Colored Random Graphs and the Order Property

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1 Introduction

In this article, a graph means an R-structure, where R is a binary symmetric irreflexive predicate. If $R(a, b)$ holds, we consider a and b are adjacent by an edge. A subgraph means a subtructure, in the graph theory terminology, it is an induced subgraph. A finite coloring of a graph G usually means a function $f : R^G \to F$, where F is a finite set of colors. However, we are going to take a slightly different setting, which will be explained later. A monochromatic subgraph is a subgraph H for which the coloring function f is constant on R^H. In general, it is an important question whether a colored graph has monochrome subgraphs of a certain kind. Here we concentrate on countable random graphs and their coloring.

A graph G is called a random graph, it satisfies the following axioms for all disjoint subsets $A \neq \emptyset$ and B,

$$\exists x(\bigwedge_{a \in A} R(a, x) \wedge \bigwedge_{b \in B} \neg R(a, b)).$$

A random graph is necessarily infinite, and is universal in the sense that it embeds all finite graphs. It is easy to see that the theory of a random graph is $\aleph_0$-categorical, and is simple. In [2], they proved:

(*) A colored countable random graph G has a subgraph H such that $H \cong G$ (as graphs) and that H is (at most) 2-colored.

They also gave an example of G without monochromatic subgraph $H \cong G$.

In this article, we study the case when G does not have a monochromatic subgraph $H \cong G$. As a main result, we state some relation between the coloring and the instability strength (see Theorem 9). We do not give a detail of the proof.

2 Definitions and Preliminaries

Let G be a countable random graph in the language $\{R\}$, where R is a binary predicate symbol for edges.

Let $N \in \omega$. An N-coloring of G means an expansion of G to the language $L \cup \{R_i\}_{i<N}$ such that R^G is the disjoint union of R_i^G $(i < N)$. For a subset $C \subset N$, $R_C(x, y)$ is an abbreviation of $\bigvee_{i \in C} R_i(x, y)$. If $R_i(a, b)$ holds, we think that the edge ab is painted in the color i.

Now we fix a countable random graph G. We assume the edges of G are N-colored.

S_{na} denotes the set of all non-algebraic types with a finite domain.

Definition 1. 1. Let $p \in S_{na}$. An infinite subset $X \subset G$ is p-large, if (1) $p(X) = X$ and (2) $q(X)$ is non-empty for all non-algebraic $q \supset p$. We say X is large, if $p(X)$ is p-large for some p.

2. We write $X \subset_{\text{lrg}} Y$, if $X \subset Y$ and X is large.

Then, we can prove the following lemmas. (Proofs are not shown here.)

Lemma 2. *Suppose that X is p-large and that $X = \bigcup_{i<n} X_i$, where $n \in \omega$. Then, there is an index $i < n$ and a non-algebraic type $q \supset p$ such that $q(X_i)$ is q-large.*

Lemma 3. *Suppose that X and Y are large. Then, there is a color $i < N$ and $X_0 \subset_{\text{lrg}} X$ such that, for all $a \in X_0$, both*

$$\{b \in Y : R_i(a, b)\} \textit{ and } \{b \in Y : \neg R(a, b)\}$$

are large.

Definition 4. Let X and Y be large.

1. $C(X, Y)$ denotes the set of all colors $i < N$ for which some $X_0 \subset_{lrg} X$ satisfies the statement of Lemma 3.

2. $C^*(X, Y) = \bigcap \{C(X', Y') : X' \subset_{\text{lrg}} X,\ Y' \subset_{\text{lrg}} Y\}$.

Lemma 5. *Let X, Y be large. Then, there is $X_0 \subset_{lrg} X$ and $Y_0 \subset_{lrg} Y$ such that $C^*(X_0, Y_0) \neq \emptyset$.*

Lemma 6. *There is a large set Z and $i^*, j^* < N$ such that for any large $W \subset Z$ there is a disjoint large sets $X, Y \subset W$ such that $i^* \in C^*(X, Y)$ and $j^* \in C^*(Y, X)$.*

3 Main Results

Now we fix a large set Z and $i^*, j^* < N$ satisfying the requirement in Lemma 6.

