$U(\mathfrak{sl}_2)$ AND THE TERWILLIGER ALGEBRAS

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ABSTRACT. The universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is a unital associative algebra over $\mathbb{C}$ generated by E, F, H subject to the relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

In 2002, Junie T. Go showed that the Terwilliger algebra of $H(D, 2)$ is a homomorphic image of $U(\mathfrak{sl}_2)$. Firstly, I will present a connection of the even subalgebra of $U(\mathfrak{sl}_2)$ with the Terwilliger algebra of $\frac{1}{2}H(D, 2)$. Secondly, I will show how the Clebsch–Gordan rule of $U(\mathfrak{sl}_2)$ is related to the Terwilliger algebra of $H(D, q)$. Thirdly, I will give an algebraic connection between the Clebsch–Gordan coefficients of $U(\mathfrak{sl}_2)$ and the Terwilliger algebra of $J(D, k)$. The first part is a joint work with Chia-Yi Wen.

1. $U(\mathfrak{sl}_2)$ AND THE TERWILLIGER ALGEBRA OF $H(D, 2)$

Definition 1.1. The *universal enveloping algebra* $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is an algebra over $\mathbb{C}$ generated by E, F, H subject to the relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

The element

$$\Lambda = EF + FE + \frac{H^2}{2}$$

is called the *Casimir element* of $U(\mathfrak{sl}_2)$.

Lemma 1.2. *For any $n \in \mathbb{N}$ there exists an $(n + 1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module L_n satisfying the following conditions:*

(i) *There exists a basis $v_0^{(n)}, v_1^{(n)}, \ldots, v_n^{(n)}$ for L_n such that*

$$Ev_i^{(n)} = iv_{i-1}^{(n)} \quad (1 \leq i \leq n), \qquad Ev_0^{(n)} = 0,$$
$$Fv_i^{(n)} = (n - i)v_{i+1}^{(n)} \quad (1 \leq i \leq n - 1), \qquad Fv_n^{(n)} = 0,$$
$$Hv_i^{(n)} = (n - 2i)v_i^{(n)} \quad (1 \leq i \leq n).$$

(ii) *The element Λ acts on L_n as scalar multiplication by $\frac{n(n+2)}{2}$.*

Note that the $U(\mathfrak{sl}_2)$-module L_n is the unique $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module up to isomorphism.

Definition 1.3. Let $D \geq 1$ denote an integer. The *D-dimensional hypercube* $H(D, 2)$ has the vertex set $X = \{0, 1\}^D$ and $x, y \in \{0, 1\}^D$ are adjacent if and only if x and y differ in exactly one coordinate.

Let $\mathbf{A}$ denote the adjacency operator of $H(D, 2)$. Let $\mathbf{A}^*(x)$ denote the dual adjacency operator of $H(D, 2)$ with respect to $x \in X$. Let $\mathbf{T}(x)$ denote the Terwilliger algebra of $H(D, 2)$ with respect to $x \in X$ [1, 7–9]. Note that $\mathbf{T}(x)$ is generated by $\mathbf{A}$ and $\mathbf{A}^*(x)$. In 2002 Junie T. Go gave the following result:

Theorem 1.4 (Theorem 13.2, [2]). *For each $x \in X$ there exists a unique algebra homomorphism $\rho(x) : U(\mathfrak{sl}_2) \to \mathbf{T}(x)$ that sends*

$$\begin{aligned} E &\mapsto \frac{\mathbf{A}}{2} - \frac{[\mathbf{A}, \mathbf{A}^*(x)]}{4}, \\ F &\mapsto \frac{\mathbf{A}}{2} + \frac{[\mathbf{A}, \mathbf{A}^*(x)]}{4}, \\ H &\mapsto \mathbf{A}^*(x). \end{aligned}$$

Moreover $\rho(x)$ is onto for each $x \in X$.

