Sparse domination of Fourier integral operators

信州大学　山本　涼介
Ryosuke Yamamoto
Shinshu University

Abstract

In this paper, we give sparse form bounds and sparse bounds for Fourier integral operators associated with the symbol belonging to Hörmander class.

1 Introduction and results

For any $m \in \mathbb{R}$ and $0 \le \rho, \delta \le 1$, Hörmander class $S^m_{\rho,\delta}$ is defined as the set of all $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \lesssim (1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

for any $(x,\xi) \in \mathbb{R}^{2n}$ and $\alpha, \beta \in \mathbb{N}_0^n$. Here, $A \lesssim B$ means $A \le CB$ with a positive constant $C > 0$. Given $a \in S^m_{\rho,\delta}$ and $\Phi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$, we define the Fourier integral operator (FIO for short) $T_{a,\Phi}$ by

$$T_{a,\Phi} f(x) = \int_{\mathbb{R}^n} e^{i\Phi(x,\xi)} a(x,\xi) \hat{f}(\xi) d\xi$$

for $f \in \mathcal{S}$. For simplicity, we write $T_{a,\Phi} = T$. We assume that

(A-1) There exists a compact set K such that $a \equiv 0$ on $K^c \times \mathbb{R}^n$,

(A-2) Φ is a real-valued function and homogeneous of degree one in ξ,

(A-3) $\inf_{x\in K, \xi \neq 0} \left| det(\frac{\partial \Phi}{\partial x_i \partial \xi_j}(x,\xi)) \right| > 0.$

In the case $\Phi(x,\xi) = x\xi$, T is a pseudodifferential operator. Pseudodifferential operators are useful for study of elliptic equations. However, the operators can not be applied to non-elliptic problems. FIOs were introduced by Hörmander [6] to consider such problems. For example, if u solves the wave equation,

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(0) = 0 & \text{in } \mathbb{R}^n, \\ \partial_t u(0) = f & \text{in } \mathbb{R}^n, \end{cases}$$

then, u is written as $u(x,t) = (T_0^t + T_1^t) f(x)$ where

$$T_0^t f(x) = \frac{1}{2i} \int e^{i(x\xi + t|\xi|)} |\xi|^{-1} \hat{f}(\xi) d\xi \ , \ T_1^t f_1(x) = -\frac{1}{2i} \int e^{i(x\xi - t|\xi|)} |\xi|^{-1} \hat{f}(\xi) d\xi.$$

Peral [11] showed that for any $t \in \mathbb{R}$, $T_0^t + T_1^t$ is bounded from L^p to H_p^s if and only if $|1/p - 1/2| \le 1/(n-1)$, where $s = 1 - (n-1)|1/p - 1/2|$. For $p \in [1, \infty)$, H^p-L^p boundedness of T with $a \in S^m_{1,0}$ was proved by Stein [13] when $m \le -(n-1)|1/p - 1/2|$. Here, H^p means the Hardy space. In the case $a \in S^m_{\rho,1-\rho}$ with $1/2 \le \rho \le 1$ and $m = -(n-\rho)|1/p - 1/2|$, L^p boundedness of T was proved by Seeger, Sogge and Stein [14]. The main purpose of this master thesis is to establish the sparse form bounds and sparse bounds for T:

$$|\langle Tf, g \rangle| \lesssim \Lambda_{\mathscr{S},r,s'}(f,g) \quad , \quad |Tf(x)| \lesssim \Lambda_{\mathscr{S},r} f(x)$$

under the conditions (A-1)–(A-3). See below for the definition of $\Lambda_{\mathscr{S},r,s'}$ and $\Lambda_{\mathscr{S},r} f$.

Definition 1.1. *Let $\eta \in (0,1)$. A collection $\mathscr{S}$ of cubes in $\mathbb{R}^n$ is η-sparse if there are pairwise disjoint subsets $\{E_Q\}_{Q\in\mathscr{S}}$ such that $E_Q \subset Q$, and $|E_Q| > \eta|Q|$.*

