Introduction to almost o-minimality

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概要

This paper is a brief survey on an almost o-minimal structure, which was proposed by the author. A locally o-minimal expansion of the set of reals has strong properties which are not possessed by other locally o-minimal structures. An almost o-minimal structure is a promising generalization of a locally o-minimal expansion of the set of reals. It is always an expansion of an o-minimal structure and admits uniform local definable cell decomposition.

1 Introduction

The purpose of this paper is an introduction to main ingredients of [9] without proofs. See the original paper for more details.

We introduce the notion of almost o-minimality in this paper. The author's motivation is first described. In real algebraic geometry [2], topological properties of semialgebraic sets have been studied. It was recognized that many of these properties are induced from a few number of axioms. An o-minimal structure [3, 15, 18] is a model-theoretic generalization of the geometry of semialgebraic sets which satisfies these axioms. An expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, \dots)$ is o-minimal if any definable subset of M is a finite union of points and open intervals.

Roughly speaking, local o-minimality [20] is a localized notion of o-minimality. The structure $\mathcal{M}$ is called locally o-minimal if, for any $x \in M$ and any definable subset X of M, there exists an open interval I containing the point x such that $X \cap I$ is a finite

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union of points and open intervals. Unfortunately, local o-minimality includes wide class of structures such as weakly o-minimal structures [16]. For instance, local definable cell decomposition theorem is unavailable in general local o-minimal structures though definable cell decomposition theorem holds true in o-minimal structures. Several relatives of local o-minimality has been suggested by adding extra requirements to the definition of local o-minimality [7, 14]. Strong local o-minimality requires the most restrictive condition among them. A locally o-minimal structure is called strongly locally o-minimal if we can choose I independently of the definable set X.

We often consider structures whose universe is the set of reals $\mathbb{R}$. When the universe is the set of reals $\mathbb{R}$, a locally o-minimal structure has good features which are not expected even in strongly locally o-minimal structures. Since any bounded closed interval is compact, $X \cap I$ is a finite union of points and open intervals if I is a bounded open interval and X is a definable subset of $\mathbb{R}$. It means that even strong local o-minimality is not a good generalization of locally o-minimal structures whose universe is $\mathbb{R}$. This is the author's motivation for introduction of almost o-minimality.

The family of subanalytic sets [1, 13] is another example many of whose topological properties are obtained from a small set of axioms like the family of subanalytic sets. The family of sets definable in a model theoretic structure is closed under coordinate projections; that is, the projection image of a definable set is again definable. Unfortunately, the projection image of a subanalytic set is not necessarily subanalytic. However, the image of subanalytic set under the projection satisfying some conditions are subanalytic. So, we cannot generalize the subanalytic category as a model-theoretic structure, but one can consider a similar generalization as o-minimal structures. The analytic-geometric category [4] and Shiota's families of $\mathfrak{X}$-sets [19] are generalizations of the family of subanalytic sets defined on the ordered field of reals.

We introduce the notions of almost o-minimality and $\mathfrak{X}$-definability in this paper. The former is a model-theoretic generalization of a locally o-minimal expansion of the set of reals, and the latter is a non-model-theoretic structure similar to Shiota's $\mathfrak{X}$-sets related to an almost o-minimal structure.

2 Main results

This section introduces the main ingredients of [9] without proofs.

2.1 Definitions

We first recall the definitions.

Definition 2.1. An expansion $\mathcal{M} = (M, <, \dots)$ of densely linearly ordered set without endpoints is *almost o-minimal* if any bounded definable set in M is a finite union of points and open intervals.

It is not difficult to prove that an almost o-minimal structure is a definably complete locally o-minimal structure. Therefore, the assertions on definably complete locally o-minimal structures given in [8, 12] are all satisfied in almost o-minimal structures.

The analytic-geometric category and Shiota's $\mathfrak{X}$-sets are defined under the assumption that the underlying set is the set of reals. They require a kind of locally finiteness property of definable sets in $\mathbb{R}$. The bounded closed interval is always compact in this case. We want to treat the case in which the underlying space is not the set of reals. We need to consider another property other than local finiteness property. So, we give another definition of 'proper' maps and generalize Shiota's $\mathfrak{X}$-sets as follows:

Definition 2.2. Let $(M, <)$ be a densely linearly ordered set without endpoints. A map p from a subset X of M^m to M^n is *proper* if the inverse image $p^{-1}(U)$ of an arbitrary bounded closed box U in M^n is bounded.

