Discrete cubical homotopy groups and real $K(\pi, 1)$ spaces

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In Brief

- Discrete cubical homotopy theory is a homotopy theory in the category of simple graphs
- Key invariants associated to Γ (finite simple graph) are groups $A_n(\Gamma, v)$ which are discrete analogues of $\Pi_n(X, x)$.
- Key concept: $\Gamma \to X_\Gamma$ top. space constructed as a cubical complex conjectured (2006) to be:

$$A_n(\Gamma, v) \overset{?}{\cong} \Pi_n(X_\Gamma, x)$$

- 2006: Proved for all n by Babson, B., de Longueville, Laubenbacher conditional on the existence a cubical approximation theorem
- 2022: Proved by Carranza and Kapulkin using categorification, circumventing need of an approximation theorem

Origins and Developments

- **Built** on Atkin works (1972-1976): on modeling of social and technological networks using simplicial complexes
- **Formalized**: Kramer, Laubenbacher (1998, $n = 1$); B., K., L.,Weaver (2001, all n): $A_n^q(\Delta, \sigma_0)$, a bi-graded family of groups
- **Cubicalized**: Babson, B., de Longueville, Laubenbacher (2006): $A_n^G(\Gamma)$
- **Generalized** to metric spaces: B., Capraro, White (2014); Delabie, Khukhro (2020)
- **Homologized**: B. Capraro, White (2014)
- **Further Developed**: Babson, B., Greene, Jarrah, Lutz, McConville, Welker (2015-)
- **Categorified**: Carranza, Kapulkin (2022, preprint)

Discrete (Cubical) Homotopy Theory for Graphs

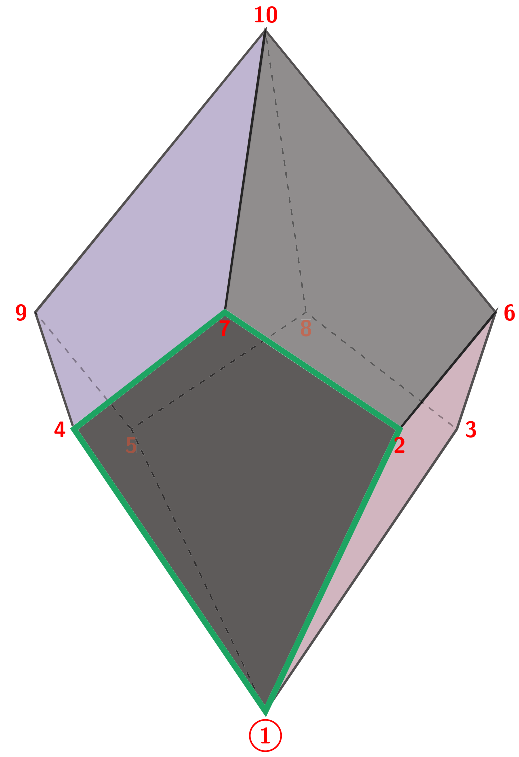
(Babson, B., Kramer, de Longueville, Laubenbacher, Severs, Weaver, White)

Definitions

1. Γ - graph (Δ simplicial complex; X metric space)
   v_0 - distinguished vertex (σ_0; x_0)
   $\mathbb{Z}^n$ - infinite lattice (usual metric)
2. $\mathcal{A}_n(\Gamma, v_0)$ - set of graph homs $f : \mathbb{Z}^n \to V(\Gamma)$, with finite support:

   if $d(\vec{a}, \vec{b}) = 1$ in $\mathbb{Z}^n$ then $d(f(\vec{a}), f(\vec{b})) = 0$ or 1, with $f(\vec{i}) = v_0$ almost everywhere
3. f, g are *discrete homotopic* if there exist $h \in \mathcal{A}_{n+1}(\Gamma, v_0)$ and $k, \ell \in \mathbb{N}$ such that for all $\vec{i} \in \mathbb{Z}^n$,
   $$h(\vec{i}, k) = f(\vec{i})$$
   $$h(\vec{i}, \ell) = g(\vec{i})$$
4. $A_n(\Gamma, v_0)$ - set of equivalence classes of maps in $\mathcal{A}_n(\Gamma, v_0)$
   Note: translation preserves discrete homotopy

