Classification of the tight Euclidean 5-designs in $\mathbb{R}^2$

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1 Introduction

The concept of Euclidean t-design stated below was defined in the paper by Neumaier and Seidel in 1988 ([10]) for a weighted finite set (X, w) in the n dimensional Euclidean space $\mathbb{R}^n$.

Definition 1 (Euclidean design, Neumier-Seidel (1988) [10]) *Let (X, w) be a weighted finite set with a positive weight function w defined on X. Assume X is supported by a union $S = S_1 \cup \cdots \cup S_p$ of p concentric spheres of $\mathbb{R}^n$ centered at the origin. Let r_i be the radius of S_i for $1 \le i \le p$. Then we call (X, w) a Euclidean t-design if the following condition*

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{x \in S_i} f(x) d\sigma_i(x) = \sum_{x \in X} w(x) f(x) \tag{1.1}$$

is satisfied for any polynomial $f(x_1, x_2, \ldots, x_n)$ with n variable $x_1, x_2, \ldots, x_n$ of degree at most t. Here we define $w(X_i) = \sum_{x \in X_i} w(x)$ and $|S_i|$ is the surface area of S_i for $1 \le i \le p$.

Delsarte and Seidel ([7]) studied more precise properties of Euclidean design on a union of several spheres centered at the origin. For more information please look at the articles [3] (Ei. Bannai, Et. Bannai, M. Hirao and M. Sawa), [4] (Ei. Bannai, Et. Bannai, D. Suprijanto), and [5] (Et. Bannai). They developed the arguments farther and gave more interesting examples.

In this talk we try to give the classification of tight Euclidean 5-design of two dimensional Euclidean space $\mathbb{R}^2$.

First we consider the general situation, i.e., (X, w) is a weighted finite set of $\mathbb{R}^n$. For a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we define norm of x by $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Let X be a finite set in $\mathbb{R}^n$ supported by p concentric spheres $S_1, \ldots, S_p$ centered at the origin $\mathbf{0}$ with positive radius $r_1, r_2, \ldots, r_p$ respectively. Let $w(x)$, $x \in X$, be a positive weight function defined on X. Thus we consider a positive weighted finite set (X, w) supported by a union of p concentric spheres. Let $S = S_1 \cup \cdots \cup S_p$. Let $\varepsilon_S \in \{0, 1\}$ be defined by $\varepsilon_S = 1$ if $0 \in S$ and $\varepsilon_S = 0$ otherwise.

Let $\mathcal{P}(\mathbb{R}^n) = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be the vector space of polynomials in n variables $x_1, x_2, \ldots, x_n$ over the field of real numbers. Let $\mathrm{Hom}_\ell(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of homogeneous polynomials of degree ℓ. Let $\mathcal{P}_\ell(\mathbb{R}^n) = \bigoplus_{i=0}^{\ell} \mathrm{Hom}_i(\mathbb{R}^n)$. Let $\mathrm{Harm}(\mathbb{R}^n)$ be the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of all the harmonic polynomials. Let $\mathrm{Harm}_\ell(\mathbb{R}^n) = \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_\ell(\mathbb{R}^n)$. Let $\mathcal{P}_\ell^*(\mathbb{R}^n) = \bigoplus_{\substack{i \equiv \ell (2), \\ 0 \le i \le \ell}} \mathrm{Hom}_i(\mathbb{R}^n)$. Let $\mathcal{P}(S)$, $\mathcal{P}_\ell(S)$, $\mathrm{Hom}_\ell(S)$, $\mathrm{Harm}(S)$, $\mathrm{Harm}_\ell(S)$ and $\mathcal{P}_\ell^*(S)$ be the sets of corresponding polynomials restricted to the union S. For example $\mathcal{P}(S)=\{f\mid_S\ \mid f\in\mathcal{P}(\mathbb{R}^n)\}$.

A finite subset $X\subset\mathbb{R}^n$ is said to be antipodal if $-x\in X$ holds for any $x\in X$. Let X^* be a subset of X satisfying $X=X^*\cup(-X^*), X^*\cap(-X^*)=\emptyset$, where $-X^*=\{-x\mid x\in X^*\}$. Also we define $A(X)=\{\ \|x-y\|\mid x,y\in X,\ x\neq y\}$.

The following basic facts are well known ([6], [7], see also [1], [2], [5]):

$$\dim(\mathcal{P}_e(\mathbb{R}^n))=\binom{n+e}{e},\qquad \dim(\mathcal{P}_e^*(\mathbb{R}^n))=\sum_{i=0}^{[\frac{e}{2}]}\binom{n+e-2i-1}{e-2i}$$

$$\dim(\mathcal{P}_e(S))=\varepsilon_S+\sum_{i=0}^{2(p-\varepsilon_S)-1}\binom{n+e-i-1}{e-i}<\binom{n+e}{e}\ \text{for}\ p\leq[\tfrac{e+\varepsilon_S}{2}],$$

$$\dim(\mathcal{P}_e(S))=\sum_{i=0}^{e}\binom{n+e-i-1}{e-i}=\binom{n+e}{e},\ \text{for}\ p\geq[\tfrac{e+\varepsilon_S}{2}]+1$$

To study Euclidean designs the following theorem, proved by Neumier and Seidel, is most fundamental and important.

Theorem 2 (Neumier and Seidel ([10],1988)) *Let (X,w) be a weighted finite subset in $\mathbb{R}^n$ which may possibly contain the origin, $\mathbf{0}\in X$. Then the following (1) and (2) are equivalent:*

(1) *(X,w) is a Euclidean t-design.*

(2) *For any polynomial $f\in\|x\|^{2j}Harm_\ell(\mathbb{R}^n)$ with $1\leq\ell\leq t$, $0\leq j\leq[\frac{t-\ell}{2}]$ the following holds:*

$$\sum_{u\in X}w(u)f(u)=0.$$

Before Neumier and Seidel gave the definition of Euclidean design as in above, there were works on cubature formulas in analysys. Among them were works by H. M. Möller. He studied cubature formulas and gave a lower bound of the number of points contained in a cubature formula. The following is a well known theorem proved by Möller written in terms of Euclidean designs.

Theorem 3 (Möller [8, 9] (1976, 1979)) *Let (X,w) be a Euclidean t-design in $\mathbb{R}^n$ supported by a union S of p concentric spheres centered at the origin. Then the followings hold.*

(1) If $t=2e$, then $|X|\geq\dim(\mathcal{P}_e(S))$.

(2) If $t=2e+1$ and e is odd, or e is even and $0\notin X$, then $|X|\geq 2\dim(\mathcal{P}_e^*(S))$,

(3) If $t=2e+1$ and e is even and $0\in X$, then $|X|\geq 2\dim(\mathcal{P}_e^*(S))-1$.

Definition 4 *If an equality holds in the above condition, then we say the weighted pair (X,w) is a tight Euclidean t-design on a union of p concentric spheres.*

Remark *For a Euclidean t-design (X, w), if origin $0 \notin X$, then. consider $\{0\}$ as a circle of radius 0 centered at the origin and define the weight with $w(0) = \alpha$, with a positive real number α. Then it is easy to see that $(X \cup \{0\}, w)$ is also a Euclidean t-design. Hence in the following we consider the case when Euclidean t-design is supported by a union of spheres with positive radii.*

For the case t is an odd integer, $2e + 1$, it is known that any tight Euclidean $(2e + 1)$-design (X, w) is antipodal and its weight function is constant on each shell. For more information see the papers [3], by Eiichi Bannai, Etsuko Bannai, Hirao and Sawa, and also [8] and [9], by Möller.