An $\mathfrak{X}$-*structure* is a triple $\mathcal{X} = (M, <, \mathcal{S} = \{\mathcal{S}_n\}_{n \in \mathbb{N}})$ satisfying the following conditions (1) through (7), where $(M, <)$ is a densely linearly ordered set without endpoints and, for all $n \in \mathbb{N}$, $\mathcal{S}_n$ is a family of subsets in M^n. In this paper, $\mathbb{N}$ denotes the set of positive integers.

(1) For all $x \in M$, the singletons $\{x\}$ belong to S_1. All open intervals also belong to S_1.
(2) The sets $\{(x, y) \in M^2 \mid x = y\}$ and $\{(x, y) \in M^2 \mid x < y\}$ belong to S_2.
(3) $\mathcal{S}_n$ is a boolean algebra and $M^n \in \mathcal{S}_n$;
(4) We have $X_1 \times X_2 \in \mathcal{S}_{m+n}$ whenever $X_1 \in \mathcal{S}_m$ and $X_2 \in \mathcal{S}_n$;
(5) For any permutation σ of $\{1, \dots, n\}$, the image $\widetilde{\sigma}(X)$ belongs to $\mathcal{S}_n$ when $X \in \mathcal{S}_n$ and the notation $\widetilde{\sigma} : M^n \to M^n$ denotes the map given by $\widetilde{\sigma}(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$;

(6) Let $\pi : M^n \to M^m$ be a coordinate projection and $X \in \mathcal{S}_n$ such that the restriction $\pi|_X$ of π to X is proper. Then, the image $\pi(X)$ belongs to $\mathcal{S}_m$.
(7) The intersection $I \cap X$ is a finite union of points and open intervals when $X \in \mathcal{S}_1$ and I is a bounded open interval.

The set M is called the *universe* and the *underlying set* of the $\mathfrak{X}$-structure $\mathcal{X}$. A subset X of M^n is called *$\mathfrak{X}$-definable* in $\mathcal{X}$ when X is an element of $\mathcal{S}_n$. A set $\mathfrak{X}$-definable in $\mathcal{X}$ is simply called $\mathfrak{X}$-definable when $\mathcal{X}$ is clear from the context. A map from a subset of M^m to M^n is *$\mathfrak{X}$-definable* if its graph is $\mathfrak{X}$-definable.

When $(M, <, 0, +)$ is an ordered divisible abelian group and the addition is $\mathfrak{X}$-definable, we call the $\mathfrak{X}$-structure an *$\mathfrak{X}$-expansion of an ordered divisible abelian group.* We define an *$\mathfrak{X}$-expansion of an ordered real closed field* in the same manner. An $\mathfrak{X}$-structure $\mathcal{X}_1$ is an *$\mathfrak{X}$-expansion* of an $\mathfrak{X}$-structure $\mathcal{X}_2$ if they have a common universe and a set $\mathfrak{X}$-definable in $\mathcal{X}_2$ is always $\mathfrak{X}$-definable in $\mathcal{X}_1$. We also say that $\mathcal{X}_2$ is an *$\mathfrak{X}$-reduct* of $\mathcal{X}_1$ when the above conditions are satisfied.

The author considers that the theory of the geometry of $\mathfrak{X}$-definable sets can be developed in the same manner as [19]. An almost o-minimal structure is obviously an $\mathfrak{X}$-structure. The following is another important example of $\mathfrak{X}$-structures.

Definition 2.3. Let $\mathcal{R} = (M, <, \dots)$ be an o-minimal structure. A subset X of M^n is *semi-definable in* $\mathcal{R}$ if the intersection $U \cap X$ is definable in $\mathcal{R}$ for any bounded open box U in M^n. A map from a subset of M^m to M^n is *semi-definable* if its graph is semi-definable. For any positive integer n, let $\mathcal{S}(\mathcal{R})_n$ denote the family of semi-definable subsets of M^n, and set $\mathcal{S}(\mathcal{R}) = \{\mathcal{S}(\mathcal{R})_n\}_{n\in\mathbb{N}}$. The family $\mathcal{S}(\mathcal{R})$ satisfies the conditions in Definition 2.2. The proof of this fact is straightforward and we omit it. The $\mathfrak{X}$ structure $\mathfrak{X}(\mathcal{R}) = (M, <, \mathcal{S}(\mathcal{R}))$ is called the *$\mathfrak{X}$-structure of semi-definable sets in* $\mathcal{R}$.