A Discrete Homotopy of Graph Homomorphisms – Step 1



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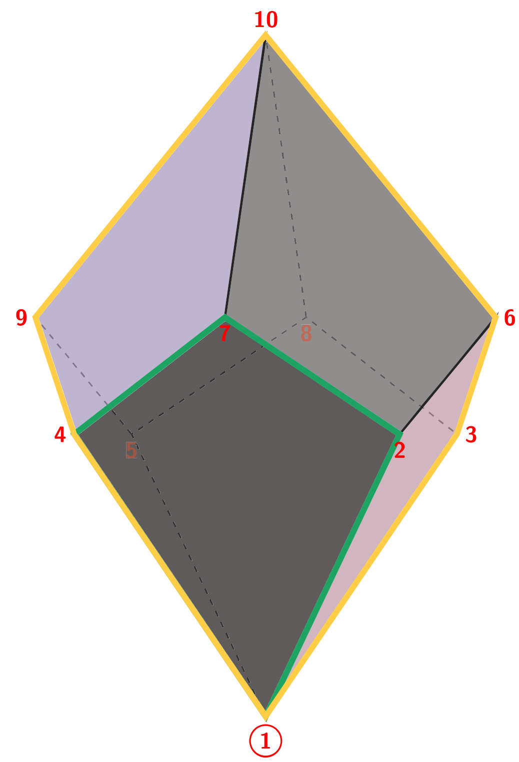
A Discrete Homotopy of Graph Homomorphisms – Step 2



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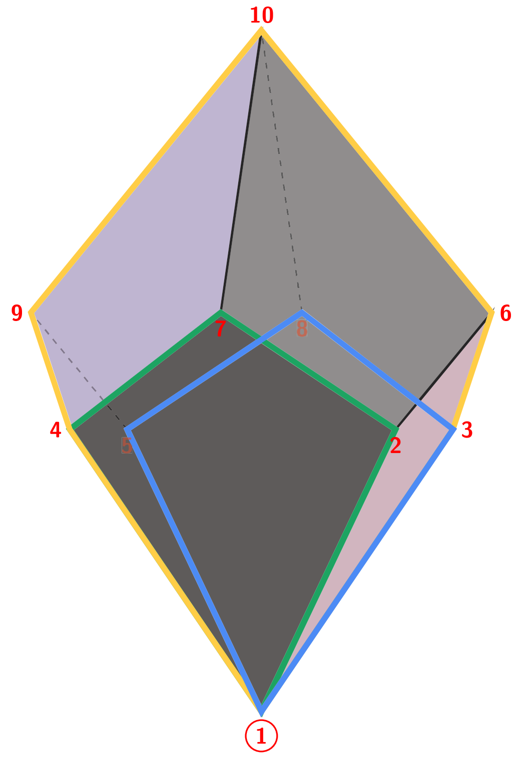
A Discrete Homotopy of Graph Homomorphims – Step 3



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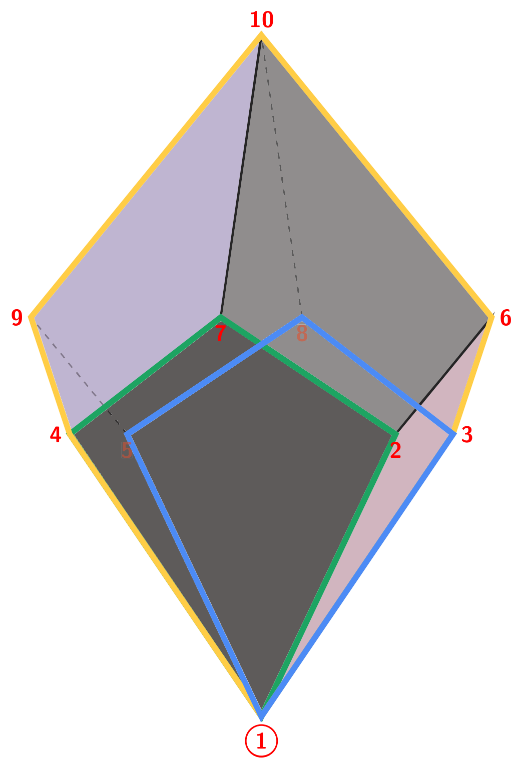
A Discrete Homotopy of Graph Homomorphims – Step 4



