MATRIX WEIGHTS, SINGULAR INTEGRALS, JONES FACTORIZATION AND RUBIO DE FRANCIA EXTRAPOLATION

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ABSTRACT. In this article we give an overview of the problem of finding sharp constants in matrix weighted norm inequalities for singular integrals, the so-called matrix A_2 conjecture. We begin by reviewing the history of the problem in the scalar case, including a sketch of the proof of the scalar A_2 conjecture. We then discuss the original, qualitative results for singular integrals with matrix weights and the best known quantitative estimates. We give an overview of new results by the author and Bownik, who developed a theory of harmonic analysis on convex set-valued functions. This led to the proof the Jones factorization theorem and the Rubio de Francia extrapolation theorem for matrix weights, two longstanding problems. Rubio de Francia extrapolation is expected to be a major tool in the proof of the matrix $\mathcal{A}_2$ conjecture, and we discuss some ideas which may lead to a complete solution.

1. INTRODUCTION

One weight norm inequalities have been extensively studied since the work of Muckenhoupt and others in the 1970s. (See [23, 26, 28] for details and references.) Central to their study is the Muckenhoupt A_p condition. It gives a necessary and sufficient condition for many of the operators of classical harmonic analysis (maximal operators, singular integral operators, square functions) to satisfy weighted norm inequalities. There are two very important results in the study of of A_p weights. The first is the Jones factorization theorem [35], which gives a complete characterization of the class A_p in terms of the simpler class A_1. The second is the Rubio de Francia extrapolation theorem [53], which, in its simplest form, shows that if an operator is bounded on weighted L^2, then it is bounded on weighted L^p for all p, $1 < p < \infty$. Rubio de Francia's colleague Antonio Cordoba [25] summarized this result by remarking that it showed that *there are no L^p spaces, only weighted L^2*.

Since the 1990s, there has been a great deal of interest in extending the theory of scalar weights to the setting of matrix weights: that is, $d \times d$ measurable matrix functions W that are self-adjoint, positive semi-definite, and act on vector-valued functions $\vec{f}$. These problems were first studied by Nazarov, Treil and Volberg [45, 55–57], who asked if the weighted norm inequalities for singular integrals could be extended to the matrix setting. They defined a class of matrix weights, $\mathcal{A}_p$, and showed that the Hilbert transform is bounded with respect to this class. This result was later extended to general singular integral operators by Christ and Goldberg [11, 27].

More recently, attention has been focused on determining the sharp constant in matrix norm inequalities. In the scalar case, Hytönen [31] proved that the sharp constant in the weighted L^p norm inequality is proportional to $[w]_{A_p}^{\max\{1,p'-1\}}$, and it is an open question as to whether the same bound is true for matrix weights. The best known result is that it is bounded above by $[W]_{\mathcal{A}_p}^{1+\frac{1}{p-1}-\frac{1}{p}}$, which, when $p = 2$ gives an exponent of $\frac{3}{2}$ rather than the conjectured value of 1. In the scalar case, most proofs of this result proceed in two steps: first, it is

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proved in the case $p = 2$, and then it is extended to all p by using a sharp version of the Rubio de Francia extrapolation theorem. (See [22, 32, 41].) For this reason, the problem is referred to as the "A_2 conjecture."

In the 1990s Nazarov and Treil [45] posed two fundamental problems: extend the Jones factorization theorem and the Rubio de Francia extrapolation theorem to matrix weights. These problems were only solved last year by the author and Bownik [7]. These results are a significant advance in the study of the matrix A_2 conjecture, since they allow the problem to be reduced to the case $p = 2$.

The goal of this article is to provide an overview of the theory of matrix weights and of the proofs of factorization and extrapolation for matrix weights. It is organized as follows. In Section 2 we provide some background about scalar weights. We focus on those results which are essential for understanding the subsequent development of the matrix theory. In Section 3 we review some of the history of matrix weights. We will concentrate on the results of Christ and Goldberg [11, 27] and the more recent work of Nazarov, Petermichl, Treil, and Volberg [43]. In Section 4 we will discuss the work of the author and Bownik. Our approach, which considerably extends the work of Nazarov, *et al.*, builds a theory of harmonic analysis on convex set-valued functions. This in turn provides the necessary tools to prove factorization and extrapolation for matrix weights. Finally, in Section 5 we make some final remarks and conjectures about the matrix A_2 conjecture.

