INVARIANT HILBERT SCHEME OF THE COX REALIZATION OF THE NILPOTENT CONE IN $\mathfrak{sl}(n)$

AYAKO KUBOTA

The *nilpotent cone* in $\mathfrak{sl}(n)$ is the closure of the regular nilpotent orbit O_n. It has symplectic singularities and admits a unique symplectic resolution, namely the Springer resolution:

$$SL(n) \times_B \mathfrak{n} \to \overline{O_n},\ (X, A) \mapsto XAX^{-1},$$

where $\mathfrak{n} = \{A = (a_{i,j}) \in \mathfrak{sl}(n) : a_{i,j} = 0\ (i \geq j)\}$, and $SL(n) \times_B \mathfrak{n}$ denotes the quotient $(SL(n) \times \mathfrak{n})/B$ under the action of the Borel subgroup B of upper triangular matrices in $SL(n)$ defined by $b \cdot (X, A) = (Xb^{-1}, bAb^{-1})$.

In this proceedings, we show that the Springer resolution of the nilpotent cone $\overline{O_n} \subset \mathfrak{sl}(n)$ can be described in terms of the invariant Hilbert scheme (Theorem 1).

When $n = 2$, Theorem 1 is classically well-known: in this case, the nilpotent cone $\overline{O_2}$ is isomorphic to the quotient of $\mathbb{C}^2$ by a natural action of the cyclic group $\mu_2 \subset SL(2)$ of order 2, and the invariant Hilbert scheme $\mathrm{Hilb}_h^{\mu_2}(\mathbb{C}^2)$ associated to the quotient morphism $\pi : \mathbb{C}^2 \to \mathbb{C}^2/\mu_2$, where h is the Hilbert function of a general fiber of π, gives the minimal (hence crepant) resolution of $\overline{O_2}$ via the Hilbert–Chow morphism $\gamma : \mathrm{Hilb}_h^{\mu_2}(\mathbb{C}^2) \to \mathbb{C}^2/\mu_2 \cong \overline{O_2}$. Since the notion of crepant resolutions coincides with symplectic resolutions for symplectic varieties, we see that γ coincides with the Springer resolution. An important observation in the case where $n = 2$ is that the quotient morphism π is the Cox realization of $\overline{O_2}$. Here we review the definition of the Cox realization (see e.g. [AG10]). Let W be a normal variety with finitely generated divisor class group $\mathrm{Cl}(W)$ and only constant invertible regular functions, and denote by $\mathrm{Cox}(W)$ the Cox ring of W. Put $S = \mathrm{Spec}(\mathrm{Cox}(W))$ and $L = \mathrm{Spec}(\mathbb{C}[\mathrm{Cl}(W)])$. If W is affine, then the categorical quotient $S/\!/L = \mathrm{Spec}(\mathbb{C}[S]^L)$ is isomorphic to W, and the quotient morphism $S \to S/\!/L \cong W$ is called the *Cox realization* of W.

In the following, we consider the case where n is arbitrary. By a theory of Cox rings, we can show that there exists a subgroup $F_1 \subset SL(n)$ such that the Cox ring $\mathrm{Cox}(\overline{O_n})$ of $\overline{O_n}$ is isomorphic to the invariant ring $\mathbb{C}[SL(n)]^{F_1}$, and that the quasi-torus $\mathrm{Spec}(\mathbb{C}[\mathrm{Cl}(\overline{O_n})])$ is isomorphic to the cyclic group μ_n of order n. We consider the invariant Hilbert scheme $\mathcal{H}_n := \mathrm{Hilb}_h^{\mu_n}(\mathrm{Spec}(\mathrm{Cox}(\overline{O_n})))$ associated to the Cox realization $\pi : \mathrm{Spec}(\mathrm{Cox}(\overline{O_n})) \to \mathrm{Spec}(\mathrm{Cox}(\overline{O_n}))/\mu_n \cong \overline{O_n}$ of $\overline{O_n}$. Here, h is the Hilbert function of a general fiber of π, which turns out to be the Hilbert function of the regular representation of μ_n, namely the constant function $h = 1$. The Hilbert–Chow morphism $\gamma : \mathcal{H}_n \to \overline{O_n}$ is an isomorphism over the regular nilpotent orbit O_n, and the Zariski closure $\mathcal{H}^{main} := \overline{\gamma^{-1}(O_n)}$ is an irreducible component of $\mathcal{H}_n$ called the *main component*. The restriction $\gamma|_{\mathcal{H}^{main}}$ of γ to the main component is projective and birational.