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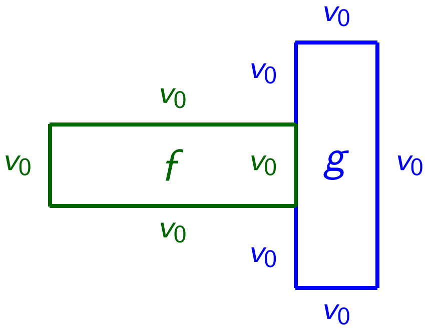
Discrete Homotopy Theory for Graphs

Group Structure

- Multiplication: for $f, g \in \mathcal{A}_n(\Gamma, v_0)$ of radius M, N,

$$f\,g\,(\vec{i}) = \begin{cases} f(\vec{i}) & i_1 \leq M \\ g(i_1 - (M+N), i_2, \dots i_n) & i_1 > M \end{cases}$$

- $n = 1$ concatenation of loops based at v_0
- $n = 2$



$$[f\,g] = [f]\,[g]$$

Discrete Homotopy Theory for Graphs

Group Structure

- Identity: $e(\vec{i}) = v_0 \quad \forall \vec{i} \in \mathbb{Z}^n$
- Inverses: $f^{-1}(\vec{i}) = f(-i_1, \dots, i_n) \quad \forall \vec{i} \in \mathbb{Z}^n$

Example ($n = 2$)

f :

2	K	N	A	H	T
1	U	O	Y		
0	N	E	M	O	W
-1	I	N			
-2	!	H	T	A	M
	-2	-1	0	1	2

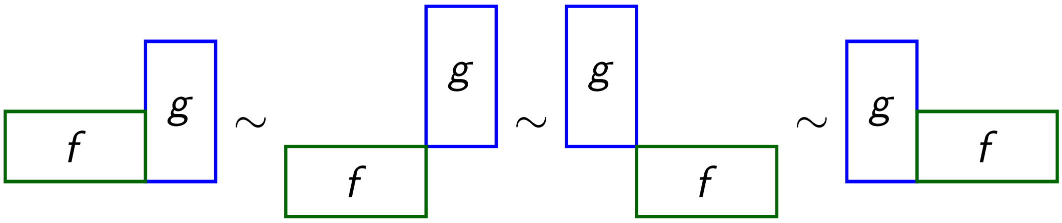
f^{-1} :

2	T	H	A	N	K
1	Y	O	U		
0	W	O	M	E	N
-1	I	N			
-2	M	A	T	H	!
	-2	-1	0	1	2

Discrete Homotopy Theory for Graphs

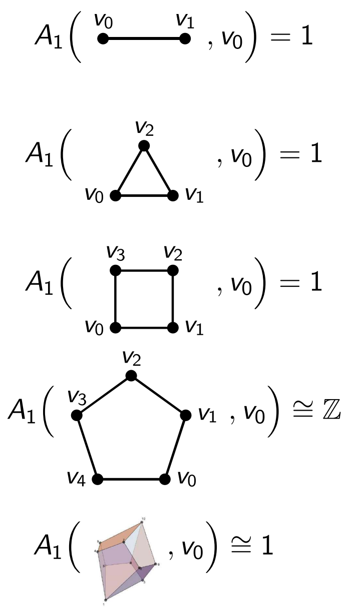
Theorem

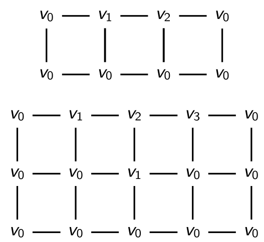
$A_n(\Gamma, v_0)$ is an abelian group $\forall\, n \geq 2$



Discrete Homotopy Theory for Graphs

Examples





$$A_1(\Gamma, v_0) \cong \pi_1(\Gamma, v_0)/N(3, 4 \text{ cycles}) \cong \pi_1(X_\Gamma, v_0)$$

(X_Γ a 2-dim cell complex: attach 2-cells to $\triangle$ and $\square$ of Γ)

Discrete Homotopy Theory: from simplices to graphs

- $A_n^q(\Delta, \sigma_0) \cong A_n(\Gamma_\Delta^q, \sigma_0)$

  q connected chains of simplices, $\sigma_0 - \sigma_1 - \sigma_2 - \cdots - \sigma_m$
  where $\dim(\sigma_i \cap \sigma_{i+1}) \geq q$

  Γ_Δ^q vertices = all maximal simplices of Δ of dim$\geq q$

  $(\sigma, \sigma') \in E(\Gamma_\Delta^q) \iff \dim(\sigma \cap \sigma') \geq q$

Is it a Good Analogy to Classical Homotopy?