Throughout this paper we will use the following notation. In Euclidean space the constant n will denote the dimension of $\mathbb{R}^n$, which will be the domain of our functions. The value d will denote the dimension of vector and set-valued functions. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^n)$ will denote the Lebesgue space of scalar functions, and $L^p(\mathbb{R}^n, \mathbb{R}^d)$ will denote the Lebesgue space of vector-valued functions.

By a cube we will always mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. The Lebesgue measure of a cube, or of any arbitrary set E, will be denoted by $|E|$. By a weight we mean a non-negative, locally integrable function that is positive except on a set of measure 0. We define $w(E) = \int_E w(x)\,dx$, and we let $⨍_Q w(x)\,dx = |Q|^{-1}w(Q)$. By $L^p(w)$ we mean the scalar weighted space with measure $w\,dx$, $L^p(\mathbb{R}^n, w\,dx)$.

Given $v = (v_1, \ldots, v_d)^t \in \mathbb{R}^d$, the Euclidean norm of v will be denoted by $|v|$; from context there should be no confusion with the notation for Lebesgue measure of a set. The closed unit ball in $\{v \in \mathbb{R}^d : |v| \leq 1\}$ will be denoted by $\mathbf{B}$. Matrices will be $d \times d$ matrices with real-valued entries. The set of all $d \times d$, symmetric, positive semidefinite matrices will be denoted by $\mathcal{S}_d$.

Given two quantities A and B, we will write $A \lesssim B$, or $B \gtrsim A$ if there is a constant $c > 0$ such that $A \leq cB$. If $A \lesssim B$ and $B \lesssim A$, we will write $A \approx B$.

2. Background: the scalar theory of weights

In harmonic analysis, a fundamental class of weights are those that satisfy the Muckenhoupt A_p condition, introduced in [42]. Given $1 < p < \infty$, $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \left(⨍_Q w(x)\,dx \right) \left(⨍_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes. A weight w is in A_1 if

$$[w]_{A_1} = \sup_Q \operatorname*{ess\,sup}_{Q \ni x} w(x)^{-1} ⨍_Q w(y)\,dy < \infty.$$

The quantity $[w]_{A_p}$ is referred to as the A_p characteristic of a weight. These weights arise naturally in the study of Hardy-Littlewood maximal operator,

$$Mf(x) = \sup_Q \fint_Q |f(y)|\,dy \cdot \chi_Q(x),$$

where the supremum is taken over all cubes. For $1 \leq p < \infty$, the A_p condition is necessary and sufficient for the maximal operator to satisfy the weak (p,p) inequality

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx,$$

and for $1 < p < \infty$ it is necessary and sufficient for the strong (p,p) inequality

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.$$

Recall that a Calderón-Zygmund singular integral T is a bounded operator on $L^2(\mathbb{R}^n)$ for which there exists a kernel $K(x,y)$, defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, where $\Delta = \{(x,x) : x \in \mathbb{R}^n\}$, such that if $f \in L^\infty_c(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$, then

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy.$$

The kernel satisfies the size and regularity conditions

$$|K(x,y)| \leq \frac{C}{|x-y|^n},$$

$$|K(x+h,y) - K(x,y)| + |K(x,y+h) - K(x,y)| \leq C\frac{|h|^\delta}{|x-y|^{n+\delta}},$$

where $|x-y| > 2|h|$. The A_p condition is also sufficient for a singular integral operator to satisfy the weak and strong (p,p) inequalities; moreover, it is necessary for non-degenerate singular integrals such as the Riesz transforms. See [23, 26, 28, 54].