2.2 On $\mathfrak{X}$-structure

We need some preparation to introduce our main results on almost o-minimal structures. We first introduce the assertion which holds true for any $\mathfrak{X}$-expansion of an ordered divisible abelian group.

The following theorem says that any $\mathfrak{X}$-expansion of an ordered divisible abelian group $(M, 0, +, <)$ has an o-minimal expansion $\mathcal{R}$ of an ordered group such that any set definable in $\mathcal{R}$ is $\mathfrak{X}$-definable and any $\mathfrak{X}$-definable set is semi-definable in $\mathcal{R}$.

Theorem 2.4. *Consider an $\mathfrak{X}$-expansion $\mathfrak{X}$ of an ordered divisible abelian group $(M, 0, +, <)$. There exists an o-minimal expansion $\mathcal{R}$ of the ordered group $(M, 0, +, <)$ satisfying the following conditions:*

(i) Any set definable in $\mathcal{R}$ is $\mathfrak{X}$-definable in $\mathcal{X}$.
(ii) Any set $\mathfrak{X}$-definable in $\mathcal{X}$ is $\mathfrak{X}$-definable in $\mathfrak{X}(\mathcal{R})$.

Here, the notation $\mathfrak{X}(\mathcal{R})$ denotes the $\mathfrak{X}$-structure of semi-definable sets in $\mathcal{R}$.

Let us consider an almost o-minimal expansion $\mathcal{M}$ of an ordered group. It is also an $\mathfrak{X}$-structure by the definition. There exists an o-minimal structure satisfying the conditions in Theorem 2.4. We fix such an o-minimal structure and denote it by $\mathcal{R}_{\text{ind}}(\mathcal{M})$.

2.3 Semi-definable connected component

We do not know whether a connected component of a set definable in a locally o-minimal expansion of the set of reals is again definable or not. We cannot consider definably connected components of a definable set differently from o-minimal structures. Instead, the notion of semi-definable (pathwise) connectedness is defined as follows and semi-definably connected components exist.

Definition 2.5. Consider an o-minimal structure $\mathcal{R} = (M, <, \ldots)$. A semi-definable subset X of M^n is *semi-definably connected* if there are no non-empty proper semi-definable closed and open subsets Y_1 and Y_2 of X such that $Y_1 \cap Y_2 = \emptyset$ and $X = Y_1 \cup Y_2$. The semi-definable set X is *semi-definably pathwise connected* if, for any $x, y \in X$, there exist elements $c_1, c_2 \in M$ and a definable continuous map $\gamma : [c_1, c_2] \to X$ with $\gamma(c_1) = x$ and $\gamma(c_2) = y$.

We then define a semi-definably connected component of a semi-definable set.

Theorem 2.6. *Consider an o-minimal expansion $\mathcal{R} = (M, <, +, 0, \ldots)$ of an or-*

dered group. Let X be a nonempty semi-definable subset of M^n. The following are equivalent:

(1) X is semi-definably connected.
(2) For any $x, y \in X$, there exists a bounded open box U in M^n such that both the points x and y are contained in some definably connected component of $X \cap U$.
(3) X is semi-definably pathwise connected.

In addition, for any $x \in X$, there exists a maximal semi-definably connected semi-definable subset Y of X containing the point x. The set Y is called the ***semi-definably connected component*** *of X containing the point x. A semi-definably connected component of X is closed and open in X.*

2.4 Decomposition into multi-cells and uniform local definable cell decomposition

We are now ready to introduce two important results on almost o-minimal structures. Let $\mathcal{M} = (M, <, +, 0, \ldots)$ be an almost o-minimal expansion of an ordered group. Since an almost o-minimal structure is an $\mathfrak{X}$-structure, there exists an o-minimal expansion of an ordered group $\mathcal{R} = \mathcal{R}_{\text{ind}}(\mathcal{M})$ satisfying the conditions (i) and (ii) in Theorem 2.4. In particular, any set X definable in $\mathcal{M}$ is semi-definable in $\mathcal{R}$. So, we can consider semi-definably connected components of X. We define a multi-cell under this condition.