1. If Γ is connected, $A_n(\Gamma, v_0)$independent of v_0
2. Siefert-van Kampen: if

   $\Gamma = \Gamma_1 \cup \Gamma_2$; Γ_i connected; $v_0 \in \Gamma_1 \cap \Gamma_2$; $\Gamma_1 \cap \Gamma_2$ connected
   $\triangle$, $\square$ lie in one of the Γ_i

   then

   $$A_1(\Gamma, v_0) \cong A_1(\Gamma_1, v_0) * A_1(\Gamma_2, v_0)/N([\ell] * [\ell]^{-1})$$

   for ℓ a loop in $\Gamma_1 \cap \Gamma_2$
3. Relative discrete homotopy theory and long exact sequences
4. Associated discrete homology theory.

Discrete Homology Theory for Graphs

(B., Capraro, White)

1. Discrete n-dim cube $Q_n = \{(a_1, \ldots, a_n) \mid a_i = 0 \text{ or } 1\}$
2. Singular n-cube $\sigma\colon Q_n \to \Gamma$ graph homomorphism
3. $\mathcal{L}_n(\Gamma) :=$ free abelian group generated by all singular n-cubes σ
   - i^{th} front and back faces of σ are singular $(n-1)$-cubes
   - Degenerate singular n-cube: if $\exists\, i$ s.t. i-front=i-back
   - $D_n(\Gamma) :=$ free abelian group generated by all degenerate singular n-cubes
4. $C_n(\Gamma) := \mathcal{L}_n(\Gamma)/D_n(\Gamma)$; n-chains
5. Boundary operators ∂_n for each $n \geq 1$

$$\partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i \left(A_i^n(\sigma) - B_i^n(\sigma)\right)$$

6. The *discrete homology groups* of Γ:

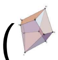
$$DH_n(\Gamma) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$$

Discrete Homology Theory for Graphs

Examples

$DH_n(-) = 0 \quad \forall\, n \geq 1$ $\qquad$ $DH_n(\triangle) = 0 \quad \forall\, n \geq 1$

$DH_n(\square) = 0 \quad \forall\, n \geq 1$ $\qquad$ $DH_1(\pentagon) = \mathbb{Z} \quad \forall\, n \geq 2, \text{ is trivial}$

$DH_1($$) = 0$ $\qquad$ $DH_2($$) = \mathbb{Z}$

$DH_3($$) = 0$

Definition

If $\Gamma' \subseteq \Gamma$, then $\partial_n(C_n(\Gamma')) \subseteq C_{n-1}(\Gamma')$ and there are maps

$$\partial_n\colon C_n(\Gamma, \Gamma') = C_n(\Gamma)/C_n(\Gamma') \to C_{n-1}(\Gamma, \Gamma')$$

The *relative homology* groups of (Γ, Γ'):

$$DH_n(\Gamma, \Gamma') = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$$

How to Judge if Homology Theory is Good?

1. Hurewicz Theorem: $DH_1(\Gamma) \cong A_1^{ab}(\Gamma)$
2. Discrete version of Mayer-Vietoris sequence
3. Eilenberg-Steenrod axioms:
   - Homotopy: If
   $$f, g\colon (\Gamma, \Gamma_1) \to (\Gamma', \Gamma_1')$$
   are discrete homotopic maps then their induced maps on homology are the same
   - Excision:
   $$DH_*(\Gamma_2, \Gamma_1 \cap \Gamma_2) \cong DH_*(\Gamma, \Gamma_1)$$
   if $\Gamma = \Gamma_1 \cup \Gamma_2$ is a discrete cover
   - Dimension:
   $$DH_n(\bullet, \emptyset) = \{0\} \quad \forall\, n \geq 1$$
   - Long exact sequence:
   $$\cdots \to DH_n(\Gamma') \hookrightarrow DH_n(\Gamma) \hookrightarrow DH_n(\Gamma, \Gamma') \xrightarrow{\partial_*} DH_{n-1}(\Gamma') \cdots$$

How to Judge if Homology Theory is Good?