When η is not important, we omit it. For any cube Q and $p \in [1,\infty)$, we define $\langle f\rangle_{p,Q} := |Q|^{-\frac{1}{p}}||f||_{L^p(Q)}$. For a sparse collection $\mathscr{S}$ and $r, s \in [1,\infty)$, the (r,s)-sparse form operator $\Lambda_{\mathscr{S},r,s}$ and r-sparse operator $\Lambda_{\mathscr{S},r}$ are defined by

$$\Lambda_{\mathscr{S},r,s}(f,g) := \sum_{Q\in\mathscr{S}} |Q|\langle f\rangle_{r,Q}\langle g\rangle_{s,Q} \quad , \quad \Lambda_{\mathscr{S},r}f(x) := \sum_{Q\in\mathscr{S}} \langle f\rangle_{r,Q}1_Q(x)$$

for $f, g \in L^1_{loc}$. If $r < p < s$, we have

$$\Lambda_{\mathscr{S},r,s'}(f,g) \lesssim ||f||_p||g||_{p'}.$$

This inequality is easily checked from the L^p-boundedness of r-Hardy Littlewood maximal operator M_r which is defined by $M_rf(x) = \sup_{Q\ni x}\langle f\rangle_{r,Q}$. Furthermore, weighted inequality with Muckenhoupt weights is deduced from sparse form bounds. Bernicot, Frey and Petermichl [2] showed

$$\Lambda_{\mathscr{S},r,s'}(f,g) \lesssim ([\omega]_{A_{p/r}}[\omega]_{RH_{(s/p)'}})^{\alpha}||f||_{L^p(\omega)}||g||_{L^{p'}(\omega^{1-p'})}$$

where $\alpha = \max(\frac{1}{p-r}, \frac{s-1}{s-p})$, $[\omega]_{A_q} = \sup_Q\langle\omega\rangle_{1,Q}\langle\omega^{1-q'}\rangle^{q-1}_{1,Q}$ and $[\omega]_{RH_q} = \sup_Q\langle\omega\rangle^{-1}_{1,Q}\langle\omega\rangle_{q,Q}$ for any $1 < q < \infty$. From this inequality, sparse form bounds is used to study weighted bounds for operators. In recent years, people are interested in establishing sparse form bounds for several operators. Sparse form bounds of rough singular integral operators and Bochner-Riesz multipliers were shown by Conde-Alonso, Culic, Plinio and Ou [4], and Lacey, Mena and Reguera [10] respectively. Beltran and Cladek [3] proved the sparse form bounds and sparse bounds for pseudodifferential operators with symbols in $S^m_{\rho,\rho}$ with $0 \le \rho < 1$ and suitable m. It is natural to ask the same problem for FIOs instead of pseudodifferential operators. Main results are the following.

Theorem 1.1. *Let $1 \le r \le s < \infty$ and $m < 0$. We assume that $a \in S^m_{1,0}$ and $\Phi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ satisfy the assumptions* (A-1)–(A-3). *Then, for any compactly supported bounded functions f, g, there is a sparse collection $\mathscr{S}$ such that*

$$|\langle Tf,g\rangle| \lesssim \Lambda_{\mathscr{S},r,s'}(f,g)$$

if

$$m < -(n-1)(1/s-1/2) - n(1/r-1/s) \quad \text{and} \quad 1 \le r \le s \le 2$$

or

$$m < -n(1/r-1/s) \quad \text{and} \quad 1 \le r \le 2 \le s \le r'.$$

Remark 1.1. *When $1 < p < \infty$, Theorem 1.1 gives us the L^p-boundedness of T with $a \in S^m_{1,0}$ and $m < -(n-1)|1/p-1/2|$. Stein [13] proved that T can be extended to a bounded operator on L^p when $m = -(n-1)|1/p-1/2|$. However, it seems that this case can not be deduced from the sparse form bounds.*

Remark 1.2. *By using duality, we can see that*

$$|\langle Tf,g\rangle| \lesssim \Lambda_{\mathscr{S},s',r}(f,g)$$

under the same condition of Theorem 1.1.

Theorem 1.1 does not include sparse form bounds in the case $1 \le r \le 2$ and $s = \infty$. However, the following theorem covers such a case.

Theorem 1.2. *Let $1 \le r \le 2$, $m < 0$ and $0 \le \rho, \delta \le 1$. We assume that $a \in S^m_{\rho,\delta}$ and $\Phi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ satisfy the assumptions* (A-1)–(A-2)*. Then, for any compactly supported bounded function f, there are sparse collections $\{\mathscr{S}_k\}_{k=1,\cdots,3^n}$ such that*

$$|Tf(x)| \lesssim \sum_{k=1}^{3^n} \Lambda_{\mathscr{S}_k,r} f(x)$$

if $m < -n/r$.

Remark 1.3. *Since*

$$|\langle \sum_{k=1}^{3^n} \Lambda_{\mathscr{S}_k,r} f, g \rangle| \le \sum_{k=1}^{3^n} \sum_{Q \in \mathscr{S}_k} \langle f \rangle_{r,Q} |\langle 1_Q, g\rangle| \le \sum_{k=1}^{3^n} \Lambda_{\mathscr{S}_k,r,1}(f,g),$$

the following bounds hold under the same condition of Theorem 1.2.

$$|\langle Tf, g\rangle| \lesssim \Lambda_{\mathscr{S},r,1}(f,g)$$

Furthermore, the Theorem 1.1 and the weighted inequality above give us weighted bounds for FIO.