Definition 2.7. Consider an almost o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, \ldots)$. Let n be a positive integer. A definable subset X of M^n is a *multi-cell* if it satisfies the following conditions:

- If $n = 1$, either X is a discrete definable set or all semi-definably connected components of the definable set X are open intervals.
- When $n > 1$, let $\pi : M^n \to M^{n-1}$ be the projection forgetting the last coordinate. The projection image $\pi(X)$ is a multi-cell and, for any semi-definably connected component Y of X, $\pi(Y)$ is a semi-definably connected component

of $\pi(X)$ and Y is one of the following forms:

$$Y = \pi(Y) \times M,$$
$$Y = \{(x,y) \in \pi(Y) \times M \mid y = f(x)\},$$
$$Y = \{(x,y) \in \pi(Y) \times M \mid y > f(x)\},$$
$$Y = \{(x,y) \in \pi(Y) \times M \mid y < g(x)\} \text{ and}$$
$$Y = \{(x,y) \in \pi(Y) \times M \mid f(x) < y < g(x)\}$$

for some semi-definable continuous functions f and g defined on $\pi(Y)$ with $f < g$.

The following theorem is one of the most important achievement in [9].

Theorem 2.8. *In an almost o-minimal expansion of an ordered group, every definable set is partitioned into finitely many multi-cells.*

The proof of this theorem is long. When we are only interested in an almost o-minimal expansion of an ordered set of reals, we do not need the introduction of notions of $\mathfrak{X}$-sets and semi-definably connected components because the notion of topological connected components work well instead of semi-definably connected components. As a corollary of this theorem, we get the following:

Theorem 2.9 (Uniformity theorem)**.** *Consider an almost o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, \ldots)$. For any definable subset X of M^{n+1} and a positive element $R \in M$, there exists a positive integer K such that, for any $a \in M^n$, the definable set $X \cap (\{a\} \times]-R, R[)$ has at most K semi-definably connected components.*

Before we give the last important theorem, let us recall the definitions of cells.

Definition 2.10 (Definable cell decomposition)**.** Consider an expansion of dense linear order without endpoints $\mathcal{M} = (M, <, \ldots)$. Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones of length n. $(i_1, \ldots, i_n)$*-cells* are definable subsets of M^n defined inductively as follows:

- A (0)-cell is a point in M and a (1)-cell is an open interval in M.
- An $(i_1, \ldots, i_n, 0)$-cell is the graph of a definable continuous function defined on an $(i_1, \ldots, i_n)$-cell. An $(i_1, \ldots, i_n, 1)$-cell is a definable set of the form $\{(x, y) \in$

$C \times M \mid f(x) < y < g(x)\}$, where C is an $(i_1, \ldots, i_n)$-cell and f and g are definable continuous functions defined on C with $f < g$.

A *cell* is an $(i_1, \ldots, i_n)$-cell for some sequence $(i_1, \ldots, i_n)$ of zeros and ones. The sequence $(i_1, \ldots, i_n)$ is called the *type* of an $(i_1, \ldots, i_n)$-cell. An *open cell* is a $(1, 1, \ldots, 1)$-cell. The dimension of an $(i_1, \ldots, i_n)$-cell is defined by $\sum_{j=1}^{n} i_j$.

We inductively define a *definable cell decomposition* of an open box $B \subseteq M^n$. For $n = 1$, a definable cell decomposition of B is a partition $B = \bigcup_{i=1}^{m} C_i$ into finite cells. For $n > 1$, a definable cell decomposition of B is a partition $B = \bigcup_{i=1}^{m} C_i$ into finite cells such that $\pi(B) = \bigcup_{i=1}^{m} \pi(C_i)$ is a definable cell decomposition of $\pi(B)$, where $\pi : M^n \to M^{n-1}$ is the projection forgetting the last coordinate. Consider a finite family $\{A_\lambda\}_{\lambda \in \Lambda}$ of definable subsets of B. A *definable cell decomposition of B partitioning* $\{A_\lambda\}_{\lambda \in \Lambda}$ is a definable cell decomposition of B such that the definable sets A_λ are unions of cells for all $\lambda \in \Lambda$.

It is well-known that an o-minimal structure admits definable cell decomposition [3, Chapter 3, Theorem 2.11].

Here is the last important theorem.