Theorem 1.5 (Theorem 10.2, [2]). *The $U(\mathfrak{sl}_2)$-module $\mathbb{C}^X$ is isomorphic to*

$$\bigoplus_{i=0}^{\lfloor \frac{D}{2} \rfloor} \frac{D-2i+1}{D-i+1} \binom{D}{i} \cdot L_{D-2i}.$$

2. The even subalgebra of $U(\mathfrak{sl}_2)$ and the Terwilliger algebra of $\frac{1}{2}H(D,2)$

Definition 2.1 (Definition 1.2, [5]). The *universal Hahn algebra* $\mathcal{H}$ is an algebra over $\mathbb{C}$ generated by A, B, C and the relations assert that $[A, B] = C$ and each of

$$\begin{aligned} \alpha &= [C, A] + 2A^2 + B, \\ \beta &= [B, C] + 4BA + 2C \end{aligned}$$

is central in $\mathcal{H}$.

Theorem 2.2 (Theorem 1.3, [5]). *There exists a unique algebra homomorphism $\natural : \mathcal{H} \to U(\mathfrak{sl}_2)$ that sends*

$$\begin{aligned} A &\mapsto \frac{H}{4}, \\ B &\mapsto \frac{E^2 + F^2 + \Lambda - 1}{4} - \frac{H^2}{8}, \\ C &\mapsto \frac{E^2 - F^2}{4}, \\ \alpha &\mapsto \frac{\Lambda - 1}{4}, \\ \beta &\mapsto 0. \end{aligned}$$

The element

$$\Omega = 4ABA + B^2 - C^2 - 2\beta A + 2(1 - \alpha)B$$

is central in $\mathcal{H}$ and it is called the *Casimir element* of $\mathcal{H}$.

Lemma 2.3 (Lemma 4.5, [5]). *The homomorphism $\natural$ maps Ω to $\frac{3}{16}(2\Lambda - 3)$.*

The algebra $U(\mathfrak{sl}_2)$ has a $\mathbb{Z}$-grading algebra structure with

$$\deg E = 1, \qquad \deg F = -1, \qquad \deg H = 0.$$

For each $n \in \mathbb{Z}$ let U_n denote the n^{th} homogeneous subspace of $U(\mathfrak{sl}_2)$. Define

$$U(\mathfrak{sl}_2)_e = \bigoplus_{n \in \mathbb{Z}} U_{2n}.$$

Since $1 \in U_0$ and by **(G2)** the space $U(\mathfrak{sl}_2)_e$ is a subalgebra of $U(\mathfrak{sl}_2)$. We call $U(\mathfrak{sl}_2)_e$ the *even subalgebra of* $U(\mathfrak{sl}_2)$.

Theorem 2.4 (Theorem 3.4, [5]). *The algebra $U(\mathfrak{sl}_2)_e$ has a presentation given by generators E^2, F^2, Λ, H and relations*

$$\begin{aligned} [H, E^2] &= 4E^2, \\ [H, F^2] &= -4F^2, \\ 16E^2F^2 &= (H^2 - 2H - 2\Lambda)(H^2 - 6H - 2\Lambda + 8), \\ 16F^2E^2 &= (H^2 + 2H - 2\Lambda)(H^2 + 6H - 2\Lambda + 8), \\ \Lambda E^2 = E^2\Lambda, &\qquad \Lambda F^2 = F^2\Lambda, \qquad \Lambda H = H\Lambda. \end{aligned}$$

Using the presentation for $U(\mathfrak{sl}_2)_e$ we found the following result:

Theorem 2.5 (Theorem 1.5, [5]). (i) $\operatorname{Im} \natural = U(\mathfrak{sl}_2)_e$.
(ii) $\operatorname{Ker} \natural$ *is the two-sided ideal of $\mathcal{H}$ generated by β and $16\Omega - 24\alpha + 3$.*

For any $U(\mathfrak{sl}_2)$-module V and any $\theta \in \mathbb{C}$ let

$$V(\theta) = \{v \in V \mid Hv = \theta v\}.$$

Proposition 2.6 (Proposition 5.1, [5]). *Let V denote a $U(\mathfrak{sl}_2)$-module. Then*

$$\bigoplus_{n\in\mathbb{Z}} V(\theta + 4n)$$

is a $U(\mathfrak{sl}_2)_e$-submodule of V for any $\theta \in \mathbb{C}$.