The weak (p,p) inequality for singular integrals can be proved using kernel estimates and the good/bad decomposition of Calderón and Zygmund (see, for instance [26]). The strong-type inequality was originally proved by comparing the norm of the singular integral operator to that of the maximal operator. Coifman and Fefferman [13] proved that given p, $0 < p < \infty$, and $w \in A_q$ for any q, $1 \leq q < \infty$, there exists a constant depending on $[w]_{A_q}$ such that

$$\int_{\mathbb{R}^n} T^*f(x)^p w(x)\,dx \leq C \int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx. \tag{2.1}$$

Here, T^* is the maximal singular integral, defined by

$$T^*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)| = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} K(x,y) f(y)\,dy \right|,$$

which dominates the singular integral pointwise. They proved this by proving a so-called good-λ inequality: there exists $\delta > 0$ such that for every γ, $\lambda > 0$ and for every cube Q,

$$w(\{x \in Q : T^*f(x) > 2\lambda, Mf(x) \leq \gamma\lambda\}) \leq C\gamma^\delta w(Q).$$

An alternative proof of (2.1) using the sharp maximal operator was given by Journé [36]; see also Alvarez and Pérez [1].

For the past three decades there has been a great deal of interest in determining the best constant in the strong (p,p) inequality for singular integrals in terms of the A_p characteristic of w. This question was first considered by Buckley in the 1990s [8]; it became the subject of concerted effort when Astala, Iwaniec and Saksman [2] proved that sharp regularity results for solutions of the Beltrami equation hold provided that the

Beurling-Ahlfors operator satisfies $\|Tf\|_{L^p(w)} \leq C[w]_{A_p}\|f\|_{L^p(w)}$ for $p \geq 2$. This problem was extended to all Calderón-Zygmund operators and all $p > 1$: it was conjectured that

$$\|Tf\|_{L^p(w)} \leq C(n,T,p)[w]_{A_p}^{\max(1,p'-1)}\|f\|_{L^p(w)}. \tag{2.2}$$

By the Rubio de Francia extrapolation theorem, which we will discuss below, it suffices to prove this when $p = 2$; for this reason this problem was referred to as the A_2 conjecture. It was studied by a number of authors, including Lacey, Petermichl, and Volberg [37, 48–50], and was finally solved in 2010 by Hytönen [31, 33]. His proof was quite difficult and we will not consider it.

Lerner and Nazarov [38–40] and Conde-Alonso and Rey [14] gave a new and simpler proof of the A_2 conjecture. As part of their proof they introduced the technique of sparse domination. Let $\mathcal{D}$ be any translation of the standard dyadic grid. A collection $\mathcal{S} \subset \mathcal{D}$ is said to be sparse if there exists a collection of pairwise disjoint sets $\{E(Q) : Q \in \mathcal{S}\}$ such that for each Q, $E(Q) \subset Q$ and $|Q| \leq 2|E(Q)|$. A sparse operator is an averaging operator of the form

$$T_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}} \left(\fint_Q f(y)\,dy\right) \cdot \chi_Q(x).$$

They showed that given a bounded function of compact support, there exist a finite collection of dyadic grids $\{\mathcal{D}_n\}_{n=1}^N$ and sparse families $\{\mathcal{S}_n\}_{n=1}^N$ such that

$$|Tf(x)| \leq C\sum_{n=1}^N T_{\mathcal{S}_n}(|f|)(x).$$

Given this, the A_2 conjecture reduces to proving the corresponding estimates for sparse operators. This is done in two steps. First, weighted L^2 bounds are proved using an argument in [17]. Let $w \in A_2$ and let $\sigma = w^{-1}$. By duality, there exists $h \in L^2(w)$, $\|h\|_{L^2(w)} = 1$, such that