Bannai-Bannai-Suprijanto [4] proved that tight Euclidean designs are not rigid so that they are deformable. So there exist infinitely many Euclidean tight designs if the number of the sphere supporting them are large. So from the combinatorial point of view it maybe interesting to study for the cases $p \leq [\frac{e+\varepsilon_S}{2}] + 1$. That means just up to when p just attains $\dim(\mathcal{P}_e(S)) = \binom{n+e}{e} = \dim(\mathcal{P}_e(\mathbb{R}^n))$.

In this talk we present the classification of Euclidean tight 5-design (X, w) of $\mathbb{R}^2$. If X is supported by a unit circle, then X is a spherical 5-design and $|X| \geq 6$. If $|X| = 6$, then X is the set of the vertices of regular 6 gon inscribed in a circle, which is a tight spherical 5-design. For the case X is supported by a union of two concentric circles, it was shown that X consists of 8 vertices of a union of two squares. So, if $p = 2$, $|X|$ attains the maximal cardinality.

In the following we go back to the general situation and give the classification of tight Euclidean 5-design (X, w) of $\mathbb{R}^2$ with $0 \notin X$. By Theorem 3 we know that $|X| = 2\dim(\mathcal{P}_2^*(\mathbb{R}^2) = 8$ and X consists of four antipodal pairs. Thus the number of circles supporting X is at most 4. Without loss of generality we may assume one of them is the unit circle and $(1, 0)$, $(-1, 0) \in X$. To investigate the structure of tight 5-designs we need to use explicit information of the vector space of harmonic polinpmials. $\dim(\mathrm{Harm}_i(\mathbb{R}^2)) = 2$ and it is well known that the basis of $\mathrm{Harm}_i(\mathbb{R}^2)$ for $i = 1, 2, \ldots, 5$ are given as below.

$$
\begin{aligned}
&\mathrm{Harm}_1(\mathbb{R}^2) : x_1, x_2; \qquad \mathrm{Harm}_2(\mathbb{R}^2) : x_1x_2, x_1^2 - x_2^2;\\
&\mathrm{Harm}_3(\mathbb{R}^2) : x_1^3 - 3x_1x_2^2, x_2^3 - 3x_1^2x_2 : \quad \mathrm{Harm}_4(\mathbb{R}^2) : x_1^4 + x_2^4 - 6x_1^2x_2^2, x_1x_2^3 - x_1^3x_2;\\
&\mathrm{Harm}_5(\mathbb{R}^2) : x_1^5 - 10x_1^3x_2^2 + 5x_1x_2^4, 5x_1^4x_2 - 10x_1^2x_2^3 + x_2^5 :
\end{aligned}
$$

For the case $p = 1$, it is known that X is similar to the set of the six vertices of the regular 6-gon inscribed in the unit circle. For the case $p = 2$, it is shown in [5], that X is similar to a set of 8 vertices of a union of 2 squares as given below

$$
\begin{aligned}
&X = X_1 \cup X_2, \quad X_1 = \{\pm(1, 0),\ \pm(0, 1)\},\\
&X_2 = \left\{\pm r_2 \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ \pm r_2 \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}, \qquad r_2 > 0,\ r_2 \neq 1.
\end{aligned}
$$

Thus X_1 is on the unit circle and X_2 is on the circle of radius $r_2 \neq 1$. The weight function is defined by $w(x) = 1$ for $x \in X_1$ and $w(x) = \frac{1}{r_2^4}$ for $x \in X_2$.

In the following we give the classification for the case $p = 3$ and $p = 4$.

2 Euclidean tight 5-design of $\mathbb{R}^2$ on 3 circles

In this subsection we will give the classification of the Euclidean tight **5**-design for the case $p = 3$. Assume that a finite weighted set (X, w) is a tight Euclidean 5-design of $\mathbb{R}^2$ supported by 3 circles. It is known that X is an antipodal set and $|X| = 2\dim(P_2(\mathbb{R}^2)) = 8$ (see [5]). We may assume $X = X_1 \cup X_2 \cup X_3$, where X_1 is a 4 points set on the unit circle of $\mathbb{R}^2$ and $|X_2| = |X_3| = 2$. So without loss of generality we may assume

$$\begin{aligned}
X_1 &= \left\{(1,0),\ (-1,0),\ \left(a_1, \sqrt{1-a_1^2}\right), \left(-a_1, -\sqrt{1-a_1^2}\right)\right\}\\
X_2 &= \{(a_2,b_2), (-a_2,-b_2)\},\\
X_3 &= \{(a_3,b_3), (-a_3,-b_3)\},
\end{aligned}$$

where $1 > a_1 \geq 0,\ a_2,\ a_3 \geq 0,\ a_i^2 + b_i^2 \neq 1\ (i = 2,\ 3),\ a_2^2 + b_2^2 > a_3^2 + b_3^2 > 0$. By definition, (X, w) is a Euclidean 5-design if and only if (X, w) satisfies the following equations

$$\sum_{x \in X_1} h(x_1,x_2) + w_2 \sum_{x \in X_2} \|x\|^{2j} h(x_1,x_2) + w_3 \sum_{x \in X_3} \|x\|^{2j} h(x_1,x_2) = 0, \tag{2.1}$$

for $x = (x_1, x_2)$, $\|x\|^2 = x_1^2 + x_2^2$, $h(x_1, x_2) \in \mathrm{Harm}_\ell(\mathbb{R}^2)$, $2j + \ell \leq 5$, $\ell \geq 1$. Then, since X is antipodal, (2.1) is automatically satisfied by any harmonic polynomial $h(x_1, x_2)$ of odd order. Hence we only need to check harmonic polynomials of even order, i.e., harmonic polynomials $h(x_1, x_2) = x_1x_2,\ x_1x_2\|x\|^2,\ x_1^2 - x_2^2,\ (x_1^2 - x_2^2)\|x\|^2,\ x_1^4 + x_2^4 - 6x_1^2x_2^2,\ x_1x_2^3 - x_1^3x_2$. Then we can prove the following Lemma 5 and Lemma 6.

Lemma 5 *Notations and definitions are given above, a positive weighted set (X, w) is a tight Euclidean 5-design of $\mathbb{R}^2$ supported by three circles if and only if the following conditions (1), (2), (3), (4), (5) and (6) hold.*

(1) $w_2a_2b_2 + w_3a_3b_3 + a_1\sqrt{1-a_1^2} = 0.$

(2) $a_2b_2(a_2^2 + b_2^2)w_2 + a_3b_3(a_3^2 + b_3^2)w_3 + a_1\sqrt{1-a_1^2} = 0.$

(3) $(a_2^2 - b_2^2)w_2 + (a_3^2 - b_3^2)w_3 + 2a_1^2 = 0.$

(4) $(a_2^4 - b_2^4)w_2 + (a_3^4 - b_3^4)w_3 + 2a_1^2 = 0.$

(5) $(a_2^4 - 6a_2^2b_2^2 + b_2^4)w_2 + (a_3^4 - 6a_3^2b_3^2 + b_3^4)w_3 + 2(4a_1^4 - 4a_1^2 + 1) = 0.$

(6) $a_2b_2(a_2^2 - b_2^2)w_2 + a_3b_3(a_3^2 - b_3^2)w_3 - a_1(1 - 2a_1^2)\sqrt{1-a_1^2} = 0.$

Lemma 6 *Definition and notation are as defined above. Let (X, w) be a tight Euclidean 5-design of $\mathbb{R}^2$ supported by 3 circles as given above. Then the followings hold.*

(1) $1 > a_1 > 0$.