For each $n \in \mathbb{N}$ let

$$L_n^{(0)} = \bigoplus_{i\in\mathbb{Z}} L_n(n - 4i).$$

For each integer $n \geq 1$ let

$$L_n^{(1)} = \bigoplus_{i\in\mathbb{Z}} L_n(n - 4i - 2).$$

Lemma 2.7 (Lemmas 5.5 and 5.8, [5]). (i) *For any $n \in \mathbb{N}$ the $U(\mathfrak{sl}_2)_e$-module $L_n^{(0)}$ is irreducible.*
(ii) *For any integer $n \geq 1$ the $U(\mathfrak{sl}_2)_e$-module $L_n^{(1)}$ is irreducible.*

Theorem 2.8 (Theorem 5.10, [5]). *The $U(\mathfrak{sl}_2)_e$-modules $L_n^{(0)}$ for all $n \in \mathbb{N}$ and the $U(\mathfrak{sl}_2)_e$-modules $L_n^{(1)}$ for all integers $n \geq 1$ are mutually non-isomorphic.*

Theorem 2.9 (Theorem 5.11, [5]). *For any $d \in \mathbb{N}$ the $U(\mathfrak{sl}_2)_e$-modules $L_{2d}^{(0)}$, $L_{2d+1}^{(0)}$, $L_{2d+1}^{(1)}$, $L_{2d+2}^{(1)}$ are all $(d+1)$-dimensional irreducible $U(\mathfrak{sl}_2)_e$-modules up to isomorphism.*

Lemma 2.10 (Lemma 6.2, [5]). *For each $x \in X$ the algebra homomorphism $\rho(x) \circ \natural : \mathcal{H} \to \mathbf{T}(x)$ maps*

$$\begin{aligned} A &\mapsto \frac{\mathbf{A}^*(x)}{4}, \\ B &\mapsto \frac{\mathbf{A}^2 - 1}{4}. \end{aligned}$$

Suppose that $D \geq 2$. Let

$$X_e = \left\{ x \in \{0,1\}^D \,\middle|\, \sum_{i=1}^{D} x_i \text{ is even} \right\}.$$

Definition 2.11. The halved graph $\frac{1}{2}H(D,2)$ of $H(D,2)$ is a finite simple connected graph with vertex set X_e and $x, y \in X_e$ are adjacent if and only if x and y differ in exactly two coordinates.

The adjacency operator of $\frac{1}{2}H(D,2)$ is equal to

$$\left.\frac{\mathbf{A}^2 - D}{2}\right|_{\mathbb{C}^{X_e}}.$$

Let $x \in X_e$ be given. The dual adjacency operator of $\frac{1}{2}H(D,2)$ with respect to x is equal to

$$\begin{cases} \frac{1}{2}\mathbf{A}^*(x)\big|_{\mathbb{C}^{X_e}} & \text{if } D = 2, \\ \mathbf{A}^*(x)\big|_{\mathbb{C}^{X_e}} & \text{if } D \geq 3. \end{cases}$$

Therefore the *Terwilliger algebra* $\mathbf{T}_e(x)$ *of* $\frac{1}{2}H(D,2)$ *with respect to* x is the subalgebra of $\mathrm{End}(\mathbb{C}^{X_e})$ generated by $\mathbf{A}^2|_{\mathbb{C}^{X_e}}$ and $\mathbf{A}^*(x)\big|_{\mathbb{C}^{X_e}}$ [1, 7–9].

Theorem 2.12 (Theorem 6.4, [5])**.** *For each $x \in X_e$ the following hold:*

(i) $\mathbf{T}_e(x) = \{M|_{\mathbb{C}^{X_e}} \mid M \in \mathrm{Im}\,(\rho(x) \circ \natural)\}$.

(ii) $\mathbf{T}_e(x) = \{M|_{\mathbb{C}^{X_e}} \mid M \in \mathrm{Im}\,(\rho(x)|_{U(\mathfrak{sl}_2)_e})\}$.