Corollary 1.1. *Let $a \in S^m_{1,0}$ with $m < 0$, Φ satisfy the assumptions* (A-1)–(A-3) *and $\omega \in A_\infty$. Then,*

$$||Tf||_{L^p(\omega)} \lesssim ||f||_{L^p(\omega)}.$$

holds in the following cases;
(1) $m < -n$, $1 < p < \infty$ and $\omega \in A_p$.
(2) $-n \le m < -n/2$, $-n/m < p < \infty$ and $\omega \in A_{-(mp)/n}$.
(3) $-n/2 \le m < 0$, $2 < p < 2n/(n+2m)$ and $\omega \in A_{p/2} \cap RH_{(2n/\{(n+2m)p\})'}$.

2 Sparse domination of Fourier integral operators

In this section, we give a proof of Theorems 1.1 and 1.2 and Corollary 1.1

2.1 Decomposition of T

Using the idea of Beltran and Cladek [3], we decompose T. We take $\psi \in C_0^\infty(\mathbb{R}^n)$ such that supp $\psi \subset B(0,2)$, $\psi = 1$ on $B(0,1)$ and $\psi \ge 0$, and denote $\psi_s(\xi) := \psi(2^{-s}\xi) - \psi(2^{-s+1}\xi)$ for $s \in \mathbb{R}$. Then T_j is defined by

$$\begin{aligned} T_j f(x) &= \int_{\mathbb{R}^n} e^{i\Phi(x,\xi)} a(x,\xi)\psi_j(\xi)\hat{f}(\xi)d\xi \\ &= \int\int_{\mathbb{R}^{2n}} e^{i\{\Phi(x,\xi)-y\xi\}} a(x,\xi)\psi_j(\xi) f(y) d\xi dy \end{aligned}$$

for any $f \in S$ and $j \in \mathbb{Z}$. Moreover, for $\varepsilon > 0$ and $\ell \in \mathbb{Z}$, we define

$$T_{j,\ell}f(x) = \begin{cases} \int\int_{\mathbb{R}^{2n}} e^{i\{\Phi(x,\xi)-y\xi\}} a(x,\xi)\psi_j(\xi)\psi_\ell(x-y) f(y) d\xi dy & (j \ge 0) \\ \int\int_{\mathbb{R}^{2n}} e^{i\{\Phi(x,\xi)-y\xi\}} a(x,\xi)\psi_j(\xi)\psi_{\ell-(1+\varepsilon)j}(x-y) f(y) d\xi dy & (j < 0) \end{cases}.$$

For any $\varepsilon > 0$, T is decomposed as follows :

$$T = \sum_{j\ge 0}\sum_{\ell > \varepsilon j} T_{j,\ell} + \sum_{j \ge 0}\sum_{\ell \le \varepsilon j} T_{j,\ell} + \sum_{j<0}\sum_{\ell \ge 0} T_{j,\ell} + \sum_{j<0}\sum_{\ell<0} T_{j,\ell}.$$

We need further decomposition of $T_{j,\ell}$. For $\nu \in \{0,1,2\}^n$ and $k \in \mathbb{Z}$, the set $\mathcal{D}_\nu^k$ of dyadic cubes is defined by

$$\mathcal{D}_\nu^k := \{2^k[m_1 + \frac{\nu_1}{3}, m_1 + 1 + \frac{\nu_1}{3}) \times \cdots \times 2^k[m_n + \frac{\nu_n}{3}, m_n + 1 + \frac{\nu_n}{3}) \ ; \ m \in \mathbb{Z}^n\}.$$

We decompose $T_{j,\ell}$ as

$$T_{j,\ell} = \sum_\nu T_{j,\ell}^\nu$$

where

$$T_{j,\ell}^\nu f = \begin{cases} \sum_{Q \in \mathcal{D}_\nu^{[\ell+10]}} T_{j,\ell}(f 1_{\frac{1}{3}Q}) & (j \geq 0, \ell > j\varepsilon) \\ \sum_{Q \in \mathcal{D}_\nu^{[\varepsilon j+10]}} T_{j,\ell}(f 1_{\frac{1}{3}Q}) & (j \geq 0, \ell \leq j\varepsilon) \\ \sum_{Q \in \mathcal{D}_\nu^{[\ell-(1+\varepsilon)j+10]}} T_{j,\ell}(f 1_{\frac{1}{3}Q}) & (j < 0, \ell \geq 0) \\ \sum_{Q \in \mathcal{D}_\nu^{[-(1+\varepsilon)j+10]}} T_{j,\ell}(f 1_{\frac{1}{3}Q}) & (j < 0, \ell < 0) \end{cases}.$$

Here, we remark that the support of $T_{j,\ell}(f 1_{\frac{1}{3}Q})$ is contained in Q. To prove Theorem 1.1, we make use of the following lemma which was given by Lacay and Mena [9].