Theorem 2.11 (Uniform local definable cell decomposition). *Consider an almost o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, \ldots)$. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of definable subsets of M^{m+n}. Take an arbitrary positive element $R \in M$ and set $B =]-R, R[^n$. Then, there exists a finite partition into definable sets*

$$M^m \times B = X_1 \cup \ldots \cup X_k$$

such that $B = (X_1)_b \cup \ldots \cup (X_k)_b$ is a definable cell decomposition of B for any $b \in M^m$ and either $X_i \cap A_\lambda = \emptyset$ or $X_i \subseteq A_\lambda$ for any $1 \leq i \leq k$ and $\lambda \in \Lambda$. Furthermore, the type of the cell $(X_i)_b$ is independent of the choice of b with $(X_i)_b \neq \emptyset$. Here, the notation S_b denotes the fiber of a definable subset S of M^{m+n} at $b \in M^m$.

It is a uniform version of the local definable cell decomposition theorem given in [7, Theorem 4.2].

Finally, the following table summarizes the decomposition theorems proven in [7, 8, 9, 12, 14]. In the table, we assume that the structures are definably complete.

Structure	local decomp.	global decomp.
loc. o-min. exp. of $\mathbb{R}$	uniform local cell decomp.	decomp. into multi-cells
almost o-min.		
str. loc. o-min.	local cell decomp.	decomp. into quasi-special submanifolds
uni. loc. o-min. 1st		
uni. loc. o-min. 2nd		
loc. o-min.	None	

Only o-minimal structure admits definable cell decomposition, and we cannot decompose a definable set into finitely may cells when the structure is not o-minimal. Two kinds of decomposition theorems are expected to hold true for locally o-minimal structures. We cannot decompose a definable set into finitely many cells globally, it may be possible locally. In other word, for any point and any definable set, there may be a definable neighborhood of the point and a decomposition of the intersection of the given definable set with the neighborhood into finitely many cells. This kind of decomposition is provided in the column of 'local decomp.' of the table. Uniformly locally o-minimal structures of the first/second kind are defined in [7, Definition 2.1]. The local definable cell decomposition theorem was first demonstrated in [7, Theorem 4.2] for uniformly locally o-minimal structures of the first/second kind. It was proven for strongly locally o-minimal structures in [14, Proposition 13] prior to [7].

Another type of decomposition theorems claim that a definable set is partitioned into finitely many good-shaped definable sets which satisfies looser constraints than cells. They are described in the column of 'global decomp.' of the table. Quasi-special submanifolds are defined in [8, Definition 4.1]. Decomposition theorem into quasi-special submanifolds is provided in [8, Theorem 4.5] under a technical assumption on structures. In [12, Theorem 2.5], it was proven that this technical condition is always satisfied. We defined special submanifolds in [11, Definition 3.1] which satisfy more severe conditions than quasi-special submanifolds. The decomposition theorem into special submanifolds was proved in [5, Theorem 5.6] when the structure is a definably complete locally o-minimal expansion of an ordered field. In [5], different terminology and definition are used, but they are same by [11, Proposition 3.13]. The decomposition theorem into special submanifolds was proved in [11, Theorem 3.19] in more general setting; namely, the case in which the structure is a definably complete locally o-minimal expansion of an ordered group. It is easy to demonstrate that a multi-cell is always a special submanifold.

3 A non-interesting application

The following lemma is a generalization of a lemma used for the study of definable topological vector bundle in o-minimal structures [6, Lemma 2.7]. We used definable cell decomposition theorem for o-minimal structures in the proof of the original lemma. The proof of the current lemma indicates that the assertions in Section 2 can be used as substitutes of definable cell decomposition theorem. The author could not find an interesting application of the following lemma, but he found several other interesting properties of almost o-minimal structures whose proofs are too long to be introduced in this paper such as [10].

Lemma 3.1. *Let $\mathcal{M}$ be an almost o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, 1, +, \dots)$ with the distinguished element 1. Let X be a definable subset of M^n and $\{V_j\}_{j=1}^p$ be a finite definable open covering of $X \times [0,1]$. Then, there exist a finite definable open covering $\{U_i\}_{i=1}^q$ of X and finite definable continuous functions $0 = \varphi_{i,0} < \cdots < \varphi_{i,k} < \cdots < \varphi_{i,r_i} = 1$ on U_i such that, for any $1 \leq i \leq q$ and $1 \leq k \leq r_i$, the definable set*

$$\{(x,t) \in U_i \times [0,1] \mid \varphi_{i,k-1}(x) \leq t \leq \varphi_{i,k}(x)\}$$

is contained in V_j for some $1 \leq j \leq p$.