(2) $a_2 > 0$ *and* $b_2 \neq 0$.

(3) $a_3 > 0$ and $b_3 \neq 0$.

Then we can show the following lemma which describes the property of tight 5-design of $\mathbb{R}^2$ supported by three circles.

Lemma 7 *Notations and definitions are given as above. Let (X, w) be a weighted subset of $\mathbb{R}^2$ supported by three circles. Then the following conditions (1) and (2) hold:*

(1) The weights w_2 and w_3 are given as below.

$$w_2 = \frac{a_1(a_3^2 + b_3^2 - 1)\sqrt{1 - a_1^2}}{a_2 b_2 (a_2^2 + b_2^2 - a_3^2 - b_3^2)}, \tag{2.2}$$

$$w_3 = -\frac{a_1(a_2^2 + b_2^2 - 1)\sqrt{1 - a_1^2}}{a_3 b_3 (a_2^2 + b_2^2 - a_3^2 - b_3^2)}. \tag{2.3}$$

(2) $(a_2 a_3 + b_2 b_3)(a_2 b_3 - a_3 b_2) = 0$.

According to Lemma 6 (2), any Euclidean tight 5-design of $\mathbb{R}^2$ supported by 3 circles must satisfy either $a_2 a_3 + b_2 b_3 = 0$ or $a_2 b_3 - a_3 b_2 = 0$. In the following Subsection 2.1 and Subsection 3.1.2 we disscuss the case $a_2 a_3 + b_2 b_3 = 0$ and the case $a_2 b_3 - a_3 b_2 = 0$ respectively.

2.1 Case $a_2 a_3 + b_2 b_3 = 0$

In this subsection we consider the case when $a_2 a_3 + b_2 b_3 = 0$ holds. So we have $b_3 = -\frac{a_2 a_3}{b_2}$. Since $\frac{a_2}{b_2}\frac{a_3}{b_3} = -1$, in this case, the line passing through the point (a_2, b_2) and the origin $(0, 0)$ and the line passing through the point (a_3, b_3) and the origin $(0, 0)$ are perpendicular to each other.

Since $a_1,\ a_2,\ a_3 > 0$, $b_2 b_3 < 0$ holds and we assumed $a_2^2 + b_2^2 > a_3^2 + b_3^2 > 0$, (2.2) implies $b_2(a_3^2 + b_3^2 - 1) > 0$ and (2.3) implies $b_3(a_2^2 + b_2^2 - 1) < 0$. So we have the following conditions between the parameters.

If $1 > a_2^2 + b_2^2 > a_3^2 + b_3^2 > 0$, then we have $b_2 < 0$ and $b_3 > 0$.

If $a_2^2 + b_2^2 > 1 > a_3^2 + b_3^2 > 0$, then we have $b_2 < 0$ and $b_3 < 0$.

If $a_2^2 + b_2^2 > a_3^2 + b_3^2 > 1$, then we have $b_2 > 0$ and $b_3 < 0$.

By assumption $b_2 b_3 = -a_2 a_3 < 0$ holds, hence we have the following two cases:

case: $1 > a_2^2 + b_2^2 > a_3^2 + b_3^2 > 0$, $b_2 < 0$, $b_3 > 0$

and

case: $a_2^2 + b_2^2 > a_3^2 + b_3^2 > 1$, $b_2 > 0$, $b_3 < 0$.

We also have $w_2 = \frac{a_1(a_3^2+b_3^2-1)\sqrt{1-a_1^2}}{a_2 b_2(a_2^2+b_2^2-a_3^2-b_3^2)}$ and $w_3 = \frac{a_1(1-a_2^2-b_2^2)\sqrt{1-a_1^2}}{a_3 b_3(a_2^2+b_2^2-a_3^2-b_3^2)}$ (see (2.2) and (2.3) respectively). Then (2.1) with $j = 0$ and $h(x_1, x_2) = x_1^2 - x_2^2$, implies

$$(b_2^2 - a_2^2)\sqrt{1 - a_1^2} + 2a_1 a_2 b_2 = 0. \tag{2.4}$$

Hence we have $b_2 = a_2\sqrt{\frac{1-a_1}{1+a_1}}$ or $-a_2\sqrt{\frac{1+a_1}{1-a_1}}$. Note that in this case we assumed that $a_2 a_3 + b_2 b_3 = 0$ holds.

Case: $a_2a_3+b_2b_3=0,\ b_2=a_2\sqrt{\frac{1-a_1}{1+a_1}},\ (a_2,\ a_3>0,\ 1>a_1>0,\ b_2>0,\ b_3<0)$

In this case we will show that we have the following design:

Theorem 8 *Assume $a_2>0,\ 1>a_1>0,\ b_2>0,\ a_2a_3+b_2b_3=0$ and $b_2=a_2\sqrt{\frac{1-a_1}{1+a_1}}$, then we have the following design:*

$(X,w),\ X=X_1\cup X_2\cup X_3$, *where*

$$X_1=\left\{(1,0),(-1,0),(a_1,\sqrt{1-a_1^2}),(-a_1,-\sqrt{1-a_1^2})\right\},$$
$$X_2=\{(a_2,b_2),\ (-a_2,-b_2)\},\quad b_2=a_2\sqrt{\frac{1-a_1}{1+a_1}},$$
$$X_3=\{(a_3,b_3),\ (-a_3,-b_3)\},$$
$$a_3=a_2(1-a_1)\sqrt{\frac{2a_1+1}{4a_1a_2^2+1-2a_1^3-3a_1^2}},\quad b_3=-\frac{a_2\sqrt{(1-a_1^2)(2a_1+1)}}{\sqrt{4a_1a_2^2-(2a_1+3)a_1^2+1}},$$
$$w_2=\frac{(a_1+1)^3(1-2a_1)}{4a_2^4},\quad w_3=\frac{\left\{(1-2a_1)(a_1+1)^2+4a_1a_2^2\right\}^2}{4(1-a_1)(2a_1+1)a_2^4},$$

Hence we must have $0<a_1<\frac{1}{2}$, $a_2>0$,

$$(2a_1^2-1)\left\{\left((2a_1-1)(a_1+1)^2-4a_1a_2^2\right)a_3^2+a_2^2(2a_1+1)(a_1-1)^2\right\}=0. \tag{2.5}$$