C. Which groups can we obtain?
   - For a fine enough rectangulation R of a compact, metrizable, smooth, path-connected manifold M, let Γ_R be the natural graph associated to R. Then
   $$\pi_1(M) \cong A_1(\Gamma_R)$$
   $$\Downarrow \text{ (+ suspension)}$$
   - For each finitely generated abelian group G and $\bar{n} \in \mathbb{N}$, there is a finite connected simple graph Γ such that
   $$DH_n(\Gamma) = \begin{cases} G & \text{if } n = \bar{n} \\ 0 & \text{if } n \leq \bar{n} \end{cases}$$
   - There is a graph S^n such that
   $$DH_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Applications ($n = 1$)

- Maurer (1971): Characterize matroid basis graphs: (connected), interval and positioning conditions and
  $A_1(\Gamma) \overset{?}{\cong} 1 \iff \Gamma$ is a matroid basis graph
  No (M. 1973), unless Γ contains at least one vertex with finitely many neighbours (2015 Chapolin et al.)
- Lovász (1977): Homology theory for spanning trees of a graph – arborescence complex
- Malle (1983): Net homotopy of graphs; String groups are $A_1(\Gamma)$ and $A_1(\Gamma) \cong 1 \iff$ each cycle has a pseudoplanar net.
- Laubenbacher et al. (2004): Time Series Analysis of data from agent-base computer simulations. Trivial A_1 correlates with high fitness of agents.

Applications ($n = 1$)

- B. Seavers, White (2011):

$$A_1^{n-k+1}(\mathbb{R}\text{-Coxeter comp W}) \cong \pi_1\big(M(k\text{-parabolic arr. W})\big)$$

  generalizing Brieskorn's results for $\mathbb{C}$-hyperbolic arrangements.

- A. Khukhro, T. Delabie (2020)

$$A_1^r(Cay(G/N, \bar{S}), e) \cong N.$$

  Uses r-Lipschitz maps, Cayley graph of a finitely generated group $G = < S >$, N a normal subgroup of G. The discrete fundamental group of a Cayley graph detects the normal subgroup used to build it.

Unexpected Application of Discrete Homotopy Theory

Complex $K(\pi,1)$ Spaces

$\mathcal{A}^{\mathbb{C}}_{n,2}$ braid arrangement:
$\{\vec{z} \in \mathbb{C}^n \mid z_i = z_j\}, \ i < j$

$M(\mathcal{A}^{\mathbb{C}}_{n,2})$ is $K(\pi,1)$
(Fadell-Neuwirth 1962)

$\pi_1(M(\mathcal{A}^{\mathbb{C}}_{n,2})) \cong$ pure braid gp.
(Fox-Fadell 1963)

$M(\mathcal{A}^{\mathbb{C}}_{n,2}(W))$ is $K(\pi,1)$
(Deligne 1972)

Real $K(\pi,1)$ Spaces

$\mathcal{A}^{\mathbb{R}}_{n,3}$ 3-equal subspace arr:
$\{\vec{x} \in \mathbb{R}^n \mid x_i = x_j = x_k\}, \ i < j < k$

$M(\mathcal{A}^{\mathbb{R}}_{n,3})$ is $K(\pi,1)$
(Khovanov 1996)

$\pi_1(M(\mathcal{A}^{\mathbb{R}}_{n,3})) \cong$ pure triplet gp.
(Khovanov 1996)

$M(\mathcal{A}^{\mathbb{R}}_{n,3}(W))$ are $K(\pi,1)$
Davis-Januszkiewicz-Scott 2008)

Unexpected Application of Discrete Homotopy Theory

Complex $K(\pi,1)$ Spaces

$\mathcal{A}^{\mathbb{C}}_{n,2}$ braid arrangement:
$\{\vec{z} \in \mathbb{C}^n \mid z_i = z_j\}, \ i < j$

$\pi_1(M(\mathcal{A}^{\mathbb{C}}_{n,2}(W))$
$\cong$ pure Artin group
$\cong \mathrm{Ker}(\phi)$
(Brieskorn 1971)

