

# Virasoro Action on Schur Q-function

Kazuya Aokage, Eriko Shinkawa, Hiro-Fumi Yamada

## Abstract

Schur Q-function was introduced by Schur as a symmetric polynomial describing the irreducible index of the projective representation of a symmetric group. A formula for Schur Q-functions is presented which describes the action of the Virasoro operators. For strict partition, we prove a formula for each  $L_k Q_\lambda$  and  $L_{-k} Q_\lambda$  ( $k \geq 1$ ), where  $L_k$  is the Virasoro operator. The present paper is a résumé of [1] and [2].

## 1 Schur Q-function

A partition is an integer sequence  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ ,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ , whose size is  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$ . The number of nonzero parts is the length of  $\lambda$ , denoted by  $\ell(\lambda)$ . Let  $\mathcal{SP}(n)$  be the set of partitions of  $n$  into distinct parts. We call a  $\lambda \in \mathcal{SP}(n)$  strict partition of  $n$ . Let  $V = \mathbb{C}[t_j; j \geq 1, \text{ odd}]$ . This is decomposed as  $V = \bigoplus_{n=0}^{\infty} V(n)$ , where  $V(n)$  is the space of homogeneous polynomials of degree  $n$ , according to the counting  $\deg t_j = j$ . An inner product of  $V$  is defined by  $\langle F, G \rangle = F(2\tilde{\partial})\bar{G}(t)\Big|_{t=0}$ , where  $2\tilde{\partial} = (2\partial_1, \frac{2}{3}\partial_3, \frac{2}{5}\partial_5, \dots)$  with  $\partial_j = \frac{\partial}{\partial t_j}$ .

Schur Q-functions are defined in our context as follows. Put  $\xi(t, u) = \sum_{j \geq 1, \text{ odd}} t_j u^j$  and define  $q_n(t) \in V(n)$  by

$$e^{\xi(t, u)} = \sum_{n=0}^{\infty} q_n(t) u^n.$$

For integers  $a, b$  with  $a > b > 0$ , define

$$Q_{ab}(t) := q_a(t)q_b(t) + 2 \sum_{i=1}^b (-1)^i q_{a+i}(t)q_{b-i}(t), \quad Q_{ba}(t) := -Q_{ab}(t).$$

Finally, the Q-function labelled by the strict partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2m})$  ( $\lambda_1 > \lambda_2 > \dots > \lambda_{2m} \geq 0$ ) is defined by

$$Q_\lambda(t) = Q_{\lambda_1 \lambda_2 \dots \lambda_{2m}}(t) = \text{Pf} (Q_{\lambda_i \lambda_j})_{1 \leq i, j \leq 2m}.$$

The Q-function  $Q_\lambda(t)$  is homogeneous of degree  $|\lambda|$ . It is known that  $\{Q_\lambda(t); |\lambda| = n\}$  forms an orthogonal basis for  $V(n)$ , with respect to the above inner product.

## 2 Action of the Virasoro algebra

For a positive odd integer  $j$ , put  $a_j = \sqrt{2}\partial_j$  and  $a_{-j} = \frac{j}{\sqrt{2}}t_j$  so that they satisfy the Heisenberg relation as operators on  $V$ :

$$[a_j, a_i] = j\delta_{j+i, 0}.$$

For an integer  $k$ , put

$$L_k = \frac{1}{2} \sum_{j \in \mathbb{Z}_{\text{odd}}} : a_{-j} a_{j+2k} : + \frac{1}{8} \delta_{k, 0},$$

where

$$: a_j a_i := \begin{cases} a_j a_i & \text{if } j \leq i, \\ a_i a_j & \text{if } j > i \end{cases}$$

is the normal ordering.

**Theorem 1.** Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2m})$  be a strict partition. Then for any  $k \geq 1$

$$L_k Q_\lambda = \sum_{i=1}^{2m} (\lambda_i - k) Q_{\lambda - 2k\epsilon_i},$$

where  $\lambda - 2k\epsilon_i = (\lambda_1, \dots, \lambda_i - 2k, \dots, \lambda_{2m})$ .

**Theorem 2.** Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  be a positive integer sequence. Then

$$L_{-k} Q_\alpha = \sum_{i=1}^{\ell} (\alpha_i + k) Q_{\alpha + 2k\epsilon_i} + \frac{1}{2} \sum_{i=0}^{k-1} (-1)^i (k - i) Q_{\alpha, 2k-i, i}, \quad k \geq 1.$$

Theorem 1 and Theorem 2 completely describe the reduced Fock representation of the Virasoro algebra. Consider the Lie subalgebra  $\mathfrak{g} = \sum_{|k| \leq 1} \mathbb{C}l_k$  which is isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$ . Let  $\mathcal{ESP}$  be the set of the strict partitions whose parts are all even numbers, and let  $V^{\text{even}}$  be the subspace of  $V$  spanned by the  $Q_\lambda$  for  $\lambda \in \mathcal{ESP}$ .

**Corollary 1.** The space  $V^{\text{even}}$  is invariant under the action of  $\mathfrak{g}$ .

## References

- [1] Kazuya Aokage, Eriko Shinkawa, and Hiro Fumi Yamada. Pfaffian identities and virasoro operators. *Letters in Mathematical Physics*, Vol. 110, No. 6, pp. 1381–1389, 2020.
- [2] Kazuya Aokage, Eriko Shinkawa, and Hiro Fumi Yamada. Virasoro action on the q-functions. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, Vol. 17, 2021.

Department of Mathematics  
National Institute of Technology  
Ariake College  
Fukuoka 836-8585  
JAPAN  
E-mail: aokage@ariake-nct.ac.jp

Advanced Institute for Materials Research  
Tohoku University  
Sendai 980-8577  
JAPAN  
E-mail: eriko.shinkawa.e8@tohoku.ac.jp

Department of Mathematics  
Okayama University  
Okayama 700-8530  
JAPAN  
E-mail: yamada@math.okayama-u.ac.jp