First we will show that $a_1\neq\frac{1}{\sqrt{2}}$ holds. So we assume $a_1=\frac{1}{\sqrt{2}}$. Then Lemma 7 (6) implies

$$\{8a_2^2-(2+\sqrt{2})\}a_3^2-(2-\sqrt{2})a_2^2=0. \tag{2.6}$$

Since $a_2,\ a_3>0$, we have $8a_2^2-(2+\sqrt{2})>0$ and

$$a_3=a_2\sqrt{\frac{2-\sqrt{2}}{8a_2^2-(2+\sqrt{2})}}. \tag{2.7}$$

However in this case (2.2) implies

$$w_2=-\frac{4+3\sqrt{2}}{16a_2^4}\ <\ 0, \tag{2.8}$$

which is a contradiction. Thus we must have $a_1\neq\frac{1}{\sqrt{2}}$ and

$$((2a_1-1)(a_1+1)^2-4a_1a_2^2)a_3^2+a_2^2(2a_1+1)(a_1-1)^2=0 \tag{2.9}$$

holds. Since $0<a_1<1$, (2.9) implies $(2a_1-1)(a_1+1)^2-4a_1a_2^2<0$ and we have

$$a_3=a_2(1-a_1)\sqrt{\frac{2a_1+1}{4a_1a_2^2-(2a_1-1)(a_1+1)^2}}. \tag{2.10}$$

By assumption $b_3 = -\frac{a_2 a_3}{b_2}$, hence we have

$$b_3 = -a_2 \sqrt{\frac{(2a_1+1)(1-a_1^2)}{4a_1 a_2^2 - (2a_1+3)a_1^2 + 1}}. \tag{2.11}$$

Then (2.2) and (2.3) imply

$$w_2 = \frac{(a_1+1)^3(1-2a_1)}{4a_2^4} \quad \text{and} \quad w_3 = \frac{\left\{(1-2a_1)(a_1+1)^2 + 4a_1 a_2^2\right\}^2}{4(1-a_1)(2a_1+1)a_2^4}. \tag{2.12}$$

Since $w_2 > 0$, we must have $0 < a_1 < \frac{1}{2}$. This completes the proof. ∎

Case: $a_2 a_3 + b_2 b_3 = 0$, $b_2 = -\sqrt{\frac{1+a_1}{1-a_1}} a_2$, $(a_2,\ a_3 > 0,\ b_2 < 0, 1 > a_1 > 0)$

In this case we have the following theorem.

Theorem 9 *Assume $1 > a_1 > 0$, $a_2 > 0$, $a_2 a_3 + b_2 b_3 = 0$, and $b_2 = -\sqrt{\frac{1+a_1}{1-a_1}} a_2$, then (X, w) is similar to the following design.*

$$\begin{aligned}
X_1 &= \left\{(1,0), (-1,0), \left(a_1, \sqrt{1-a_1^2}\right), \left(-a_1, -\sqrt{1-a_1^2}\right)\right\}, \\
X_2 &= \left\{a_2\left(1, -\sqrt{\frac{1+a_1}{1-a_1}}\right), a_2\left(-1, \sqrt{\frac{1+a_1}{1-a_1}}\right)\right\}, \quad a_2 > 0 \\
X_3 &= \left\{a_3\left(1, \sqrt{\frac{1-a_1}{1+a_1}}\right),\ -a_3\left(1, \sqrt{\frac{1-a_1}{1+a_1}}\right)\right\}, \\
& \qquad\qquad a_3 = a_2(1+a_1)\sqrt{\frac{1-2a_1}{(2a_1+1)(a_1-1)^2 - 4a_1 a_2^2}}, \\
w_2 &= \frac{(1-a_1)^3(2a_1+1)}{4a_2^4}, \quad w_3 = \frac{\left(4a_1 a_2^2 - (2a_1+1)(a_1-1)^2\right)^2}{4a_2^4(1-2a_1)(1+a_1)}, \\
0 &< a_1 < \frac{1}{2},\ 0 < a_2 < \sqrt{\frac{(2a_1+1)(1-a_1)^2}{4a_1}}.
\end{aligned} \tag{2.13}$$

Proof: By assumption we have $b_3 = -\frac{a_2 a_3}{b_2}$ and $b_2 = -a_2\sqrt{\frac{1+a_1}{1-a_1}}$, hence $b_3 = a_3\sqrt{\frac{1-a_1}{1+a_1}}$. Then Lemma 5 (5) implies

$$(2a_1^2 - 1)\left\{(2a_1^3 - 4a_1 a_2^2 - 3a_1^2 + 1)a_3^2 + a_2^2(2a_1 - 1)(a_1+1)^2\right\} = 0. \tag{2.14}$$

If $a_1 = \frac{1}{\sqrt{2}}$, then Lemma 5 (6) implies

$$2a_2^2 a_3^2 + 2a_3^2 b_2^2 - a_3^2 - b_2^2 = 0. \tag{2.15}$$

Then we have

$$w_2 = -\frac{(4+3\sqrt{2})}{16a_2^4} < 0, \tag{2.16}$$

which is a contradiction. Hence we must have $a_2 \neq \frac{1}{\sqrt{2}}$.
Then (2.14) implies

$$-\{4a_1a_2^2 - (2a_1+1)(a_1-1)^2\}a_3^2 + a_2^2(2a_1-1)(a_1+1)^2 = 0. \tag{2.17}$$

Hence we have $4a_1a_2^2 - (2a_1+1)(a_1-1)^2 > 0$. And then we have the following:

$$\begin{aligned}
w_2 &= \frac{(1-a_1)^3(2a_1+1)}{4a_2^4}, \\
w_3 &= \frac{\left(4a_1a_2^2 - (2a_1+1)(a_1-1)^2\right)^2}{4a_2^4(1-2a_1)(1+a_1)}, \\
a_3 &= a_2(1+a_1)\sqrt{\frac{1-2a_1}{(2a_1+1)(a_1-1)^2 - 4a_1a_2^2}}.
\end{aligned}$$

Therefore we have $0 < a_1 < \frac{1}{2}$ and $a_2 < \sqrt{\frac{(2a_1+1)(a_1-1)^2}{4a_1}}$. ∎

2.2 Case $a_2b_3 - a_3b_2 = 0$

In this subsection we consider the case case when $a_2b_3 - a_3b_2 = 0$ holds. So we assume $a_2b_3 - a_3b_2 = 0$. Then we obtain the following theorem.

Theorem 10 *Assume $a_2b_3 - a_3b_2 = 0$ holds. Then we (X, w) is similar to the following design:*

$$\begin{aligned}
&X_1 = \{(1,0), (-1,0), \frac{1}{2}(1,\sqrt{3}), -\frac{1}{2}(1,\sqrt{3})\}, \\
&X_2 = \{a_2(1,-\sqrt{3}), a_2(-1,\sqrt{3})]\}, \\
&X_3 = \{a_3(1,-\sqrt{3}), a_3(-1,\sqrt{3})\}, \\
&\quad a_2 > \frac{1}{2} > a_3 > 0 \quad \textit{or} \quad a_3 > \frac{1}{2} > a_2 > 0, \\
&w_2 = \frac{1-4a_3^2}{16(a_2^2-a_3^2)a_2^2}, \\
&w_3 = \frac{4a_2^2-1}{16(a_2^2-a_3^2)a_3^2}.
\end{aligned}$$

Proof: By assumption we have $a_2b_3 - a_3b_2 = 0$. Hence $b_3 = \frac{a_3b_2}{a_2}$. Then Lemma 5 (6) implies

$$a_1^2a_2^2 + a_1^2b_2^2 - a_2^2 = 0. \tag{2.18}$$

Hence we must have

$$b_2 = \pm\frac{a_2\sqrt{1-a_1^2}}{a_1}. \tag{2.19}$$

If $b_2 = \frac{a_2\sqrt{1-a_1^2}}{a_1}$, then Lemma 5 (3) implies $\sqrt{1-a_1^2} = 0$, which is a contradiction.