Lemma 2.1. *Suppose $\eta \in (0,1)$ and $r, s \in [1,\infty)$. For any compactly supported bounded functions f, g, there is a sparse collection $\mathscr{S}_0$ such that*

$$\Lambda_{\mathscr{S},r,s}(f,g) \lesssim \Lambda_{\mathscr{S}_0,r,s}(f,g)$$

for any η-sparse collection $\mathscr{S}$.

2.2 Estimates of $T_{j,l}$

We estimate $||T_{j,l}||_{r,s} := \sup_{||f||_{L^r}=1} ||T_{j,\ell} f||_{L^s}$ with $1 \leq r \leq s \leq \infty$ to prove Theorem 1.1.

Lemma 2.2. *For $1 \leq r \leq s \leq 2$ and $j \geq 0$, we have the followings.*

(1) *For $\ell \in \mathbb{Z}$ with $\ell > j\varepsilon$, $||T_{j,\ell}||_{r,s} \lesssim 2^{10n(m-n)(j+\ell)}$.*

(2) $||\sum_{\ell \leq \varepsilon j} T_{j,\ell}||_{r,s} \lesssim 2^{jm + j(n-1)(1/s-1/2) + jn(1/r-1/s)}$.

Proof. (1) Let N be a positive integer and $\Psi(x,\xi) = \Phi(x,\xi) - x\xi$. We integrate by parts in ξ to obtain

$$\begin{aligned} |T_{j,\ell} f(x)| &= \left| \iint e^{i(x-y)\xi} e^{i\Psi(x,\xi)} a(x,\xi) \psi_j(\xi) \psi_\ell(x-y) f(y) d\xi dy \right| \\ &\leq \int |\Delta_\xi^N (e^{i\Psi(x,\xi)} a(x,\xi)\psi_j(\xi))| d\xi \int |x-y|^{-2N} |\psi_\ell(x-y) f(y)| dy. \end{aligned}$$

Since Ψ is homogeneous of degree one in ξ and ξ is localized in the annulus $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, it holds that $|\partial_\xi^\alpha \Psi(x,\xi)| \lesssim |\xi|^{1-|\alpha|} \lesssim 2^{(1-|\alpha|)j}$ for any $\alpha \in \mathbb{N}_0^n$, and $|\partial_\xi^\alpha e^{i\Psi(x,\xi)}| \lesssim 1$. Therefore we obtain

$$\sup_{x,\xi} |\Delta_\xi^N (e^{i\Psi(x,\xi)} a(x,\xi) \psi_j(\xi))| \lesssim 2^{jm} \leq 1,$$

which yields

$$|T_{j,\ell} f(x)| \lesssim 2^{jn - \ell N - j\varepsilon N} \int |\psi_\ell(x-y)||f(y)| dy.$$

By using Young's inequality and taking sufficiently large N, we have

$$||T_{j,\ell}||_{r,s} \lesssim 2^{10n(m-n)(j+\ell)}$$

(2) Since $\sum_{\ell \leq j\varepsilon} T_{j,\ell} = T_j - \sum_{\ell > j\varepsilon} T_{j,\ell}$, it suffices to prove

$$||T_j||_{r,s} \lesssim 2^{jm+j(n-1)(1/s-1/2)+jn(1/r-1/s)}.$$

Since $|\partial_x^\beta \partial_\xi^\alpha \{a(x,\xi)\psi_j(\xi)\}| \lesssim 2^{jm}(1+|\xi|)^{-|\alpha|}$ and L^2-boundedness of FIO [13], it holds $||T_j||_{2,2} \lesssim 2^{jm}$. Let K_j be the kernel of T_j, i.e.