Theorem 2.13 (Theorem 6.5, [5])**.** *The $U(\mathfrak{sl}_2)_e$-module $\mathbb{C}^{X_e}$ is isomorphic to*

$$\bigoplus_{\substack{k=0 \\ k \text{ is even}}}^{\lfloor \frac{D}{2} \rfloor} \frac{D-2k+1}{D-k+1}\binom{D}{k} \cdot L^{(0)}_{D-2k} \oplus \bigoplus_{\substack{k=1 \\ k \text{ is odd}}}^{\lfloor \frac{D-1}{2} \rfloor} \frac{D-2k+1}{D-k+1}\binom{D}{k} \cdot L^{(1)}_{D-2k}.$$

3. The Clebsch–Gordan rule for $U(\mathfrak{sl}_2)$ and the Terwilliger algebra of $H(D,q)$

Definition 3.1 (Definition 1.6, [4])**.** Given any scalar $\omega \in \mathbb{C}$ the *Krawtchouk algebra* $\mathfrak{K}_\omega$ is an algebra over $\mathbb{C}$ generated by A and B subject to the relations

$$A^2B - 2ABA + BA^2 = B + \omega A,$$
$$B^2A - 2BAB + AB^2 = A + \omega B.$$

Theorem 3.2 ([4, 6])**.** *For any $\omega \in \mathbb{C}$ there exists a unique algebra homomorphism $\zeta : \mathfrak{K}_\omega \to U(\mathfrak{sl}_2)$ that sends*

$$\begin{aligned} A &\mapsto \tfrac{1+\omega}{2}E + \tfrac{1-\omega}{2}F - \tfrac{\omega}{2}H, \\ B &\mapsto \tfrac{1}{2}H, \\ C &\mapsto -\tfrac{1+\omega}{2}E + \tfrac{1-\omega}{2}F. \end{aligned}$$

Moreover, if $\omega^2 \neq 1$ then ζ is an isomorphism and its inverse sends

$$\begin{aligned} E &\mapsto \tfrac{1}{1+\omega}A + \tfrac{\omega}{1+\omega}B - \tfrac{1}{1+\omega}C, \\ F &\mapsto \tfrac{1}{1-\omega}A + \tfrac{\omega}{1-\omega}B + \tfrac{1}{1-\omega}C, \\ H &\mapsto 2B. \end{aligned}$$

Let $D \geq 1$ denote an integer. Let $q \geq 2$ denote an integer. Set

$$X = \{\hat{i} \mid i = 0, 1, \ldots, q-1\}.$$

Definition 3.3. The *D-dimensional Hamming graph $H(D,q)$ over X* has the vertex set X^D and $x, y \in X^D$ are adjacent if and only if x and y differ in exactly one coordinate.

Let $\mathbf{A}$ denote the adjacency operator of $H(D,q)$. Let $\mathbf{A}^*(x)$ denote the dual adjacency operator of $H(D,q)$ with respect to $x \in X^D$. Let $\mathbf{T}(x)$ denote the Terwilliger algebra of $H(D,q)$ with respect to x [1, 7–9]. Without loss of generality we fix $x = (\hat{0}, \hat{0}, \ldots, \hat{0}) \in X^D$. Set

$$\omega = 1 - \frac{2}{q}.$$

Definition 3.4. Let $\mathbb{C}_0^X$ denote the subspace of $\mathbb{C}^X$ consisting of all vectors $\sum_{i=1}^{q-1} c_i \hat{i}$ where $c_1, c_2, \ldots, c_{q-1} \in \mathbb{C}$ with $\sum_{i=1}^{q-1} c_i = 0$. Let $\mathbb{C}_1^X$ denote the subspace of $\mathbb{C}^X$ spanned by $\hat{0}$ and $\sum_{i=1}^{q-1} \hat{i}$. Note that $\mathbb{C}^X = \mathbb{C}_0^X \oplus \mathbb{C}_1^X$.