Definition 7. Let A be a finite subset of Z, and $D \supset A$ a finite subset of G. Let $\mathfrak{X} = \{X_p\}_{p \in S_{na}(A)}$ be a set of large subsets of Z and let $\mathfrak{T} = \{p^*\}_{p \in S_{na}(A)}$ be a set of types. We say the tuple $(A, D, \mathfrak{X}, \mathfrak{T})$ is good, if the following are true: For all $p \neq q \in S_{na}(A)$, 1. $p \subset p^* \in S_{na}(D)$; 2. X_p is p^*-large;3. (i^*, j^*) or (j^*, i^*) belongs to $C^*(X_p, X_q) \times C^*(X_q, X_p)$. 4. For all $a \in A$ and $b \in AX_p$, $R(a, b) \iff R_{\{i^*, j^*\}}(a, b)$.

Proposition 8. *Suppose that $\left(A, D, \{X_p\}_{p \in S_{na}(A)}, \{p^*\}_{p \in S_{na}(A)}\right)$ is good. Then, for all $s \in S_{na}(A)$, we can find $d \in X_s$, $D' \supset D$, $\{X_q\}_{q \in S_{na}(Ad)}$ and $\{q^*\}_{q \in S_{na}(Ad)}$ such that*

- *$\left(Ad, D', \{X_q\}_{q \in S_{na}(Ad)}, \{q^*\}_{q \in S_{na}(Ad)}\right)$ is also good;*
- *$p^* \subset q^*$ and $X_q \subset X_p$, if $p \in S_{na}(A)$, $q \in S_{na}(Ad)$ and $p \subset q$.*

Theorem 9. *Let G be a random graph and suppose that an N-coloring is given on G by $L^* = \{R, R_1, \ldots, R_N\}$. Then the following conditions are equivalent:*

(a) G does not have a monochromatic generic subgraph;

(b) For any generic subgraph $G_0 \subset G$, there is a generic $H \subset G_0$ having the strict order property in the expanded language L^*.

Sketch of Proof. (b) $\Rightarrow$ (a): This is trivial since a monochromatic subgraph is a mere random graph. (a) $\Rightarrow$ (b): We assume (a). For simplicity of the notation, we can assume $G_0 = G$. We choose $i^*, j^* < N$ and Z as in Lemma 6. Since G does not have a monochromatic generic subgraph, we have $i^* \neq j^*$. So, for simplicity, we assume $i^* = 0$ and $j^* = 1$. Let $\{g_i\}_{i \in \omega}$ be an enumeration of G such that for all $i > 0$,

1. $R(g_0, g_i)$ if and only if i is even;

2. $R(g_{4i}, g_j)$ for all odd numbers $j < 4i$.

Notice that such an enumeration does exist. Choose disjoint large subsets $X_0, X_1 \subset Z$ such that $0 \in C^*(X_0, X_1)$ and $1 \in C^*(X_1, X_0)$. We are going to define h_i $(i < \omega)$ such that $(g_i)_{i\in\omega} \cong (h_i)_{i\in\omega}$. By symmetry, shrinking X_0 and X_1, we may assume $\forall x \in X_0(R(h_0, x))$ and $\forall x \in X_1(\neg R(h_0, x))$ hold for some $h_0 \in G$. In this proof, we inductively choose elements $h_i \in X_0 X_1$ $(i > 0)$ such that, by letting

$$D_n := \{h_m : h_m \neq h_0, \neg R(h_0, h_m) \text{ and } R_1(h_{4n}, h_m)\},$$

$\{D_n : n \in \omega\}$ forms a strictly increasing sequence of uniformly defined sets. Thus, $H := \{h_i\}_{i\in\omega}$ has the strict order property. □

References

[1] Chang-Keisler, Model Theory

[2] Maurice Pouzet and Norbert Sauer, Edge Partitions of the Rado Graph, Combinatorica 16 (4) (1996) 505–520.

[3] Takeuchi and Tsuboi, Infinite subgraphs with monochromatic edges, Unpublished.