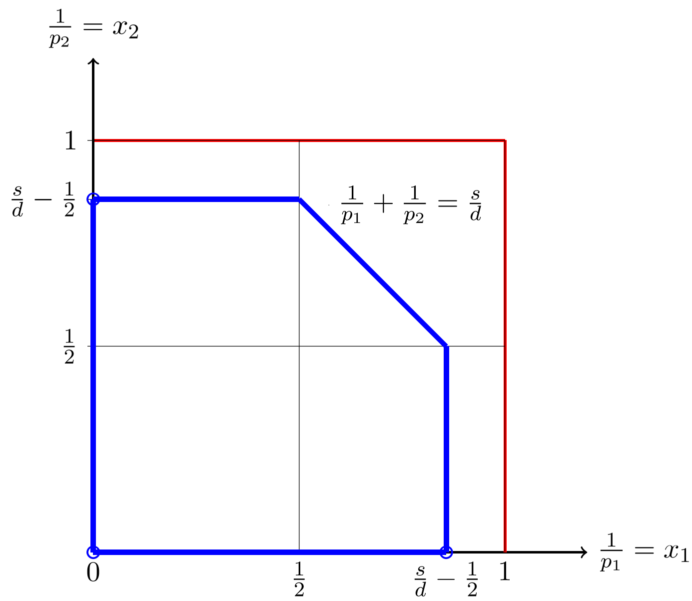
Then for any $0 \le \delta < 1$, if $\big(\frac{1}{p_1}, \dots, \frac{1}{p_n}\big) \in \mathrm{B_n}(\frac{s}{d})$, then

$$\big\|\mathrm{T}_{\mathbf{m}}(f_1, \dots, f_n)\big\|_{L^p(\mathbb{R}^d)} \le C\,\|\mathbf{m}\|_{\mathcal{L}^2_{s,\delta}} \prod_{i=1}^{n} \|f_i\|_{H^{p_i}(\mathbb{R}^d)}$$

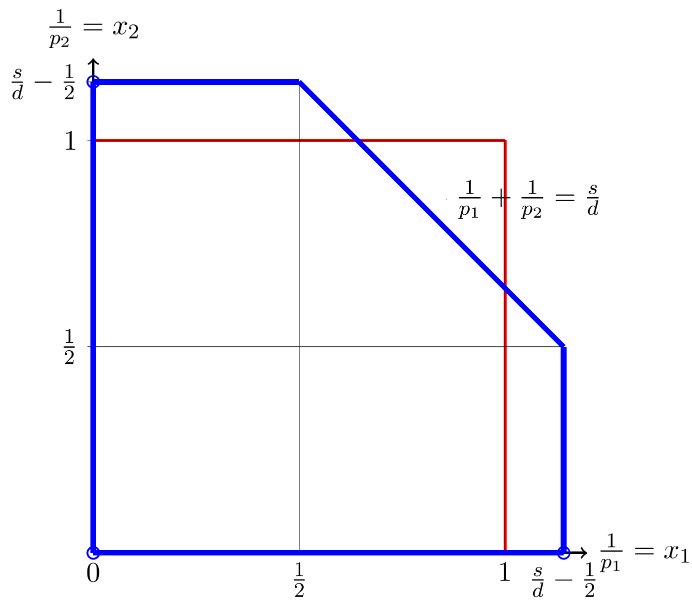
for Schwartz functions $f_1, \dots, f_n$ on $\mathbb{R}^d$, where

$$\|\mathbf{m}\|_{\mathcal{L}^2_{s,\delta}} := \sup_{x\in\mathbb{R}^d} \bigg(\sum_{|\alpha|\le 1} \big\|\partial_x^\alpha \mathbf{m}(x,\ \vec{\cdot}\ )\widehat{\Phi}(\ \vec{\cdot}\ )\big\|_{L^2_s((\mathbb{R}^d)^n)} \bigg) + \sup_{j\ge 0}\sup_{x\in\mathbb{R}^d} \bigg(\sum_{|\alpha|\le 1} 2^{-j\delta|\alpha|} \big\|\partial_x^\alpha \mathbf{m}(x, 2^j\ \vec{\cdot}\ )\widehat{\Psi}(\ \vec{\cdot}\ )\big\|_{L^2_s((\mathbb{R}^d)^n)} \bigg).$$

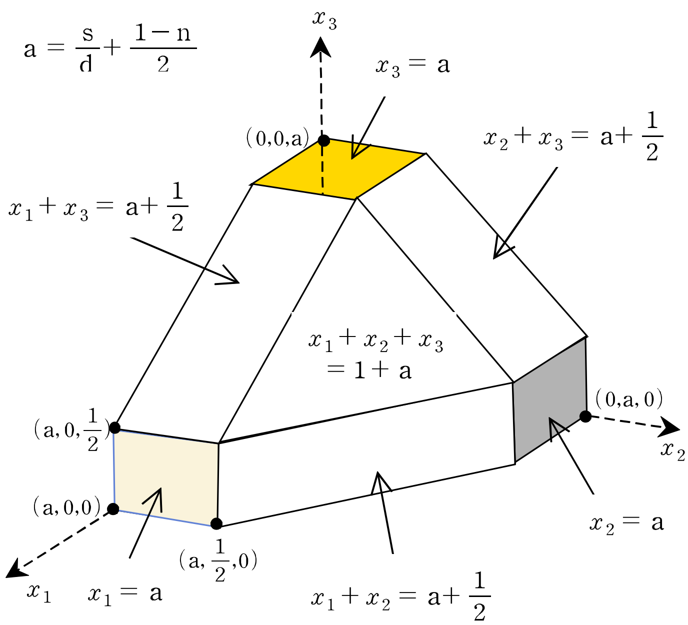
(1) A Figure for $\mathrm{B}_2(\frac{s}{d})$ for $1 < \frac{s}{d} \le \frac{3}{2}$.



(2) A Figure for $\mathrm{B}_2(\frac{s}{d})$ for $\frac{3}{2} < \frac{s}{d}$.



(3) A Figure for $B_3(\frac{s}{d})$.



Remark. Although Theorem 2.3 holds for $\left(\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right) \in B_n(\frac{s}{d} + \frac{1}{2})$, we obtain Theorem 7.1 for $\left(\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right) \in B_n(\frac{s}{d})$. At present we do not know whether our results can be extended to $B_n(\frac{s}{d} + \frac{1}{2})$ or not.

References

[1] Jongho Lee, Yaryong Heo, Sunggeum Hong, Jin Bong Lee, Bae Jun Park, Yejune Park and Chan Woo Yang, *The Hörmander multiplier theorem for n-linear operators.* Math. Ann. 381 (2021), no. 1-2, 499–555.

[2] Yaryong Heo, Sunggeum Hong and Chan Woo Yang, *A Hörmander type multiplier theorem for multilinear pseudo-differential operators*, preprint.

Sunggeum Hong, Department of Mathematics, Chosun University, Gwangju 61452, S. Korea
Email address: skhong@chosun.ac.kr