Definition 3.5. For any $s \in \{0,1\}^D$ we define the subspace $\mathbb{C}_s^{X^D}$ of $\mathbb{C}^{X^D}$ by

$$\mathbb{C}_s^{X^D} = \mathbb{C}_{s_1}^X \otimes \mathbb{C}_{s_2}^X \otimes \cdots \otimes \mathbb{C}_{s_D}^X.$$

Note that $\mathbb{C}^{X^D} = \bigoplus_{s \in \{0,1\}^D} \mathbb{C}_s^{X^D}$.

Proposition 3.6 (Proposition 3.12, [4]). *For any $s \in \{0,1\}^D$ there exists a $\mathfrak{K}_\omega$-module structure on $\mathbb{C}_s^{X^D}$ given by*

$$A = \frac{\mathbf{A}}{q}\Big|_{\mathbb{C}_s^{X^D}} + \frac{D}{q} - \frac{1}{2}\sum_{i=1}^{D} s_i,$$

$$B = \frac{\mathbf{A}^*(x)}{q}\Big|_{\mathbb{C}_s^{X^D}} + \frac{D}{q} - \frac{1}{2}\sum_{i=1}^{D} s_i.$$

In particular $\mathbb{C}^{X^D}$ is a $\mathfrak{K}_\omega$-module.

The Clebsch–Gordan rule for $U(\mathfrak{sl}_2)$ is as follows:

Theorem 3.7. *For any $m, n \in \mathbb{N}$ the $U(\mathfrak{sl}_2)$-module $L_m \otimes L_n$ is isomorphic to*

$$\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}.$$

The $U(\mathfrak{sl}_2)$-module $\mathbb{C}_0^X$ is isomorphic to $(q-2) \cdot L_0$. The $U(\mathfrak{sl}_2)$-module $\mathbb{C}_1^X$ is isomorphic to L_1. Hence the $U(\mathfrak{sl}_2)$-module $\mathbb{C}_s^{X^D}$ ($s \in \{0,1\}^D$) is isomorphic to $(q-2)^{D-p} \cdot L_1^{\otimes p}$ where $p = \sum_{i=1}^{D} s_i$.

Theorem 3.8 (Theorem 1.10, [4]). *The $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{X^D}$ is isomorphic to*

$$\bigoplus_{p=0}^{D} \bigoplus_{k=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p-2k+1}{p-k+1} \binom{D}{p} \binom{p}{k} (q-2)^{D-p} \cdot L_{p-2k}.$$

Here 0^0 is defined as 1.

4. The Clebsch–Gordan coefficients for $U(\mathfrak{sl}_2)$ and the Terwilliger algebra of $J(D,k)$

Inspired by the Clebsch–Gordan coefficients for $U(\mathfrak{sl}_2)$ the following result was discovered in [3]:

Theorem 4.1 (Theorem 1.4, [3])**.** *There exists a unique algebra homomorphism $\natural : \mathcal{H} \to U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$ that sends*

$$\begin{aligned} A &\mapsto \frac{H\otimes 1 - 1\otimes H}{4}, \\ B &\mapsto \frac{\Delta(\Lambda)}{2}, \\ C &\mapsto E\otimes F - F\otimes E, \\ \alpha &\mapsto \frac{\Lambda\otimes 1 + 1\otimes\Lambda}{2} + \frac{\Delta(H)^2}{8}, \\ \beta &\mapsto \frac{(\Lambda\otimes 1 - 1\otimes\Lambda)\Delta(H)}{2}. \end{aligned}$$

By pulling back via $\natural$ every $U(\mathfrak{sl}_2)\otimes U(\mathfrak{sl}_2)$-module can be considered as an $\mathcal{H}$-module. Let V denote a $U(\mathfrak{sl}_2)\otimes U(\mathfrak{sl}_2)$-module. For any $\theta\in\mathbb{C}$ we define

$$V(\theta) = \{v \in V \mid \Delta(H)v = \theta v\}.$$

It can be shown that $V(\theta)$ is an $\mathcal{H}$-submodule of V for any $\theta\in\mathbb{C}$.