If $b_2 = -\frac{a_2\sqrt{1-a_1^2}}{a_1}$, then Lemma 5 (3) implies

$$(2a_1 - 1)(2a_1 + 1)\sqrt{1 - a_1^2} = 0. \tag{2.20}$$

Hence we must have $a_1 = \frac{1}{2}$ and (2.2) and (2.3) implies

$$w_2 = \frac{1 - 4a_3^2}{16a_2^2(a_2^2 - a_3^2)} \tag{2.21}$$

and

$$w_3 = \frac{4a_2^2 - 1}{16a_3^2(a_2^2 - a_3^2)} \tag{2.22}$$

respectively. Since $a_2,\ a_3,\ w_2,\ w_3 > 0$, (2.21) and (2.22) imply $(1 - 4a_3^2)(4a_2^2 - 1) > 0$. Hence we must have $a_2 > \frac{1}{2} > a_3 > 0$ or $a_3 > \frac{1}{2} > a_2 > 0$. We also have $b_2 = -\sqrt{3}a_2,\ b_3 = -\sqrt{3}a_3$. This completes the proof. ∎

3 Euclidean tight 5-design of $\mathbb{R}^2$ on 4 circles

In this section we will give the classification of tight Euclidean 5-designs (X, w) supported by 4 circles $S_i(r_i)$, of radius $r_i > 0,\ 1 \le i \le 4$. Let $X_i = X \cap S_i(r_i),\ 1 \le i \le 4$. Let $w(x) = w_i$ for $x \in X_i$ and $i = 1, \ldots, 4$. We may assume $r_1 = 1$ and $w_1 = 1$. Also, we may assume that X is the union of the following four sets $X_i, i = 1, 2, 3, 4$.

$$\begin{aligned}
X_1 &= \{(1, 0),\ (-1, 0)\},\\
X_2 &= \{(a_2, b_2), (-a_2, -b_2)\},\\
X_3 &= \{(a_3, b_3), (-a_3, -b_3)\},\\
X_4 &= \{(a_4, b_4), (-a_4, -b_4)\},
\end{aligned}$$

where $a_2, a_3, a_4 \ge 0,\ r_i^2 = a_i^2 + b_i^2 \ne 1\ (2 \le i \le 4), a_2^2 + b_2^2 > a_3^2 + b_3^2 > a_4^2 + b_4^2 > 0$.
Then similar as before (X, w) is a Euclidean tight 5-design of $\mathbb{R}^2$ if and only if (X, w) satisfies the following equations

$$\sum_{\boldsymbol{x} \in X_1} h(x_1, x_2) + w_2 \sum_{x \in X_2} \|x\|^{2j} h(x_1, x_2) + w_3 \sum_{x \in X_3} \|x\|^{2j} h(x_1, x_2) + w_4 \sum_{x \in X_4} \|x\|^{2j} h(x_1, x_2) = 0, \tag{3.1}$$

for $x = (x_1, x_2),\ \|x\|^2 = x_1^2 + x_2^2,\ h(x_1, x_2) \in \mathrm{Harm}_\ell(\mathbb{R}^2),\ 2j + \ell \le 5,\ \ell \ge 1$. Similar as before we can show the following Lemma.

Lemma 11 *Definition and notation are given above. Then (X, w) is a tight Euclidean 5-design of $\mathbb{R}^2$ if and only if the following (1), …, (6) hold.*

(1) $a_2b_2w_2 + a_3b_3w_3 + a_4b_4w_4 = 0$,

(2) $a_2b_2(a_2^2 + b_2^2)w_2 + a_3b_3(a_3^2 + b_3^2)w_3 + a_4b_4(a_4^2 + b_4^2)w_4 = 0$,

(3) $(a_2^2 - b_2^2)w_2 + (a_3^2 - b_3^2)w_3 + (a_4^2 - b_4^2)w_4 + 1 = 0$,

(4) $(a_2^4 - b_2^4)w_2 + (a_3^4 - b_3^4)w_3 + (a_4^4 - b_4^4)w_4 + 1 = 0$,

(5) $(a_2^4 + b_2^4 - 6a_2^2b_2^2)w_2 + (a_3^4 + b_3^4 - 6a_3^2b_3^2)w_3 + (a_4^4 + b_4^4 - 6a_4^2b_4^2)w_4 + 1 = 0$,

(6) $a_2b_2(a_2^2 - b_2^2)w_2 + a_3b_3(a_3^2 - b_3^2)w_3 + a_4b_4(a_4^2 - b_4^2)w_4 = 0$.

Then we can show that $a_i > 0$ holds for $i = 2, 3, 4$, and $b_i \neq 0$ holds for $i = 2, 3, 4$.

3.1 Explicit formulae for the weight functions w_2, w_3, w_4 of the Euclidean tight 5-design on 4 circles in $\mathbb{R}^2$

In this subsection we give the explicit formulae for the weight functions w_2, w_3, w_4. Notation and Definition are as given before we have the folowing theorem. Note that by assumption $a_2^2 + b_2^2 > a_3^2 + b_3^2 > a_4^2 + b_4^2$ holds.

Theorem 12 *Let (X, w) be a tight 5-design supported by 4 circles. We assume $w(x) = 1$ for $x \in X_1$, $w(x) = w_i$, for $x \in X_i$, $i = 2, 3, 4$. Then $b_2b_3 > 0$, $b_2b_4 < 0$ hence $b_3b_4 < 0$ and the following equalities for w_2, w_3, w_4 fold.*

$$w_2 = \frac{a_3a_4b_3b_4(a_3^2 + b_3^2 - a_4^2 - b_4^2)}{G(a_2, a_3, a_4, b_2, b_3, b_4)}, \tag{3.2}$$

$$w_3 = \frac{a_2a_4b_2b_4(a_4^2 + b_4^2 - a_2^2 - b_2^2)}{G(a_2, a_3, a_4, b_2, b_3, b_4)}, \tag{3.3}$$

$$w_4 = \frac{a_2a_3b_2b_3(a_2^2 + b_2^2 - a_3^2 - b_3^2)}{G(a_2, a_3, a_4, b_2, b_3, b_4)} \tag{3.4}$$

where $G(a_2, a_3, a_4, b_2, b_3, b_4)$ is defined as below:

$$\begin{aligned} G(a_2, a_3, a_4, b_2, b_3, b_4) &= a_4b_4(a_2a_3 + b_2b_3)(a_2b_3 - a_3b_2)(a_4^2 + b_4^2) \\ &- a_2a_3b_2b_3(a_4^2 - b_4^2)(a_2^2 + b_2^2 - a_3^2 - b_3^2) + a_2a_4b_2b_4(a_3^2 - b_3^2)(a_2^2 + b_2^2) \\ &- a_3a_4b_3b_4(a_2^2 - b_2^2)(a_3^2 + b_3^2), \end{aligned} \tag{3.5}$$

and satisfies

$$G(a_2, a_3, a_4, b_2, b_3, b_4) < 0. \tag{3.6}$$

3.2 Toward the classification of Euclidean tight 5-designs on 4 circles in $\mathbb{R}^2$

Here we note again that (X, w) is a weighted finite subset in $\mathbb{R}^2$ and X is supported by a union of 4 circles and expressed in the following way. $X = X_1 \cup X_2 \cup X_3 \cup X_4$ where $X_1 = \{(1, 0), (-1, 0)\}$, $X_2 = \{(a_2, b_2), (-a_2, -b_2)\}$, $X_3 = \{(a_3, b_3), (-a_3, -b_3)\}$, $X_4 = \{(a_4, b_4), (-a_4, -b_4)\}$, where