$$\begin{aligned}
\|T_{\mathcal{S}}f\|_{L^2(w)} &= \int_{\mathbb{R}^n} T_{\mathcal{S}}f(x)h(x)w(x)\,dx \\
&\leq 2\sum_{Q\in\mathcal{S}} \fint_Q f(x)\,dx \fint_Q h(x)w(x)\,dx |E(Q)| \\
&= 2\sum_{Q\in\mathcal{S}} \frac{w(Q)}{|Q|}\frac{\sigma(Q)}{|Q|}\frac{1}{\sigma(Q)}\int_Q f(x)w(x)\sigma(x)\,dx \frac{1}{w(Q)}\int_Q h(x)w(x)\,dx|E(Q)| \\
&\leq 2[w]_{A_2}\sum_{Q\in\mathcal{S}}\int_{E(Q)} \tilde{M}^d_\sigma(fw)(x)\sigma(x)\tilde{M}^d_w h(x)w(x)\,dx \\
&\leq 2[w]_{A_2}\int_{\mathbb{R}^n} \tilde{M}^d_\sigma(fw)(x)\sigma(x)\tilde{M}^d_w h(x)w(x)\,dx \\
&\leq 2[w]_{A_2}\|\tilde{M}^d_\sigma(fw)\|_{L^2(\sigma)}\|\tilde{M}^d_w h\|_{L^2(w)} \\
&\leq 8[w]_{A_2}\|fw\|_{L^2(\sigma)}\|h\|_{L^2(w)} \\
&= 8[w]_{A_2}\|f\|_{L^2(w)}.
\end{aligned}$$

Here, $\tilde{M}^d_w$ is the "universal" dyadic maximal operator defined with respect to the grid $\mathcal{D}$ and the measure $w\,dx$:

$$\tilde{M}^d_w f(x) = \sup_{Q\in\mathcal{D}} \frac{1}{w(Q)}\int_Q |f(y)|w(y)\,dy \cdot \chi_Q(x),$$

where the supremum is taken over all dyadic cubes in the grid $\mathcal{D}$. This operator is bounded on $L^2(w)$ with a constant independent of w (see [26]). Similarly, $\tilde{M}^d_\sigma$ is bounded on $L^2(\sigma)$.

The proof that sparse operators satisfy weighted L^p bounds with the desired constant follows at once from the Rubio de Francia extrapolation theorem. Here we state it in a "sharp constant" version first proved by Dragičević, *et al.* [22]. (See also [18].)

Theorem 2.1. *Given p_0, $1 \le p_0 < \infty$, suppose that for some operator T and for all $w_0 \in A_{p_0}$, the inequality*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w_0(x)\,dx \le N_{p_0}([w_0]_{A_{p_0}}) \int_{\mathbb{R}^n} |f(x)|^{p_0} w_0(x)\,dx$$

holds. Then for all p, $1 < p < \infty$, and all $w \in A_p$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p} w(x)\,dx \le N_{p}([w]_{A_{p}}) \int_{\mathbb{R}^n} |f(x)|^{p} w(x)\,dx.$$

The proof of extrapolation ultimately depends on three things:

(1) The duality of A_p weights: for $1 < p < \infty$, $w \in A_p$ if and only if $\sigma = w^{1-p'} \in A_{p'}$, and $[\sigma]_{A_{p'}} = [w]_{A_p}^{p'-1}$. This is an immediate consequence of the definition of A_p weights.
(2) A quantitative bound for the maximal operator: for $1 < p < \infty$, $\|Mf\|_{L^p(w)} \le C[w]_{A_p}^{p'-1}\|f\|_{L^p(w)}$. This was first explicitly proved by Buckley [8] but was implicit in Christ and Fefferman [10].
(3) The Jones factorization theorem: for $1 < p < \infty$, $w \in A_p$ if and only if there exist w_0, $w_1 \in A_1$ such that $w = w_0 w_1^{1-p}$. This was first proved by Jones [35], but a much more elementary proof was given by Coifman, Jones, and Rubio de Francia [12]. (See also [15].)

With the first two facts we can define the Rubio de Francia iteration operator, which is also fundamental to the proof of the Jones factorization theorem. Let M be the Hardy-Littlewood maximal operator and $w \in A_p$, $1 < p < \infty$. Given a non-negative function h, define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L^p(w)}^k}.$$

It then follows from the definition and the properties of A_p weights that

(1) $h(x) \le \mathcal{R}h(x)$;
(2) $\|\mathcal{R}h\|_{L^p(w)} \le 2\|h\|_{L^p(w)}$;
(3) $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \le 2\|M\|_{L^p(w)} \le C[w]_{A_p}^{p'-1}$.