Theorem 4.2 (Theorem 1.6, [3])**.** *Suppose that $m,n\in\mathbb{N}$ and ℓ is an integer with $0\le\ell\le m+n$. Then the following hold:*

- (i) *The $(\min\{m,\ell\}+\min\{n,\ell\}-\ell+1)$-dimensional $\mathcal{H}$-module $(L_m\otimes L_n)(m+n-2\ell)$ is irreducible.*
- (ii) *Suppose that $m',n'\in\mathbb{N}$ and ℓ' is an integer with $0\le\ell'\le m'+n'$. The the $\mathcal{H}$-module $(L_{m'}\otimes L_{n'})(m'+n'-2\ell')$ is isomorphic to $(L_m\otimes L_n)(m+n-2\ell)$ if and only if*

$$(m',n',\ell')\in\{(m,n,\ell),(m+n-\ell,\ell,n),(\ell,m+n-\ell,m),(n,m,m+n-\ell)\}.$$

Let Ω denote a finite set with size D and let $\lessdot$ denote the covering relation in 2^Ω.

Theorem 4.3. *There exists a $U(\mathfrak{sl}_2)$-module structure on $\mathbb{C}^{2^\Omega}$ given by*

$$\begin{aligned} Ex &= \sum_{y\lessdot x} y \qquad \textit{for all } x\in 2^\Omega, \\ Fx &= \sum_{x\lessdot y} y \qquad \textit{for all } x\in 2^\Omega, \\ Hx &= (D-2|x|)x \qquad \textit{for all } x\in 2^\Omega. \end{aligned}$$

For notational convenience we define

$$m_i(n) = \frac{n-2i+1}{n-i+1}\binom{n}{i}$$

for all integers i,n with $0\le i\le\lfloor\frac{n}{2}\rfloor$.

Theorem 4.4. *The $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^\Omega}$ is isomorphic to*

$$\bigoplus_{i=0}^{\lfloor \frac{D}{2} \rfloor} m_i(D) \cdot L_{D-2i}.$$

Fix an element $x_0 \in 2^\Omega$. The spaces $\mathbb{C}^{2^{\Omega \setminus x_0}}$ and $\mathbb{C}^{2^{x_0}}$ are $U(\mathfrak{sl}_2)$-modules. Hence $\mathbb{C}^{2^{\Omega \setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ has a $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$-module structure. Consider the linear isomorphism $\iota(x_0) : \mathbb{C}^{2^\Omega} \to \mathbb{C}^{2^{\Omega \setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ given by

$$x \quad \mapsto \quad (x \setminus x_0) \otimes (x \cap x_0) \qquad \text{for all } x \in 2^\Omega.$$

By identifying $\mathbb{C}^{2^\Omega}$ with $\mathbb{C}^{2^{\Omega \setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ via $\iota(x_0)$, this induces a $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$-module structure on $\mathbb{C}^{2^\Omega}$.

Lemma 4.5 (Lemma 5.5, [3])**.** *The $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^\Omega}$ is isomorphic to*

$$\bigoplus_{i=0}^{\lfloor \frac{D-|x_0|}{2} \rfloor} \bigoplus_{j=0}^{\lfloor \frac{|x_0|}{2} \rfloor} m_i(D-|x_0|)\, m_j(|x_0|) \cdot L_{D-|x_0|-2i} \otimes L_{|x_0|-2j}.$$

Theorem 4.6 (Theorem 5.8, [3])**.** *For any $x_0 \in 2^\Omega$ the actions of A and B on the $\mathcal{H}$-module $\mathbb{C}^{2^\Omega}$ are as follows:*

$$Ax = \left(\frac{D}{4} - \frac{|x_0 \setminus x| + |x \setminus x_0|}{2} \right) x \qquad \text{for all } x \in 2^\Omega,$$

$$Bx = \left(\frac{D}{2} + \frac{(D-2|x|)^2}{4} \right) x + \sum_{\substack{|y|=|x| \\ x \cap y \subset x}} y \qquad \text{for all } x \in 2^\Omega.$$