$a_2,\ a_3,\ a_4 > 0$ and $a_i^2+b_i^2=1$ $(2\le i\le 4)$, $a_2^2+b_2^2>a_3^2+b_3^2>a_4^2+b_4^2>0$. Also $w(x)=1$ for $x\in X_1$ and $w(x)=w_i$ for any $x\in X_i$, $(2\le i\le 4)$.

In the previous subsection we gave a very explicit description of Euclidean tight 5-design on four circles in $\mathbb{R}^2$ using all the 8 parameters a_2,a_3,a_4,b_2,b_3,b_4. In this section we will show that four parameters a_2,a_3,b_2,b_3 determine all the Euclidean tight 5-design on four circles in $\mathbb{R}^2$.

Then we can show the following proposition

Proposition 13 *Lemma 11 (4), (5) and (6) are equivalent to the following (1), (2) and (3) respectively.*

(1) $F_4(a_2,a_3,a_4,b_2,b_3,b_4)=0$, *where*

$$\begin{aligned}
&F_4(a_2,a_3,a_4,b_2,b_3,b_4)\\
&=a_2a_3b_2b_3\{a_2^2+b_2^2-a_3^2-b_3^2\}a_4^4+b_4\{a_2a_3^4b_2-a_2a_3^2b_2-b_3(a_2^2+b_2^2-1)(a_2^2-b_2^2)a_3\\
&-a_2b_2b_3^2(b_3^2-1)\}a_4^3-a_2a_3b_2b_3\{a_2^2+b_2^2-a_3^2-b_3^2\}a_4^2-b_4\{a_2b_2(a_2^2+b_2^2-b_4^2)a_3^4\\
&-b_3(a_2^2+b_2^2-1)(a_2^2-b_2^2)a_3^3-a_2b_2(a_2^2+b_2^2-b_4^2)a_3^2-b_3(b_3^2-b_4^2)(a_2^2+b_2^2-1)\times\\
&(a_2^2-b_2^2)a_3-a_2b_2b_3^2(b_3^2-1)(a_2^2+b_2^2-b_4^2)\}a_4-a_2a_3b_2b_3b_4^2(b_4^2-1)(a_2^2+b_2^2-a_3^2-b_3^2).
\end{aligned} \tag{3.7}$$

(2) $F_5(a_2,a_3,a_4,b_2,b_3,b_4)=0$, *where*

$$\begin{aligned}
&F_5(a_2,a_3,a_4,b_2,b_3,b_4)\\
&=a_2a_3b_2b_3\left\{a_2^2+b_2^2-a_3^2-b_3^2\right\}a_4^4+b_4\left\{a_2b_2a_3^4-a_2b_2(6b_3^2+1)a_3^2-b_3(a_2^4-6a_2^2b_2^2+b_2^4\right.\\
&\left.-a_2^2+b_2^2)a_3+a_2b_2b_3^2(b_3^2+1)\right\}a_4^3-a_2a_3b_2b_3\left\{(6b_4^2+1)(a_2^2+b_2^2-a_3^2-b_3^2)\right\}a_4^2\\
&-b_4\left\{a_2b_2(a_2^2+b_2^2-b_4^2)a_3^4-b_3\big(a_2^4-6a_2^2b_2^2+b_2^4-a_2^2+b_2^2\big)a_3^3\right.\\
&-a_2b_2(6b_3^2+1)(a_2^2+b_2^2-b_4^2)a_3^2-b_3(b_3^2-b_4^2)(a_2^4-6a_2^2b_2^2+b_2^4-a_2^2+b_2^2)a_3\\
&\left.+a_2b_2b_3^2(b_3^2+1)(a_2^2+b_2^2-b_4^2)\right\}a_4\\
&+a_2a_3b_2b_3(b_4^2+1)(a_2^2+b_2^2-a_3^2-b_3^2)b_4^2.
\end{aligned} \tag{3.8}$$

(3)

$$(a_3^2-a_2^2)b_4^2+(a_2^2-a_4^2)b_3^2+(a_4^2-a_3^2)b_2^2=0. \tag{3.9}$$

In the following, we divide the situation into two cases $a_3=a_2$, $a_3\neq a_2$ and for each case we try to give the expression of the weights $w_2,\ w_3$ and w_4 with parameters a_2,a_3,a_4,b_2,b_3 and a_4.

Case: $\boldsymbol{a_3 = a_2}$:

Theorem 14 *If $a_3 = a_2$ holds, then we must have $a_4 = a_2$, $a_2^2(b_2 + b_3 + b_4) + b_2b_3b_4 = 0$, $b_2^2 > b_3^2 > b_4^2 > 0$, and $b_2, b_4 > 0$ $b_3 < 0$ or $b_2, b_4 < 0$, $b_3 > 0$ holds. the following holds:*

$$(b_2 + b_3)^2 b_4^3 + (b_2 + b_3)(b_2^2 - 5b_2b_3 + b_3^2 + 3)b_4^2 + \left\{2b_2b_3(b_2 - b_3)^2 + 3(b_2^2 + b_3^2)\right\}b_4 + b_2b_3(b_2 + b_3)(b_2b_3 + 3) = 0. \tag{3.10}$$

The weight functions are given as below:

$$w_1 = 1$$

$$w_2 = \frac{3(b_3 + b_4)b_3b_4}{(b_2 - b_4)(b_2 - b_3)\{(b_3 + b_4)b_2^2 + (b_4 + b_2)b_3^2 + (b_2 + b_3)b_4^2\}} \tag{3.11}$$

$$w_3 = \frac{3(b_2 + b_4)b_2b_4}{(b_3 - b_2)(b_3 - b_4)\{(b_3 + b_4)b_2^2 + (b_4 + b_2)b_3^2 + (b_2 + b_3)b_4^2\}} \tag{3.12}$$

$$w_4 = \frac{3(b_2 + b_3)b_2b_3}{(b_4 - b_3)(b_4 - b_2)\{(b_3 + b_4)b_2^2 + (b_4 + b_2)b_3^2 + (b_2 + b_3)b_4^2\}} \tag{3.13}$$