We briefly sketch the proof of extrapolation. For simplicity, we will only consider the case when $p_0 = 2$, and we will not give the proof that yields the best possible constant. (For complete details, plus references to other proofs, see [15, 18].) Fix p, $p \ne 2$, and $w \in A_p$. Let $\mathcal{R}_1$ be the Rubio de Francia iteration operator defined above, and let $\mathcal{R}_2$ be the operator corresponding to $\sigma = w^{1-p'} \in A_{p'}$. Then by duality there exists $h \in L^{p'}(w)$, $\|h\|_{L^{p'}(w)} = 1$, such that

$$\begin{aligned}
\|Tf\|_{L^p(w)} &= \int_{\mathbb{R}^n} |Tf(x)|h(x)w(x)\,dx \\
&\le \int_{\mathbb{R}^n} |Tf(x)|\mathcal{R}_1 f(x)^{-\frac12}\mathcal{R}_1 f(x)^{\frac12}\mathcal{R}_2(hw)(x)\,dx \\
&\le \left(\int_{\mathbb{R}^n} |Tf(x)|^2 \mathcal{R}_1 f(x)^{-1}\mathcal{R}_2(hw)(x)\,dx\right)^{\frac12} \left(\int_{\mathbb{R}^n} \mathcal{R}_1 f(x)\mathcal{R}_2(hw)(x)\,dx\right)^{\frac12} \\
&= I_1^{\frac12} I_2^{\frac12}.
\end{aligned}$$

We estimate each term separately. The estimate of I_2 uses the properties of the iteration operators:

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^n} \mathcal{R}_1 f(x) w(x)^{\frac1p} \mathcal{R}_2(hw)(x) w(x)^{-\frac1p}\,dx \\
&\le \|\mathcal{R}_1 f\|_{L^p(w)} \|\mathcal{R}_2(hw)\|_{L^{p'}(\sigma)} \\
&\le 4\|f\|_{L^p(w)}\|hw\|_{L^{p'}(\sigma)}
\end{aligned}$$

$$\leq 4\|f\|_{L^p(w)}.$$

To estimate I_1, we use our hypothesis, the fact that by the Jones factorization theorem, $\mathcal{R}f_1(f)^{-1}\mathcal{R}f_2 \in A_2$, and the estimate for I_2:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} |Tf(x)|^2 \mathcal{R}_1 f(x)^{-1} \mathcal{R}_2(hw)(x)\,dx \\ &\leq C \int_{\mathbb{R}^n} |f|^2 \mathcal{R}_1 f(x)^{-1} \mathcal{R}_2(hw)(x)\,dx \\ &\leq C \int_{\mathbb{R}^n} |f(x)| \mathcal{R}_2(hw)(x)\,dx \\ &\leq C I_2. \end{aligned}$$

If we combine these inequalities, we get the desired result.

Remark 2.2. Note that in this proof of extrapolation, we only use the easier half of the Jones factorization theorem: that if $w_0,\, w_1 \in A_1$, then $w_0 w_1^{1-p} \in A_p$. This property, referred to as "reverse factorization" (see [18]) follows at once from the definition of the A_p and A_1 conditions. We will return to this fact below.

3. Matrix weights and matrix-weighted norm inequalities

In the 1990s, Nazarov, Treil and Volberg in a series of papers [45,55–57] considered the question of whether the theory of Muckenhoupt weights could be extended to matrix weights applied to vector-valued functions. Beyond the intrinsic interest of this problem, their original motivation came from problems in the study of multivariate random stationary processes, and from the study of Toeplitz operators acting on vector-valued Hardy spaces.

To describe the problem, we define some notation. Let $\vec{f} = (f_1, \dots, f_d)$. Given a singular integral T, define it acting on a vector-valued function by

$$T\vec{f} = (Tf_1, \dots, Tf_d).$$

It is immediate that if $1 < p < \infty$ and $\vec{f} \in L^p(\mathbb{R}^n, \mathbb{R}^d)$, then $\|T\vec{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leq C\|\vec{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)}$.