Proof. Let $\pi : M^n \times M \to M^n$ be the projection onto the first n coordinates. In this proof, the notation A_x denotes the fiber $\{t \in M \mid (x,t) \in A\}$ for any subset A of a Cartesian product M^{k+1} and any $x \in M^k$. Since $(V_j)_x$ is open in $[0,1]$ for any $x \in X$ and $\mathcal{M}$ is almost o-minimal, the fiber $(V_j)_x$ is the union of finitely many intervals which are open in $[0,1]$. Put

$$E = \{(x,t) \in X \times [0,1] \mid t \text{ is an endpoint of a maximal interval contained in } (V_j)_x \text{ for some } 1 \leq j \leq p\}.$$

The fiber E_x at $x \in X$ is a finite set for any $x \in X$. Since $0, 1 \in E_x$, the cardinarity $|E_x|$ of the set E_x is at least two. We can find a positive integer K such that $|E_x| \leq K$

for all $x \in X$ by Theorem 2.9. Set $F_l = \{x \in X \mid |E_x| = l\}$ for all $2 \leq l \leq K$. They are definable and we have $X = \bigcup_{l=2}^{K} F_l$.

Fix $1 \leq l \leq K$. For any $1 \leq i \leq l$ and $x \in F_l$, we set $f_{l,i}(x)$ as the i-th smallest element in the fiber E_x. It defines the definable map $f_{l,i} : F_l \to [0,1]$. We set

$$G_{l,\mathfrak{S}} = \{x \in F_l \mid f_{l,i}(x) \in V_j \ (\forall (i,j) \in \mathfrak{S}) \text{ and } f_{l,i}(x) \notin V_j \ (\forall (i,j) \notin \mathfrak{S})\}$$

for all subsets $\mathfrak{S}$ of $\{1,\ldots,l\} \times \{1,\ldots,p\}$. We next fix a subset $\mathfrak{S}$ of $\{1,\ldots,l\} \times \{1,\ldots,p\}$. Using [9, Proposition 4.16(2)], by induction on dimension, we can construct a decomposition $G_{l,\mathfrak{S}} = \bigcup_{k=1}^{m_{l,\mathfrak{S}}} H_{l,\mathfrak{S},k}$ into finitely many definable sets such that they are disjoint each other and $f_{l,i}$ are continuous on $H_{l,\mathfrak{S},k}$ for all $1 \leq i \leq l$ and $1 \leq k \leq m_{l,\mathfrak{S}}$. Finally, we apply Theorem 2.8 to $H_{l,\mathfrak{S},k}$ and decompose it into multi-cells.

Consequently, we obtain

- multi-cells $D_1, \ldots, D_q$,
- positive integers l_i for all $1 \leq i \leq q$ and
- definable continuous functions $0 = \psi_{i,0} < \psi_{i,1} < \ldots < \psi_{i,l_i} = 1$ defined on D_i

satisfying the following:

(a) $D_i \cap D_j = \emptyset$ if $i \neq j$;
(b) $X = \bigcup_{i=1}^{q} D_i$;
(c) $E_x = \{\psi_{i,0}(x), \psi_{i,1}(x), \ldots, \psi_{i,l_i}(x)\}$ for any $x \in D_i$;
(d) the graph of $\psi_{i,k}$ either has an empty intersection with V_j or entirely contained in V_j for any $1 \leq i \leq q$, $0 \leq k \leq l_i$ and $1 \leq j \leq p$.

Set $r_i = 2l_i$ and define definable continuous functions $\Psi_{i,k}$ on D_i by

$$\Psi_{i,k}(x) = \begin{cases} \psi_{i,k/2}(x) & \text{if } k \text{ is even,} \\ \frac{\psi_{i,(k-1)/2}(x) + \psi_{i,(k+1)/2}(x)}{2} & \text{otherwise} \end{cases}$$

for all $0 \leq k \leq r_i$. We show the following claim:

Claim. For any $1 \leq i \leq q$ and $1 \leq k \leq r_i$, there exists a positive integer $j(i,k)$ such that the definable set

$$\{(x,t) \in D_i \times [0,1] \mid \Psi_{i,k-1}(x) \leq t \leq \Psi_{i,k}(x)\}$$

is contained in $V_{j(i,k)}$.