$$T_j f(x) = \int K_j(x,y) f(y) dy.$$

The inequality

$$\sup_{y\in\mathbb{R}^n} \int |K_j(x,y)| dx \lesssim 2^{jm+j(n-1)/2}$$

was proved by Stein [13]. Hence, one obtains $||T_j||_{1,1} \lesssim 2^{jm+j(n-1)/2}$. Interpolating this and L^2-bound above, we have

$$||T_j||_{s,s} \lesssim 2^{jm+j(n-1)(1/s-1/2)}.$$

Let $\tilde{\psi}_j = \psi_{j-1} + \psi_j + \psi_{j+1}$. From the L^s-bound, we deduce

$$\begin{aligned} ||T_j f||_{L^s} &= ||T_j(\mathcal{F}^{-1}[\tilde{\psi}_j] * f)||_{L^s} \\ &\lesssim 2^{jm+j(n-1)(1/s-1/2)} ||\mathcal{F}^{-1}[\tilde{\psi}_j] * f||_{L^s} \\ &\lesssim 2^{jm+j(n-1)(1/s-1/2)+jn(1/r-1/s)} ||f||_{L^r}. \end{aligned}$$

□

Lemma 2.3. *For $1 \leq r \leq 2 \leq s \leq r'$ and $j \geq 0$, we get the followings.*

(1) *For $\ell \in \mathbb{Z}$ with $\ell > j\varepsilon$,* $||T_{j,\ell}||_{r,s} \lesssim 2^{10n(m-n)(j+\ell)}$.

(2) $||\sum_{\ell \leq \varepsilon j} T_{j,\ell}||_{r,s} \lesssim 2^{jm+jn(1/r-1/s)}$.

Proof. First claim is the same as (1) of Lemma 2.2. Let us prove the second inequality. We can easily check that $||T_j||_{1,\infty} \lesssim 2^{jm+jn}$. Interpolating this and L^2-bound of T_j, we can see

$$||T_j||_{r,r'} \lesssim 2^{jm+jn(2/r-1)}.$$

On the other hand, we have $||T_j||_{r,2} \lesssim 2^{jm+jn(1/r-1/2)}$ from the proof of Lemma 2.2. Interpolating this and L^r-$L^{r'}$ bounds above, we obtain

$$||T_j||_{r,s} \lesssim 2^{jm+jn(1/r-1/s)}.$$

□

Lemma 2.4. *For $1 \leq r \leq s < \infty$ and $j < 0$, it holds the followings.*

(1) *For $\ell \in \mathbb{N}$,* $||T_{j,\ell}||_{r,s} \lesssim 2^{10n(j-\ell)}$.

(2) $||\sum_{\ell<0} T_{j,\ell}||_{r,s} \lesssim 2^{jn/r}$.

Proof. (1) For $|\alpha| \geq 1$, we obtain $|\partial_\xi^\alpha e^{i\Psi(x,\xi)}| \lesssim |\xi|^{1-|\alpha|}$, and $|\Delta_\xi^N(e^{i\Psi(x,\xi)}a(x,\xi)\psi_j(\xi))| \lesssim 2^{-2jN+j} \leq 2^{-2jN}$ for any $N \in \mathbb{N}$. By using integration by parts, one has

$$\begin{aligned} |T_{j,\ell}f(x)| &\lesssim \int |\Delta_\xi^N(e^{i\Psi(x,\xi)}a(x,\xi)\psi_j(\xi))|d\xi \int |x-y|^{-2N}|\psi_{\ell-(1+\varepsilon)j}(x-y)f(y)|dy \\ &\lesssim 2^{-2\ell N+2\varepsilon jN}\int |\psi_{\ell-(1+\varepsilon)j}(x-y)||f(y)|dy. \end{aligned}$$

By taking sufficiently large N, we have

$$||T_{j,\ell}||_{r,s} \lesssim 2^{10n(j-\ell)}.$$

(2) To prove the second estimate, we prove $||T_j||_{r,s} \lesssim 2^{jn/r}$. It is not hard to see that

$$||T_jf||_{L^\infty} \lesssim \min(2^{jn}||f||_{L^1}, 2^{jn/2}||f||_{L^2}).$$

Since the support of $a(\cdot,\xi)$ is compact set, we obtain $||T_j||_{2,2} \lesssim 2^{jn/2}$ and $||T_j||_{1,1} \lesssim 2^{jn}$. Therefore, we can see $||T_j||_{r,s} \lesssim 2^{jn/r}$ and $||\sum_{\ell<0} T_{j,\ell}||_{r,s} \lesssim 2^{jn/r}$ in the same way as Lemma 2.2. □

2.3 Proof of Theorem 1.1

Proof. Recall the decomposition of T in Subsection 2.1:

$$T = \sum_{j\geq 0}\sum_{\ell>\varepsilon j} T_{j,\ell} + \sum_{j\geq 0}\sum_{\ell\leq\varepsilon j} T_{j,\ell} + \sum_{j<0}\sum_{\ell\geq 0} T_{j,\ell} + \sum_{j<0}\sum_{\ell<0} T_{j,\ell}.$$