To define weights, recall that $\mathcal{S}_d$ is the set of $d \times d$, self-adjoint, positive semi-definite matrices. A matrix weight is a measurable function $W : \mathbb{R}^n \to \mathcal{S}_d$. We define an associated scalar weight using the operator norm on W:

$$|W(x)|_{\mathrm{op}} = \sup_{\substack{\xi \in \mathbb{R}^d \\ |\xi|=1}} |W(x)\xi|.$$

We define the matrix weighted space $L^p(W) = L^p(W, \mathbb{R}^n, \mathbb{R}^d)$ with the norm

$$\|\vec{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^n} |W(x)^{\frac{1}{p}} \vec{f}(x)|^p\,dx \right)^{\frac{1}{p}}.$$

Note that when $d = 1$, this reduces to the scalar space $L^p(w)$. Also note that $W^{\frac{1}{p}}$ is well-defined since W is positive semi-definite. (We note in passing that in [7] we defined this space with $W^{\frac{1}{p}}$ replaced by W. We believe that this is the correct way to define matrix weighted spaces, but here, for consistency with the earlier literature, we use the historical definition.)

With this notation, the problem posed by Nazarov, Treil and Volberg is the following: given $1 < p < \infty$, prove there is a Muckenhoupt-type condition on matrix weights so that the inequality

$$\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x)Tf(x)|^p\,dx \leq C \int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x)f(x)|^p\,dx$$

holds for singular integrals. Treil and Volberg [56] first solved this problem on the real line for the Hilbert transform when $p = 2$. They showed that this inequality holds if W satisfies an analog of the A_2 condition:

$$[W]_{A_2} = \sup_Q \left| \left(\fint_Q W(x)\, dx \right)^{\frac{1}{2}} \left(\fint_Q W^{-1}(x)\, dx \right)^{\frac{1}{2}} \right|_{\mathrm{op}} < \infty. \tag{3.1}$$

This condition, however, does not extend to the case $p \neq 2$. An equivalent, but more technical definition of matrix A_p in terms of norm functions was conjectured by Treil and used by Nazarov and Treil [45] and by Volberg [57] to prove matrix-weighted norm inequalities for the Hilbert transform. Their idea was to replace matrices with norms on $\mathbb{R}^d$. Here we sketch their definition; for complete details see the above references or [7].

Let $\rho : \mathbb{R}^n \times \mathbb{R}^d \to [0, \infty)$ be a measurable function such that for a.e. $x \in \mathbb{R}^n$, $\rho(x, \cdot)$ is a norm on $\mathbb{R}^d$: that is, given v, $w \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$,

(1) $\rho(x, v) = 0$ if and only if $v = 0$;
(2) $\rho(x, v + w) \leq \rho(x, v) + \rho(x, w)$;
(3) $\rho(x, \alpha v) = |\alpha|\rho(x, v)$.

For instance, given a matrix weight W, we can define a norm by $\rho_W(x, v) = |W(x)v|$. Define the dual of a norm function ρ^* by

$$\rho^*(x, v) = \sup_{w \in \mathbb{R}^d, \rho(x,w) \leq 1} |\langle v, w \rangle|.$$

Finally, given a cube Q, and $1 \leq p < \infty$, define the average of a norm function on Q by

$$\langle \rho \rangle_{p,Q}(v) = \left(\fint_Q \rho(x, v)^p\, dx \right)^{\frac{1}{p}}.$$

With this notation, they defined a norm ρ to be in $\mathcal{A}_p$ if for every $v \in \mathbb{R}^d$,

$$\langle \rho^* \rangle_{p',Q}(v) \leq C \langle \rho \rangle_{p,Q}(v)^*$$

When $d = 1$ this immediately reduces to the Muckenhoupt A_p condition.