We demonstrate the above claim. One of $k-1$ and k is an even number. We assume that $k-1$ is even. We can show the claim in the same way in the other case. Fix a point $x_0 \in D_i$. Since $\{V_j\}_{j=1}^p$ is an open covering, there exists $1 \leq j(i,k) \leq p$ such that $V_{j(i,k)}$ contains the point $(x_0, \psi_{i,(k-1)/2}(x_0))$. By (d), the set $\{(x,t) \in D_i \times [0,1] \mid t = \Psi_{i,k-1}(x) = \psi_{i,(k-1)/2}(x)\}$ is contained in $V_{j(i,k)}$. The set $\{(x,t) \in D_i \times [0,1] \mid \psi_{i,(k-1)/2}(x) < t < \psi_{i,(k+1)/2}(x)\}$ is contained in $V_{j(i,k)}$ by (c) and the definition of the set E. Hence, the definable set $\{(x,t) \in D_i \times [0,1] \mid \Psi_{i,k-1}(x) \leq t \leq \Psi_{i,k}(x)\}$ is contained in $V_{j(i,k)}$ because $\Psi_{i,k}(x) = \frac{\psi_{i,(k-1)/2}(x)+\psi_{i,(k+1)/2}(x)}{2} < \psi_{i,(k+1)/2}(x)$. We have demonstrated the claim.

Let $\pi_l : M^n \to M^l$ be the projection onto the first l coordinates. Fix $1 \leq i \leq q$. We inductively define definable open subsets $W_{i,l}$ of M^l and definable continuous maps $\eta_{i,l} : W_{i,l} \to \pi_l(D_i)$ as follows: When $l = 1$, the definable set $\pi_1(D_i)$ is either a discrete definable closed set or the union of open intervals. We first consider the former case. Set

$$\begin{aligned} B_{i,1} = \{(x_1+x_2)/2 \in M \mid &(x_1, x_2 \in \pi_1(D_i)) \wedge (x_1 < x_2) \\ &\wedge (\nexists x,\ (x_1 < x < x_2) \wedge (x \in \pi_1(D_i)))\} \text{ and} \\ W_{i,1} = M \setminus B_{i,1}&. \end{aligned}$$

For any $x \in W_{i,1}$, we can uniquely find the point in $\pi_1(D_i)$ nearest to x. We denote this point $\eta_{i,1}(x)$. We have defined the definable map $\eta_{i,1} : W_{i,1} \to \pi_1(D_1)$. It is obviously continuous. In the latter case, we set $W_{i,1} = \pi_1(D_i)$ and $\eta_{i,1}$ is defined as the restriction of the identity map to $\pi_1(D_i)$.

When $l > 1$, any semi-definably connected component Y of the definable set $\pi_l(D_i)$ is one of the following forms:

$$\begin{aligned} &\{(x,t) \in \rho_l(Y) \times M \mid t = f(x)\}, \\ &\{(x,t) \in \rho_l(Y) \times M \mid f_1(x) < t < f_2(x)\} \end{aligned}$$

because D_i are multi-cells. Here, f, f_1 and f_2 are semi-definable continuous functions on $\rho_l(Y)$. The notation ρ_l denotes the coordinate projection $M^l \to M^{l-1}$ forgetting the last coordinate. In the former case, set

$$\begin{aligned} B_{i,l} = \{&(x,(y_1+y_2)/2) \in W_{i,l-1} \times M \mid (y_1, y_2 \in (\pi_l(D_i))_{\eta_{i,l-1}(x)}) \wedge (y_1 < y_2) \\ &\wedge (\nexists y,\ (y_1 < y < y_2) \wedge (y \in (\pi_l(D_i))_{\eta_{i,l-1}(x)}))\} \text{ and} \\ W_{i,l} = &(W_{i,l-1} \times M) \setminus B_{i,l}. \end{aligned}$$

The definable continuous map $\eta_{i,l} : W_{i,l} \to \pi_l(D_i)$ is defined by

$$\eta_{i,l}(x,t) = (\eta_{i,l-1}(x),\ \text{the point in } (\pi_l(D_i))_{\eta_{i,l-1}(x)} \text{ nearest to } t).$$

Set $W_{i,l} = \{(x,t) \in W_{i,l-1} \times M \mid t \in (\pi_l(D_i))_{\eta_{i,l-1}(x)}\}$ and $\eta_{i,l}(x,t) = (\eta_{i,l-1}(x), t)$ in the latter case.