Proof: If $a_3 = a_2$, then Proposition 13 (3) implies

$$(a_2^2 - a_4^2)(b_3^2 - b_2^2) = 0. \tag{3.14}$$

On the other hand by assumption we have $a_2^2 + b_2^2 > a_3^2 + b_3^2 > a_4^2 + b_4^2$, hence we have $b_2^2 > b_3^2$. Therefore we must have $a_2 = a_4$, hence $a_2 = a_3 = a_4$ holds. This implies $b_2^2 > b_3^2 > b_4^2$. Then Proposition 13 (1) and (2) imply

$$F_5(a_2, a_2, a_2, b_2, b_3, b_4) - F_4(a_2, a_2, a_2, b_2, b_3, b_4) = 2a_2^2 b_2b_3b_4(b_3 - b_4)(b_2 - b_4)(b_2 - b_3)(3a_2^2 + b_2b_3 + b_2b_4 + b_3b_4), \tag{3.15}$$

and Proposition 13 (2) implies

$$F_5(a_2, a_2, a_2, b_2, b_3, b_4) + F_4(a_2, a_2, a_2, b_2, b_3, b_4) = 2a_2^2(b_3 - b_4)(b_2 - b_4)(b_2 - b_3)\left\{(a_2^4 - a_2^2)(b_2 + b_3 + b_4) + (3a_2^2 - 1)b_2b_3b_4\right\}. \tag{3.16}$$

Hence we have

$$a_2^2 = -\frac{1}{3}(b_2b_3 + b_3b_4 + b_4b_2), \tag{3.17}$$

and

$$(b_2 + b_3 + b_4)a_2^2(a_2^2 - 1) - b_2b_3b_4(b_2b_3 + b_2b_4 + b_3b_4 + 1) = 0. \tag{3.18}$$

Hence we have $b_2b_3 + b_3b_4 + b_4b_2 < 0$. Then (3.18) implies

$$(b_2 + b_3)^2 b_4^3 + (b_2 + b_3)(b_2^2 - 5b_2b_3 + b_3^2 + 3)b_4^2 + \left\{2b_2b_3(b_2 - b_3)^2 + 3(b_2^2 + b_3^2)\right\}b_4 + b_2b_3(b_2 + b_3)(b_2b_3 + 3) = 0. \tag{3.19}$$

Here we note that $b_2^2 > b_3^2 > b_4^2$, $b_2 b_4 > 0$ and $b_3 b_4 < 0$ hold. Hence we have $b_2 b_3 < 0$. That is, we must have $b_2, b_4 > 0, b_3 < 0$ or $b_2, b_4 < 0, b_3 > 0$.
Then (3.2), (3.3) and (3.4) imply (3.11), (3.12) and (3.13), respectively. ∎

Case $\boldsymbol{a_3 \neq a_2}$:

In this case Proposition 13 (3) implies

$$b_4^2 = \frac{(a_2^2 - a_4^2)b_3^2 + (a_4^2 - a_3^2)b_2^2}{a_2^2 - a_3^2}. \tag{3.20}$$

In the following we use the case $b_4 = \sqrt{\frac{(a_2^2 - a_4^2)b_3^2 + (a_4^2 - a_3^2)b_2^2}{a_2^2 - a_3^2}} > 0$ and show that there are infinitely many tight Euclidean 5-design of $\mathbb{R}^2$ with parameters a_2, a_3, a_4, b_2, b_3. Note that here $a_2, a_3, a_4 > 0$. Here we asume $b_4 > 0$, we must have $b_2, b_3 < 0$ (see Theorem 12). Then the condition $F_4(a_2, a_3, a_4, b_2, b_3, b_4) = F_5(a_2, a_3, a_4, b_2, b_3, b_4) = 0$ implies $P_4(a_2, a_3, a_4, b_2, b_3) = P_5(a_2, a_3, a_4, b_2, b_3) = 0$ respectively, where

$$\begin{aligned}
P_4(a_2, a_3, a_4, b_2, b_3) = \Bigg[& a_2^2 a_3^9 b_2^3 + a_2^3 a_3^8 b_2^2 b_3 - a_2^2 a_3^7 b_2 \Big\{ a_2^2(b_2^2 + 2b_3^2) + 2(b_2^2 - b_3^2) \Big\} - a_2 a_3^6 b_2^2 b_3 \times \\
& \Big\{ 3a_2^4 - 2(b_2^2 - 1)(b_2^2 - b_3^2) \Big\} + a_2^2 a_3^5 b_2 \Big\{ 3a_2^4 b_3^2 + 2a_2^2(b_2^2 - b_3^2) + b_2^2 - b_3^2 \Big\} + a_2 a_3^4 b_3 \Big\{ (2b_2^2 + b_3^2)a_2^6 \\
& + 2(b_2^2 - b_3^2)a_2^4 - (2b_2^4 - 2b_2^2 + 1)(b_2^2 - b_3^2)a_2^2 - 2b_2^2(b_2^2 - 1)(b_2^2 - b_3^2) \Big\} \\
& - a_3^3 b_2 \Big\{ a_2^8 b_3^2 + (2b_3^4 - 2b_3^2 + 1)(b_2^2 - b_3^2)a_2^4 - 2b_3^2(b_3^2 - 1)(b_2^2 - b_3^2)a_2^2 - b_2^2 b_3^2 (b_2^2 - 1)^2 (b_2^2 - b_3^2) \Big\} \\
& - a_2 b_3 \Big\{ a_2^8 b_3^2 + 2(b_2^2 - b_3^2)a_2^6 - (b_2^2 - b_3^2)a_2^4 - 2b_2^2(b_2^2 - 1)(b_2^2 - b_3^2)a_2^2 - b_2^2 b_3^2 (b_2^2 - 1)(b_2^2 - b_3^2) \times \\
& (b_2^2 - 2b_3^2 + 1) \Big\} a_3^2 + a_2^2 b_2 b_3^2 (b_3^2 - 1)(b_2^2 - b_3^2)(2a_2^4 - 2a_2^2 - 2b_2^4 + b_2^2 b_3^2 + b_2^2)a_3 + a_2^3 b_2^2 b_3^3 (b_3^2 - 1)^2 \times \\
& (b_2^2 - b_3^2) \Bigg] a_4^4 + (a_2 b_3 + a_3 b_2) \Bigg[- a_2^4 a_3^8 b_2^2 + 2a_2 b_2 b_3 (a_2^2 + b_2^2 - 1)(a_2^2 - b_2^2)a_3^7 + a_2^2 \Big\{ a_2^4(b_2^2 - b_3^2) \\
& + 2a_2^2(b_2^2 + b_3^2) + 2b_2^4 b_3^2 - 2b_2^2 b_3^2 - b_3^2 \Big\} a_3^6 - 2a_2 b_2 b_3 (a_2^2 + 1)(a_2^2 + b_2^2 - 1)(a_2^2 - b_2^2)a_3^5 \\
& + \Big\{ a_2^8 b_3^2 - 2(b_2^2 + b_3^2)a_2^6 - (2b_2^2 b_3^2 + 1)(b_2^2 - b_3^2)a_2^4 - 2b_2^2 b_3^2 (b_2^2 + b_3^2 - 2)a_2^2 - b_2^2 b_3^2 (b_2^2 - 1)^2 \times \\
& (2b_2^2 - b_3^2) \Big\} a_3^4 + 2a_2 b_2 b_3 (a_2^2 - b_3^4 + b_3^2)(a_2^2 + b_2^2 - 1)(a_2^2 - b_2^2)a_3^3 - b_2^2 a_2^2 \Big\{ 2a_2^4(b_3^4 - b_3^2) - 5b_2^4 b_3^4 \\
& + 5b_2^2 b_3^6 - 2a_2^2 b_2^2 b_3^2 - a_2^4 - 2a_2^2 b_3^2 (b_3^2 - 2) + 4b_2^4 b_3^2 - 4b_3^6 - 3b_2^2 b_3^2 + 3b_3^4 \Big\} a_3^2 + 2a_2^3 b_2 b_3^3 (b_3^2 - 1) \times \\
& (a_2^2 + b_2^2 - 1)(a_2^2 - b_2^2)a_3 - a_2^4 b_2^2 b_3^2 (b_3^2 - 1)^2 (b_2^2 - 2b_3^2) \Bigg] a_4^2 \\
& - b_2^2 b_3^2 (a_2 b_3 - a_3 b_2)(a_2 b_3 + a_3 b_2)^2 \Big\{ (b_2^2 - 1)a_3^2 - (b_3^2 - 1)a_2^2 \Big\}^2,
\end{aligned} \tag{3.21}$$