First, we concider the case $1 \leq r \leq s \leq 2$. Since supp $T_{j,\ell}(f\chi_{\frac{1}{3}Q}) \subset Q$, (1) of Lemma 2.2 yields the following estimate for the first term.

$$\begin{aligned} &|\langle \sum_{j\geq 0}\sum_{\ell>j\varepsilon}\sum_{\nu}\sum_{Q\in\mathcal{D}_\nu^{[\ell+10]}} T_{j,\ell}(f\chi_{\frac{1}{3}Q}), g\rangle| \\ \lesssim\ & \sum_{j\geq 0}\sum_{\ell>j\varepsilon}\sum_{\nu}\sum_{Q\in\mathcal{D}_\nu^{[\ell+10]}} ||T_{j,\ell}||_{r,s}||f||_{L^r(Q)}||g||_{L^{s'}(Q)} \\ \lesssim\ & \sum_{j\geq 0} 2^{10n(m-n)j}\sum_{\ell>j\varepsilon} 2^{10n(m-n)\ell+\ell n(1/r-1/s)}\sum_{\nu}\sum_{Q\in\mathcal{D}_\nu^{[\ell+10]}} |Q|\langle f\rangle_{r,Q}\langle g\rangle_{s',Q} \\ \lesssim\ & \left(\sum_{j\geq 0} 2^{10n(m-n)j}\right)\left(\sum_{\ell>0} 2^{10n(m-n)\ell+\ell n(1/r-1/s)}\right)\Lambda_{\mathscr{S}_0,r,s'}(f,g) \end{aligned}$$

where $\mathcal{S}_0$ is the sparse collection in Lemma 2.1. Since the both series converge whenever $m < 0$ and $1 \leq r \leq s < 2$. Therefore, we have

$$|\langle \sum_{j\geq 0}\sum_{\ell>j\varepsilon} T_{j,\ell}f, g\rangle| \lesssim \Lambda_{\mathcal{S}_0,r,s'}(f,g).$$

On the other hand, (2) of Lemma 2.2 gives the following estimate for the second term.

$$\begin{aligned} &|\langle \sum_{j\geq 0}\sum_{\ell\leq j\varepsilon}\sum_{\nu}\sum_{Q\in\mathcal{D}_\nu^{[\varepsilon j+10]}} T_{j,\ell}(f\chi_{\frac{1}{3}Q}), g\rangle| \\ \lesssim\ & \sum_{j\geq 0}\sum_{\nu}\sum_{Q\in\mathcal{D}_\nu^{[\varepsilon j+10]}} ||\textstyle\sum_{\ell\leq j\varepsilon} T_{j,\ell}||_{r,s}|Q|^{1/r-1/s}|Q|\langle f\rangle_{r,Q}\langle g\rangle_{s',Q} \\ \lesssim\ & \sum_{j\geq 0} 2^{jm+j(n-1)(1/s-1/2)+jn(1/r-1/s)+j\varepsilon n(1/r-1/s)}\sum_{\nu}\sum_{Q\in\mathcal{D}_\nu^{[\varepsilon j+10]}} |Q|\langle f\rangle_{r,Q}\langle g\rangle_{s',Q} \\ \lesssim\ & \sum_{j\geq 0} 2^{jm+j(n-1)(1/s-1/2)+jn(1/r-1/s)+j\varepsilon n(1/r-1/s)}\Lambda_{\mathscr{S}_0,r,s'}(f,g). \end{aligned}$$

If $m < -(n-1)|(1/s-1/2)| - n(1/r-1/s)$, we can take ε such that the last geometric series absolutely converges. Consequently, we obtain the desired sparse form bound for the second term. For the rest of terms, we can get same bounds in the same way as that of Lemma 2.4. If $1 \le r \le 2 \le s \le r'$, the inequality is shown from the same argument by using Lemmas 2.3. and 2.4. □

2.4 Proof of Theorem 1.2

In this part, we prove Theorem 1.2. Hytönen, Lacey and Pérez [7] showed the following property of shifted dyadic cubes.

Proposition 2.1. *For any cube Q, there exist $\nu \in \{0,1,2\}^n$ and $Q_0 \in \mathcal{D}_\nu$ such that $Q \subset Q_0$ and $|Q| \sim |Q_0|$.*

To prove Theorem 1.2, we give a sparse bounds for Hardy-Littlewood maximal operator by using Proposition 2.1 and the idea of Pérez [12].