Set $W_i = W_{i,n}$ and $\eta_i = \eta_{i,n}$. It is obvious that D_i is contained in W_i and the restriction of η_i to D_i is the identity map. For any $1 \leq i \leq q$ and $1 \leq k \leq r_i$, we define a definable set $X_{i,k}$ as follows:

$$X_{i,k} = \{x \in W_i \mid (x,t) \in V_{j(i,k)} \text{ for all } t \in M \text{ with } \Psi_{i,k-1}(\eta_i(x)) \leq t \leq \Psi_{i,k}(\eta_i(x))\}.$$

It is obvious that D_i is contained in $X_{i,k}$ by the above claim. We show that $X_{i,k}$ is an open set. Let $x \in X_{i,k}$ be fixed. Consider the closed definable subset

$$Y_{i,k}(x) = \{t \in M \mid \Psi_{i,k-1}(\eta_i(x)) \leq t \leq \Psi_{i,k}(\eta_i(x))\}$$

of M. We also set $Z_{i,k}(x) = \{x\} \times Y_{i,k}(x)$. The definable continuous function ρ on the closed definable set $Y_{i,k}(x)$ is defined as the distance between the point (x,t) and the closed set $V^c_{j(i,k)}$. Since $Y_{i,k}(x)$ is closed and bounded, the function ρ takes the minimum m. The minimum m is positive because the intersection of $Z_{i,k}(x)$ with $V^c_{j(i,k)}$ is empty. Take an arbitrary $y \in W_i$ sufficiently close to x. We may assume the following inequalities:

- $d(x,y) < \frac{m}{2}$,
- $\Psi_{i,k-1}(\eta_i(y)) < \Psi_{i,k}(\eta_i(x))$,
- $\Psi_{i,k}(\eta_i(y)) > \Psi_{i,k-1}(\eta_i(x))$,
- $|\Psi_{i,k-1}(\eta_i(y)) - \Psi_{i,k-1}(\eta_i(x))| < \frac{m}{2}$ and
- $|\Psi_{i,k}(\eta_i(y)) - \Psi_{i,k}(\eta_i(x))| < \frac{m}{2}$.

Here, the notation $d(x,y)$ denotes the distance of $x = (x_1, \ldots, x_n)$ to $y = (y_1, \ldots, y_n)$ given by $d(x,y) = \max\{|x_i - y_i| \mid 1 \leq i \leq n\}$. We lead to a contradiction assuming that $y \notin X_{i,k}$. There exists an element $t \in M$ with $\Psi_{i,k-1}(\eta_i(y)) \leq t \leq \Psi_{i,k}(\eta_i(y))$ and $(y,t) \notin V_{j(i,k)}$ by the assumption. If $t \in Y_{i,k}(x)$, the distance between the point (y,t) and $Z_{i,k}(x)$ is $d(x,y)$ and less than m. It is a contradiction to the assumption that $(y,t) \notin V_{j(i,k)}$. In the other case, we get either

$$|t - \Psi_{i,k-1}(\eta_i(x))| \leq |\Psi_{i,k-1}(\eta_i(y)) - \Psi_{i,k-1}(\eta_i(x))| < \frac{m}{2}$$

or

$$|t - \Psi_{i,k}(\eta_i(x))| \leq |\Psi_{i,k}(\eta_i(y)) - \Psi_{i,k}(\eta_i(x))| < \frac{m}{2}$$

because we have either $\Psi_{i,k-1}(\eta_i(y)) \leq t < \Psi_{i,k-1}(\eta_i(x))$ or $\Psi_{i,k}(\eta_i(x)) < t \leq \Psi_{i,k}(\eta_i(y))$ by the assumption. Easy computations imply that $\rho(y,t) < m/2$ in both cases. For instance, in the former case, we have $\rho(y,t) = \min\{d(x,y), |t - \Psi_{i,k-1}(\eta_i(x))|\} < m/2$. It contradicts the definition of m. We have demonstrated that $X_{i,k}$ is open.

Set $U_i = \bigcap_{k=1}^{r_i} X_{i,k}$ and $\varphi_{i,k} = \Psi_{i,k} \circ \eta_i|_{U_i}$ for $1 \leq i \leq q$. The set U_i is a definable open set and $\varphi_{i,k}$ is a definable continuous function on U_i. The set $\{(x,t) \in U_i \times [0,1] \mid \varphi_{i,k-1}(x) \leq t \leq \varphi_{i,k}(x)\}$ is contained in $V_{j(i,k)}$ by the definition of $X_{i,k}$. Since $X = \bigcup_{i=1}^{q} D_i$ and $D_i \subset U_i$, we have $X \subseteq \bigcup_{i=1}^{q} U_i$. □

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