Real $K(\pi,1)$ Spaces

$\mathcal{A}^{\mathbb{R}}_{n,3}$ 3-equal subspace arr:
$\{\vec{x} \in \mathbb{R}^n \mid x_i = x_j = x_k\}, \ i < j < k$

$\pi_1(M(\mathcal{A}^{\mathbb{R}}_{n,3}(W)) \cong \mathrm{Ker}(\phi')$
where $\mathcal{A}^{\mathbb{R}}_{n,3}(W)$ is a 3-parabolic subsp. arr. of type W
(B-Severs-White 2009)

Theorem

$$A_1^{n-k+1}(\textit{Coxeter complex } W) \cong \pi_1(M(\mathcal{A}^{\mathbb{R}}_{n,k}(W))) \quad 3 \leq k \leq n$$

Note: $A_1^{n-k+1} \cong \pi_1 \cong 1$ for $k > 3$

Essence of the Proof

1. Presentation of a Coxeter group (W, S) subject to
   (i) $s^2 = 1$ for $s \in S$
   (ii) $(st)^2 = 1$ for s, t such that $m(s,t) = 2$
   (iii) $(st)^3 = 1$ for s, t such that $m(s,t) = 3$
   $\vdots$
2. Artin group: "$W -$ (i)" i.e.

$$(st)^2 = 1, \qquad (st)^3 = 1, \qquad \cdots$$

   ($W = S_n$ represent the braid group)
3. Pure Artin gp: Ker(ϕ), where ϕ: "$W -$ (i)" $\to W$ by $\phi(s_i) = s_i$

$$\pi_1(M(\mathcal{A}^{\mathbb{C}}_{n,2})) \cong \mathrm{Ker}(\phi)$$

Essence of the Proof

4. 3-parabolic arrangement of type W, subspaces invariant under the action of irreducible parabolic subgroups of rank 2 (closed under conjugation).
5. Real-Artin group "$W' = (W - \{$(iii),(iv),$\ldots\}, S)$," i.e.: keep only:
   (i) $s^2 = 1$ for $s \in S$
   (ii) $(st)^2 = 1$ for s, t such that $m(s,t) = 2$ ($W = S_n$ represent the triplet group (Khovanov))
6. ϕ': $W' \to W$ with $\phi'(s) = s, \forall s \in S$

$$\pi_1(M(\mathcal{A}^{\mathbb{R}}_{n,3}(W))) \cong \mathrm{Ker}(\phi') \cong A_1^{n-3+1}(\text{Coxeter complex } W)$$

Essence of Proof

- The *W-permutahedron* is the Minkowski sum of unit line segments $\perp$ to hyperplanes of W
- Its 2-skeleton has:

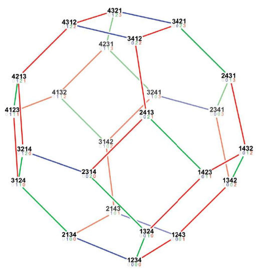
  vertices $w \in W$
  edges (w, ws), where s is a simple reflection
  2-faces are bounded by cycles $(w, ws, wst, \dots, w(st)^{m(s,t)})$

4-cycles	$(st)^2 = 1$	(s and t commute)
6-cycles	$(st)^3 = 1$	
8-cycles	$(st)^4 = 1$	

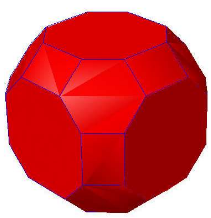
- The complement of the 3-parabolic subspace arrangement of type W is homotopy equivalent to the space obtained from the (dual) W-permutahedron by removing the faces bounded by 6-cycles, 8-cycles,. . .

Unexpected Application of Discrete Homotopy Theory

- (Dual) Coxeter complex for S_n is the permutahedron



- (Dual) Coxeter complex for B_n



Conclusion

We have replaced a group (π_1) defined in terms of the topology of a space with a group (A_1) defined in terms of the combinatorial structure of the space.

"The further a mathematical theory is developed, the more harmoniously and uniformly does its construction proceed, and unsuspected relations are disclosed between hitherto separated branches of the science." — David Hilbert

THANK YOU!