Later, Roudenko [52] gave an equivalent definition of $\mathcal{A}_p$ that more closely resembled the scalar definition: for $1 < p < \infty$, $W \in \mathcal{A}_p$ if

$$[W]_{A_p} = \sup_Q \fint_Q \left(\fint_Q |W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y)|_{\mathrm{op}}^{p'}\, dy \right)^{\frac{p}{p'}} dx < \infty. \tag{3.2}$$

Frazier and Roudenko [24] also introduced the concept of matrix $\mathcal{A}_1$ weights: $W \in \mathcal{A}_1$ if

$$[W]_{A_1} = \sup_Q \operatorname*{ess\,sup}_{x \in Q} \fint_Q |W^{-1}(x) W(y)|_{\mathrm{op}}\, dy < \infty. \tag{3.3}$$

Norm inequalities for Calderón-Zygmund singular integrals in $\mathbb{R}^n$ were proved by Christ and Goldberg [11, 27]. Treil and Volberg had earlier noted that a key obstacle to proving matrix-weighted inequalities was the lack of a maximal operator that did not lose the geometric information embedded in a vector-valued function. A key component of the proofs in [11, 27] is a scalar-valued, matrix weighted maximal operator (now referred to as the Christ-Goldberg maximal operator):

$$M_W \vec{f}(x) = \sup_Q \fint_Q |W(x) W^{-1}(y) \vec{f}(y)|\, dy \cdot \chi_Q(x).$$

The motivation for this definition comes from the following observation: if T is a singular integral operator (or indeed any linear operator), then T satisfies $\|T\vec{f}\|_{L^p(W)} \leq C\|\vec{f}\|_{L^p(W)}$ if and only if the operator T_W, defined by

$$T_W \vec{f}(x) = W(x)^{\frac{1}{p}} T(W^{-\frac{1}{p}} \vec{f})(x)$$

satisfies $\|T\vec{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leq C\|\vec{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)}$.

The first step was to prove that if $W \in \mathcal{A}_p$, $1 < p < \infty$, then M_W is bounded from $L^p(\mathbb{R}^n,\mathbb{R}^d)$ into $L^p(\mathbb{R}^n)$. This required the introduction of an auxiliary maximal operator M'_W, defined by

$$M_W\vec{f}(x) = \sup_Q \fint_Q |\mathcal{W}_Q^p W^{-1}(y)\vec{f}(y)|\,dy \cdot \chi_Q(x).$$

Here, $\mathcal{W}_Q^p$ is a constant matrix defined as follows: given the norm $\langle \rho_W\rangle_{p,Q}$, let K_W^p be the closed unit ball in $\mathbb{R}^d$ with respect to this norm:

$$K_W^p = \{v \in \mathbb{R}^d : \langle \rho_W\rangle_{p,Q}(v) \leq 1\}.$$

Then K_W^p is a convex set in $\mathbb{R}^d$, so there exists a unique ellipsoid of maximal volume, called the John ellipsoid and that we denote by E_Q^p, such that $E_Q^p \subset K_W^p \subset \sqrt{d}E_Q^p$. Finally, there exists a matrix $\mathcal{W}_Q^p$ such that $E_Q^p = \mathcal{W}_Q^p\mathbf{B}$, and the norm induced by this matrix is equivalent to $\langle \rho_W\rangle_{p,Q}$. The matrix $\mathcal{W}_Q^p$ is referred to as the reducing matrix associated to W on Q. Note that when $d = 1$, $\mathcal{W}_Q^p$ is just the p-average of the weight.

They showed that the auxiliary maximal operator maps $L^p(\mathbb{R}^n,\mathbb{R}^d)$ into $L^p(\mathbb{R}^n)$; this can be done using Calderón-Zygmund cubes and an argument analogous to that for the maximal operator. Then, via a stopping time argument, they showed that $\|M_W\vec{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)} \leq C\|M_W\vec{f}\|_{L^p(\mathbb{R}^n,\mathbb{R}^d)}$.