Theorem 1 ([Kub])**.** *With the notation above, the main component $\mathcal{H}^{main}$ of $\mathcal{H}_n$ is non-singular, and the restriction $\gamma|_{\mathcal{H}^{main}} : \mathcal{H}^{main} \to \overline{O_n}$ coincides with the Springer resolution.*

Sketch of proof. Let V denote the standard representation of $SL(n)$. We first construct an $SL(n)$-equivariant morphism (an evaluation map) $\varphi_d : \mathcal{H}_n \to \mathrm{Gr}(1, \bigwedge^d V^*) \cong \mathbb{P}(\bigwedge^d V^*)$ for each $1 \leq d \leq n-1$. Next we describe explicit generators of the defining ideals of points in $\gamma^{-1}(O_n)$ to show that the image of the main component $\mathcal{H}^{main}$ under $\varphi := \prod_{1\leq d\leq n-1} \varphi_d$ is isomorphic to the full flag variety $SL(n)/B$. The most technical part of the proof is to construct an equivariant bijective morphism $\psi : SL(n) \times_B \mathfrak{n} \to \mathcal{H}^{main}$ such that the following $SL(n)$-equivariant diagram commutes:

$$\begin{array}{ccccc}
\mathcal{H}^{main} & \xrightarrow{\gamma\times\varphi} & \overline{O_n}\times(SL(n)/B) & \hookrightarrow & \overline{O_n}\times\prod_{1\leq d\leq n-1}\mathbb{P}(\bigwedge^d V^*)\,. \\
\psi \uparrow & \nearrow Q & \downarrow pr_1 & & \\
SL(n)\times_B \mathfrak{n} & \xrightarrow[\eta]{} & \overline{O_n} & &
\end{array}$$

Here, η is the Springer resolution, and Q is defined by $Q((g, A)) = (\eta(g, A), \overline{g})$. Since Q is a closed immersion ([Slo80]), we deduce from the Zariski's main theorem that $\mathcal{H}^{main}$ is isomorphic to $SL(n) \times_B \mathfrak{n}$. Moreover, by the commutativity of the diagram, $\gamma|_{\mathcal{H}^{main}}$ coincides with η. Q.E.D.

Remark 2. In the conventional construction of the invariant Hilbert scheme, the problem of resolution of singularities via the Hilbert–Chow morphism is considerable only when a singularity is given as a quotient variety. The advantage of considering the Cox realization of a normal variety is that this construction can be applied as long as the divisor class group is finitely generated and the only regular functions are constants. Therefore, it is worth asking the following

Question 3 ([Kub])**.** *Let W, L, and S be as above. Let $\pi : S \to S/\!/L \cong W$ be the Cox realization of W, and let h be the Hilbert function of a general fiber of π. Consider the associated Hilbert–Chow morphism $\gamma : \mathrm{Hilb}_h^L(S) \to S/\!/L$. Then, does the restriction $\gamma|_{\mathcal{H}^{main}}$ of γ to the main component $\mathcal{H}^{main}$ of $\mathrm{Hilb}_h^L(S)$ give a resolution of singularities of W?*

Remark 4. By Theorem 1, the answer to Question 3 is positive if $W = \overline{O_n}$.

Since the setting of Question 3 works for a larger class of singularities, we could expect that it opens up possibilities for the study of the invariant Hilbert schemes.

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Department of Mathematics, School of fundamental Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjuku, Tokyo 169-8555, Japan

E-mail address: kubota.a@aoni.waseda.jp