Proposition 2.2. *Let M_r denotes r-Hardy Littlewood maximal operator. Then for any $f \in L^p$ for some $1 < p \le \infty$, there exist sparse collections $\mathscr{S}_\nu \subset \mathcal{D}_\nu$ such that*

$$M_r f(x) \lesssim \sum_{\nu \in \{0,1,2\}^n} \Lambda_{\mathscr{S}_\nu, r} f(x) \quad a.e.\ x \in \mathbb{R}^n.$$

Proof. From Proposition 2.1,

$$M_r f(x) \lesssim \sum_{\nu \in \{0,1,2\}^n} M_r^{\mathcal{D}_\nu} f(x).$$

where $M_r^{\mathcal{D}_\nu} f(x) := \sup_{\mathcal{D}_\nu \ni Q \ni x} \langle f \rangle_{Q,r}$. Fix $a \gg 1$ and let

$$\mathscr{S}_\nu^k := \{Q \in \mathcal{D}_\nu \ ; \ a^k < \langle f \rangle_{Q,r} \ \&\ \textit{maximal with inclusion}\}$$

for any $k \in \mathbb{Z}$ and let $\mathscr{S}_\nu = \cup_{k \in \mathbb{Z}} \mathscr{S}_\nu^k$. From the maximality, one has

$$a^k < \langle f \rangle_{Q,r} \le 2^n a^k$$

for each $Q \in \mathscr{S}_\nu^k$. First, we prove that $\mathscr{S}_\nu$ is a sparse collection. For each $Q \in \mathscr{S}_\nu^k$, let

$$E_Q := \{x \in Q \ ; \ a^k < M_r^{\mathcal{D}_\nu}(f 1_Q)(x) \le a^{k+1}\}.$$

From weak type (r,r) boundedness of M_r, we obtain

$$\begin{aligned} |Q \setminus E_Q| &\le ||M_r||_{L^r \to L^{r,\infty}} a^{-kr-r} ||f||_{L^r(Q)}^r \\ &\le 2^{nr} ||M_r||_{L^r \to L^{r,\infty}} a^{-r} |Q| \\ &\le \frac{1}{2}|Q| \end{aligned}$$

by taking sufficiently large a. Next we prove that $\{E_Q\}_{Q \in \mathscr{S}_\nu^k}$ is disjoint. Let $Q_k \in \mathscr{S}_\nu^k$, $Q_s \in \mathscr{S}_\nu^s$ and $Q_k \ne Q_s$. We may assume $k \le s$. If $Q_k \cap Q_s \ne \emptyset$, then it holds that $Q_k \subset Q_s$ or $Q_s \subset Q_k$. Hence, one has $k < s$ and $Q_s \subset Q_k$ from maximality of cubes in $\mathscr{S}_\nu^k$. If there exists $x \in E_{Q_k} \cap E_{Q_s}$, it holds

$$a^s < M_r^{\mathcal{D}_\nu}(f 1_{Q_s})(x) \le M_r^{\mathcal{D}_\nu}(f 1_{Q_k})(x) \le a^{k+1}$$

which contradicts $k < s$. Therefore $\mathscr{S}_\nu$ is a (1/2)-sparse collection. To complete the proof, it is sufficient to prove

$$M_r^{\mathcal{D}_\nu} f(x) \lesssim \Lambda_{\mathscr{S}_\nu, r} f(x) \quad a.e.\ x \in \mathbb{R}^n.$$

For any $x \in \mathbb{R}^n$ such that $M_r^{\mathcal{D}_\nu} f(x) \neq \infty$, there is a $k_0 \in \mathbb{Z}$ so that

$$a^{k_0} < M_r^{\mathcal{D}_\nu} f(x) \leq a^{k_0+1}.$$

From the definition of $M_r^{\mathcal{D}_\nu}$, we can take a dyadic cube $Q_0 \ni x$ which satisfies $Q_0 \in \mathscr{S}_\nu^{k_0}$ that means $a^{k_0} < \langle f \rangle_{Q_0,r}$. Therefore we obtain

$$M_r^{\mathcal{D}_\nu} f(x) \leq a \langle f \rangle_{Q_0,r} \leq a \Lambda_{\mathscr{S}_\nu,r} f(x).$$

□

From Proposition 2.2, it is sufficient to prove following lemma.

Lemma 2.5. *Let $1 \leq r \leq 2$, $m < 0$ and $0 \leq \rho, \delta \leq 1$. We assume that $a \in S^m_{\rho,\delta}$ and $\Phi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ satisfy the assumptions* (A-1)–(A-2). *Then,*

$$|Tf(x)| \lesssim M_r f(x) \quad a.e.\ x \in \mathbb{R}^n$$

holds if $m < -n/r$.