Let k denote an integer with $0 \leq k \leq D$. The notation $\binom{\Omega}{k}$ denotes the set of all k-element subsets of Ω. It follows from the above theorem that $\mathbb{C}^{\binom{\Omega}{k}}$ is an $\mathcal{H}$-submodule of $\mathbb{C}^{2^\Omega}$. Let

$$\mathbf{P}(k) = \left\{ (i,j) \in \mathbb{Z}^2 \,\middle|\, 0 \leq i \leq \frac{D-k}{2}, 0 \leq j \leq \min \left\{ D-k-i, k-i, \frac{k}{2} \right\} \right\}.$$

Theorem 4.7 (Theorem 5.7, [3])**.** *Suppose that k is an integer with $0 \leq k \leq D$. For any $x_0 \in \binom{\Omega}{k}$ the following statements hold:*

(i) *Suppose that $k \neq \frac{D}{2}$. Then the $\mathcal{H}$-module $\mathbb{C}^{\binom{\Omega}{k}}$ is isomorphic to*

$$\bigoplus_{(i,j) \in \mathbf{P}(k)} m_i(D-k)\, m_j(k) \cdot (L_{D-k-2i} \otimes L_{k-2j})(D-2k).$$

Moreover the irreducible $\mathcal{H}$-modules $(L_{D-k-2i} \otimes L_{k-2j})(D-2k)$ for all $(i,j) \in \mathbf{P}(k)$ are mutually non-isomorphic.

(ii) *Suppose that $k = \frac{D}{2}$. Then the $\mathcal{H}$-module $\mathbb{C}^{\binom{\Omega}{k}}$ is isomorphic to*

$$\bigoplus_{i=0}^{\lfloor \frac{D}{4} \rfloor} m_i \left(\frac{D}{2} \right)^2 \cdot (L_{\frac{D}{2}-2i} \otimes L_{\frac{D}{2}-2i})(0)$$

$$\oplus \bigoplus_{i=0}^{\lfloor \frac{D}{4} \rfloor} \bigoplus_{j=i+1}^{\lfloor \frac{D}{4} \rfloor} 2m_i \left(\frac{D}{2}\right) m_j \left(\frac{D}{2}\right) \cdot (L_{\frac{D}{2}-2i} \otimes L_{\frac{D}{2}-2j})(0).$$

Now we assume that $D \geq 2$ and k is an integer with $1 \leq k \leq D-1$.

Definition 4.8. The *Johnson graph* $J(D,k)$ is a finite simple connected graph whose vertex set is $\binom{\Omega}{k}$ and two vertices x, y are adjacent whenever $x \cap y \Subset x$.

The adjacency operator $\mathbf{A}$ of $J(D,k)$ is a linear endomorphism of $\mathbb{C}^{\binom{\Omega}{k}}$ given by

$$\mathbf{A}x = \sum_{\substack{|x|=|y| \\ x \cap y \Subset x}} y \qquad \text{for all } x \in \tbinom{\Omega}{k}.$$

The dual adjacency operator $\mathbf{A}^*(x_0)$ of $J(D,k)$ with respect to $x_0 \in \binom{\Omega}{k}$ is a linear endomorphism of $\mathbb{C}^{\binom{\Omega}{k}}$ given by

$$\mathbf{A}^*(x_0)x = (D-1)\left(1 - \frac{D(|x_0 \setminus x| + |x \setminus x_0|)}{2k(D-k)}\right)x$$

for all $x \in \binom{\Omega}{k}$. Let $\mathbf{T}(x_0)$ denote the Terwilliger algebra of $J(D,k)$ with respect to x_0 [1,7–9].

Theorem 4.9 (Theorem 5.9, [3])**.** *For any $x_0 \in \binom{\Omega}{k}$ the following equation holds:*

$$\mathbf{T}(x_0) = \operatorname{Im}\left(\mathcal{H} \to \operatorname{End}(\mathbb{C}^{\binom{\Omega}{k}})\right).$$

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