and

$$
\begin{aligned}
&P_5(a_2,a_3,a_4,b_2,b_3) = \bigg[a_2^2a_3^9b_2^3 + a_2^3a_3^8b_2^2b_3 - a_2^2b_2\bigg\{(b_2^2+2b_3^2)a_2^2 + 2(b_2^2-b_3^2)\bigg\}a_3^7 - a_2b_2^2b_3\bigg\{3a_2^4 \\
&-12(b_2^2-b_3^2)a_2^2 + 2(b_2^2+1)(b_2^2-b_3^2)\bigg\}a_3^6 + a_2^2b_2\bigg\{3a_2^4b_3^2 - 2(6b_3^2-1)(b_2^2-b_3^2)a_2^2 + \big\{12(3b_2^2+1)b_3^2 \\
&+1\big\}(b_2^2-b_3^2)\bigg\}a_3^5 + a_2b_3\bigg\{(2b_2^2+b_3^2)a_2^6 - 2(6b_2^2-1)(b_2^2-b_3^2)a_2^4 - (b_2^2-b_3^2)\big\{36b_2^2b_3^2 - 2b_2^2(b_2^2-5) \\
&+1\big\}a_2^2 + 2b_2^2(6b_3^2+1)(b_2^2+1)(b_2^2-b_3^2)\bigg\}a_3^4 \\
&-b_2\bigg\{a_2^8b_3^2 - 12b_3^2(b_2^2-b_3^2)a_2^6 - \big\{2b_3^4 - 2(18b_2^2+5)b_3^2 - 1\big\}(b_2^2-b_3^2)a_2^4 \\
&-2b_3^2(b_3^2+1)(b_3^2-b_2^2)(6b_2^2+1)a_2^2 - b_2^2b_3^2(b_2^2+1)^2(b_2^2-b_3^2)\bigg\}a_3^3 \\
&-a_2b_3\bigg\{a_2^8b_3^2 + 2(b_2^2-b_3^2)a_2^6 - (36b_2^2b_3^2+12b_2^2+1)(b_2^2-b_3^2)a_2^4 + 2b_2^2(6b_3^2+1)(b_2^2+1)(b_2^2-b_3^2)a_2^2 \\
&-b_2^2b_3^2(b_2^2+1)(b_3^2-b_2^2)(2b_3^2-b_2^2+1)\bigg\}a_3^2 - a_2^2b_2b_3^2(b_3^2+1)(b_2^2-b_3^2)\bigg\{2a_2^4 - 2(6b_2^2+1)a_2^2 \\
&+b_2^2(2b_2^2-b_3^2+1)\bigg\}a_3 + a_2^3b_2^2b_3^3(b_3^2+1)^2(b_2^2-b_3^2)\bigg]a_4^4 \\
&+(a_2b_3+a_3b_2)\bigg[-a_2^4a_3^8b_2^2 + 2b_2b_3a_2\bigg\{a_2^4 - (6b_2^2+1)a_2^2 + b_2^4 + b_2^2\bigg\}a_3^7 \\
&+a_2^2\bigg\{(b_2^2-b_3^2)a_2^4 + 2\bigg((12b_2^2+1)b_3^2 + b_2^2\bigg)a_2^2 - b_3^2(38b_2^4+14b_2^2+1)\bigg\}a_3^6 \\
&-2a_2b_2b_3(a_2^2+6b_3^2+1)\bigg\{a_2^4 - (6b_2^2+1)a_2^2 + b_2^4 + b_2^2\bigg\}a_3^5 + \bigg\{a_2^8b_3^2 - 2(b_3^2+12b_2^2b_3^2+b_2^2)a_2^6 \\
&+(38b_2^2b_3^2-1)(b_2^2-b_3^2)a_2^4 + 2b_2^2b_3^2\bigg((12b_2^2+7)b_3^2 + 7b_2^2 + 2\bigg)a_2^2 - b_2^2b_3^2(b_2^2+1)^2(2b_2^2-b_3^2)\bigg\}a_3^4 \\
&+2a_2b_2b_3\bigg((6b_3^2+1)a_2^2 + b_3^4 + b_3^2\bigg)\bigg\{a_2^4 - (6b_2^2+1)a_2^2 + b_2^4 + b_2^2\bigg\}a_3^3 + a_2^2b_2^2\bigg\{(38b_3^4+14b_3^2+1)a_2^4 \\
&-2b_3^2\bigg((12b_2^2+7)b_3^2 + 7b_2^2 + 2\bigg)a_2^2 + b_3^2\bigg((5b_2^2+4)b_3^2 + 4b_2^2 + 3\bigg)(b_2^2-b_3^2)\bigg\}a_3^2 \\
&-2b_2b_3^3(b_3^2+1)a_2^3\bigg\{a_2^4 - (6b_2^2+1)a_2^2 + b_2^4 + b_2^2\bigg\}a_3 - b_2^2b_3^2(b_3^2+1)^2(b_2^2-2b_3^2)a_2^4\bigg]a_4^2 \\
&-b_2^2b_3^2(a_2b_3-a_3b_2)(a_2b_3+a_3b_2)^2\bigg\{a_2^2(b_3^2+1) - a_3^2(b_2^2+1)\bigg\}^2. \qquad (3.22)
\end{aligned}
$$

Both of the polynomials $P_4(a_2,a_3,a_4,b_2,b_3)$ and $P_5(a_2,a_3,a_4,b_2,b_3)$ are, as polynomials respect to ${a_4}^2$, of order 2. Therefore we can determine the conditions to have tight Euclidean 5-designs on 4 circles in $\mathbb{R}^2$. Thus we can show explicitly that there exist continuously many tight Euclidean 5-designs of $\mathbb{R}^2$, that was shown in the paper by Bannai-Bannai-Suprijanto ([4], 2007). Though, here, we are not trying to give the list of explicit solutions for the formulae given above.

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