Proof. We use the decomposition of $T = \sum_{j \in \mathbb{Z}} T_j$ in Subsection 2.1. Let N be a positive integer such that $2N > n$ and $\Psi(x,\xi) = \Phi(x,\xi) - x\xi$.
(1) When $j \geq 0$, we integrate by parts in ξ and use the Hausdorff-Young's inequality to obtain

$$\begin{aligned}
|T_j f(x)| &= \frac{1}{(2\pi)^n} \left| \int\int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} (I-\Delta)^N \{e^{i\Psi(x,\xi)} a(x,\xi)\psi_j(\xi)\} \langle x-y \rangle^{-2N} f(y) d\xi dy \right| \\
&\lesssim 2^{jm} \int_{|\xi| \sim 2^j} |\mathcal{F}[\langle x - \cdot \rangle^{-2N} f](\xi)| d\xi \\
&\lesssim 2^{jm+jn/r} ||\mathcal{F}[\langle x - \cdot \rangle^{-2N} f]||_{L^{r'}} \\
&\lesssim 2^{jm+jn/r} ||\langle x - \cdot \rangle^{-2N} f||_{L^r}.
\end{aligned}$$

Since $m < -n/r$, we have

$$\sum_{j \geq 0} |T_j f(x)| \lesssim M_r f(x).$$

(2) When $j < 0$, we define the self-adjoint differential operator L as

$$L = 2^{-2jN+j} I + (-\Delta_\xi)^N.$$

From

$$e^{i(x-y)\xi} = \frac{1}{2^{-2jN+j} + |x-y|^{2N}} L e^{i(x-y)\xi}$$

and

$$|L(e^{i\Psi(x,\xi)} a(x,\xi)\psi_j(\xi))| \lesssim 2^{-2Nj+j},$$

$T_j f$ is dominated by Mf as follows.

$$\begin{aligned}
|T_j f(x)| &\lesssim \int\int_{\mathbb{R}^{2n}} |L(e^{i\Psi(x,\xi)} a(x,\xi)\psi_j(\xi))| \frac{1}{2^{-2jN+j} + |x-y|^{2N}} |f(y)| d\xi dy \\
&\lesssim \int \frac{2^{-2jN+j+jn}}{2^{-2jN+j} + |x-y|^{2N}} |f(y)| dy \\
&= 2^{jn} \sum_{k \in \mathbb{Z}} \int_{|x-y| \sim 2^k 2^{-j+\frac{j}{2N}}} \frac{1}{1+2^{2kN}} |f(y)| dy \\
&\lesssim 2^{\frac{n}{2N}j} \sum_{k \in \mathbb{Z}} \frac{2^{kn}}{1+2^{2kN}} Mf(x) \\
&\sim 2^{\frac{n}{2N}j} Mf(x).
\end{aligned}$$

Therefore, it holds that

$$\sum_{j<0} T_j f(x) \lesssim Mf(x) \le M_r f(x).$$

□

2.5 Proof of Corollary 1.1

In this subsection, we prove Corollary 1.1 by using weighted bounds for sparse form. Bernicot, Frey and Petermichl [2] showed following weighted bounds for sparse form.

Proposition 2.3. *Suppose $1 \le r < p < s \le \infty$ and $\mathscr{S}$ is a sparse collection. Then, it holds that*

$$\Lambda_{\mathscr{S},r,s'}(f,g) \lesssim ([\omega]_{A_{p/r}}[\omega]_{RH_{(s/p)'}})^{\alpha} ||f||_{L^p(\omega)} ||g||_{L^{p'}(\omega^{-p/p'})}$$

for any $\omega \in A_{p/r} \cap RH_{(s/p)'}$ where

$$\alpha = \max\left(\frac{1}{p-r}, \frac{s-1}{s-p}\right).$$

Let us prove Corollary 1.1.

Proof. (1) It is clear from Lemma 2.5.
(2) We take $\delta > 0$ such that

$$\omega \in A_{(-m/n-\delta)p}$$

and let $r = (-m/n - \delta)^{-1}$. By taking sufficiently small δ, one obtains $r < \min\{p, 2\}$. Furthermore, it holds that

$$-n/r = -n(-m/n - \delta) > m$$

which yields

$$Tf(x) \lesssim M_r f(x).$$

The statement follows from this.
(3) We can take $\delta > 0$ such that $\omega \in RH_{(2n/\{(n+2m)p\})'+\delta}$. Let s denote a number which satisfies

$$(s/p)' = (2n/\{(n+2m)p\})' + \delta,$$

and then it holds that

$$2 < s < \frac{2n}{n+2m}$$

which implies

$$m < -n(1/2 - 1/s).$$

From Theorem 1.1 and Remark 1.2, we obtain

$$|\langle Tf, g\rangle| \lesssim \Lambda_{\mathcal{S},2,s'}(f,g).$$

This bounds and Proposition 2.3 complete the proof. □

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