Definable proper quotients

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Abstract

Consider a definably complete locally o-minimal expansion $\mathcal{F} = (F, +.\cdot, <, 0, 1, \dots)$ of an ordered field. We prove the existence of definable quotients of definable sets by definable equivalence relations when curtain conditions are satisfied. These conditions are satisfied when X is a locally closed definable subset of F^n and there is a definable proper action of a definable group G on X. We give its application.

1 Introduction

We study definable proper quotients of definable sets by definable equivalence relations for a definably complete locally o-minimal expansion of an ordered field in this paper [3].

An expansion $\mathcal{M} = (M, <, \dots)$ of dense linear order $<$ without endpoints is *locally* o-*minimal* if for any point $x \in M$ and for any definable subset A of M, there exists an open interval $I \ni x$ such that $A \cap I$ is a finite union of open intervals and points [6]. We say that $\mathcal{M}$ is *definably complete* if any nonempty subset A of M has $\sup A, \inf A \in M \cup \{-\infty, \infty\}$ [4].

This paper considers a definable proper quotients and its application.

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2 History of definable quotients

Definition 2.1. *A semialgebraic equivalence relation E on a semialgebraic set X is a semialgerbraic subset of $X \times X$ such that the binary relation $\sim_E$ defined by $x \sim_E y \Leftrightarrow (x, y) \in E$ is an equivalence relation defined on X.*

We say that E is *proper* over X if the projection $p : E \to X$ onto the first factor is proper. That is the inverse image of every bounded closed subset C in the whole space F^n with $C \subset X$ is bounded and closed in F^{2n}.

In 1987, Brumfiel [1] considers semialgebraic proper quotients. Let X be a semialgebaric set

Theorem 2.2 (1.4 [1])**.** *Suppose that $E \subset X \times X$ is a closed semialgebraic equivalence relation such that the projection $p : E \to X$ onto the first factor is proper. Then there exists a topological map $f : X \to Y$ with $E = E_f$. Moreover any such f is proper, and conversely, any proper $f : X \to Y$ induces an equivalence reration $E_f \subset X \times X$ proper over X.*

Consider an o-minimal expansion $\mathcal{F} = (F, +, \cdot, <, \dots)$ of a real closed field F.

Theorem 2.3 (van den Dries [2])**.** *Let E be an equivalence relation on a definable set X which is definably proper over X. Then X/E exists as a definable proper quotient of X.*

Scheiderer [5] proves the following theorem.

Theorem 2.4 ([5])**.** *Let M be a locally complete semialgebraic space and $E \subset M \times M$ a closed semialgebraic equivalence relation. Then the following are equivalent.*

(1) The geometric quoitents M/E exists.

(2) There exists a semialgebriac subspace $K \subset M$ such that $p_1|p_2^{-1}(K) : p_2^{-1}(K) \to X$ is proper and surjective.

3 Our results

Let $\mathcal{F} = (F, +, \cdot, <, \dots)$ be a definably complete locally o-minimal expansion of an ordered field F. Suppose that $X \subset F^m, Y \subset F^n$ are definable sets and $f : X \to Y$ is a definable continuous map.

Definition 3.1. *The map f is definably proper if for any closed bounded definable subset K in F^m with $K \subset Y$, the inverse image $f^{-1}(K)$ is closed and bounded in F^m. It is called definably indentifying if it is surjective and for any definable subset Y , K is closed in Y whenever $f^{-1}(K)$ is closed in X.*

Let $\mathcal{F} = (F, +, \cdot, <, \dots)$ be a definably complete locally o-minimal expansion of an ordered field F, X a definable set and E a definable equivalence reration. A *definably quotient* of X by E is a definable identifying map $f : X \to Y$ such that $f(x) = f(x')$ if and only if $(x, x') \in E$. A *definably proper quotient* of X be E is a definable quotient which is definably proper. The target space Y is denoted by X/E.

Let $\mathcal{F} = (F, +, \cdot, <, \dots)$ be a definably complete locally o-minimal expansion of an ordered field F, X a definable set and E a definable equivalence relation. Suppose that $p_i : E \to X$ is the restriction of the projection $X \times X \to X$ onto the i-th factor X for $i = 1, 2$. The equivalence reration E is *definably proper over* X if p_1 is a definably proper map.

Let $G \subset F^m$ be a definable set. We say that G is a *definable group* if it is a group and the group operations $G \times G \to G$ and $G \to G$ are definable continuous maps. A *definable G set* is a pair (X, ϕ) consisting of a definable set X and a group action $\phi : G \times X \to X$ is a definable continuous map. We simply write X instead of (X, ϕ). A G invariant definable subset A of a definable G set is called *definable G subset.* The G action on X is *definably proper* if the map $G \times X \to X \times X, (g, x) \mapsto (gx, x)$ is a definable proper map.

Theorem 3.2 ([3])**.** *Consider a definably complete locally o-minimal expansion $\mathcal{F} = (F, +, \cdot, <, \dots)$ of an ordered field F. Suppose that X is a nonempty locally closed definable set and E is a definable equivalence relation on X which is proper over X. Then there exists a definable proper quotient $\pi : X \to X/E$.*

Theorem 3.3 ([3])**.** *Consider a definably complete locally o-minimal expansion $\mathcal{F} = (F, +, \cdot, <, \dots)$ of an ordered field F. Suppose that G is a definable group and X is a definable G set which is locally closed. Assume that the action on X is definable proper. Then there exists a definable quotinet $X \to X/G$.*

Theorem 3.4 ([3])**.** *Consider a definably complete locally o-minimal expansion $\mathcal{F} = (F, +, \cdot, <, \dots)$ of an ordered field F. Suppose that G is a definable group, X is a definable G set which is locally closed and A is a closed definable G subset of X. Assume that the action on X is definable proper. Then there exists a G invariant definable continuous function $f : X \to F$ with $A = f^{-1}(0)$.*

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