Given this maximal operator, to prove weighted norm inequalities for singular integrals they adapted the ideas of Coifman and Fefferman to prove a good-λ inequality. Define

$$T_W^*\vec{f}(x) = \sup_{\epsilon>0} |W^{\frac{1}{p}}(x)T_\epsilon(W^{-\frac{1}{p}}\vec{f})(x)|.$$

Then for every smooth function $\vec{f}$ with compact support, they proved that there exist constants $0 < b < 1$ and $c > 0$ such that for all $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : T_W^*\vec{f}(x) > \lambda, \max\{M'_W\vec{f}(x), M_W\vec{f}(x)\} < c\lambda\}| \leq \frac{1}{2}b^p|\{x \in \mathbb{R}^n : T_W^*\vec{f}(x) > b\lambda\}|.$$

As in the scalar case, this approach to proving matrix weighted norm inequalities does not yield quantitative estimates on the constant in terms of the $[W]_{\mathcal{A}_p}$ characteristic. After the proof of the scalar A_2 conjecture by Hytönen, it was conjectured that the corresponding result holds for matrix weights: for $1 < p < \infty$,

$$\|T\vec{f}\|_{L^p(W)} \leq C[W]_{\mathcal{A}_p}^{\max\{1,p'-1\}}\|\vec{f}\|_{L^p(W)}.$$

This problem was first considered by Bickel, Petermichl and Wick [6] and by Pott and Stoica [51] when $p = 2$. More recently, Nazarov, Petermichl, Treil and Volberg [43] showed that when $p = 2$, they can get a constant of the form $C(n,d,T)[W]_{A_2}^{\frac{3}{2}}$. (Also see [20].) Their proof is based on a deep generalization of the sparse domination estimates described above in the scalar case; they then estimate their sparse operator using square function estimates.

The sparse operator introduced in [43] replaces vector-valued functions by averages that are convex sets. They show that given $\vec{f} \in L_c^\infty(\mathbb{R}^n,\mathbb{R}^d)$, there exists a finite collection of sparse sets $\{\mathcal{S}_n\}_{n=1}^N$ such that

$$T\vec{f}(x) \in C\sum_{n=1}^N \sum_{Q\in\mathcal{S}_n} \langle\langle \vec{f}\rangle\rangle_Q \chi_Q(x), \tag{3.4}$$

where $\langle\langle f\rangle\rangle_Q$ is the convex set

$$\langle\langle \vec{f}\rangle\rangle_Q = \left\{\fint_Q k(y)\vec{f}(y)\,dy : k \in L^\infty(Q), \|k\|_\infty \leq 1\right\},$$

and the sum is the (infinite) Minkowski sum of convex sets (see below for a definition). They referred to this estimate as a convex body sparse domination. To complete their proof, however, they did not work directly with these convex set-valued functions, but rather replaced them by vector-valued sparse operators of the form

$$T^S\vec{f}(x) = \sum_{Q\in\mathcal{S}} \fint_Q \varphi_Q(x,y)\vec{f}(y)\,dy,$$

where for each Q, φ_Q is a real-valued function supported on $Q \times Q$ such that, for each x, $\|\varphi_Q(x,\cdot)\|_\infty \leq 1$. These they estimated using square function estimates.

In the case $p \neq 2$, quantitative results were proved by the author, Isralowitz and Moen [16], who got a constant of the form

$$C(n,d,p,T)[W]_{A_p}^{1+\frac{1}{p-1}-\frac{1}{p}}.$$

They used the sparse domination result of Nazarov, *et al.* and reduced to the vector-valued sparse operators, but instead of using square function estimates, they adapted techniques from the theory of A_p bump conditions in the study of scalar, two-weight norm inequalities. In both proofs there is a loss of information: even in the scalar case these techniques do not yield the A_2 conjecture. So the problem becomes to work directly with the convex set-valued sparse operator.

4. Convex set-valued functions and matrix weights

The convex body sparse operator of Nazarov *et al.* introduced a powerful new tool into the study of matrix weights. Building upon these ideas, we further developed the theory of convex set-valued functions to prove Jones factorization and Rubo de Francia extrapolation for matrix weights. This problem was first raised by Nazarov and Treil [45]:

> *Actually, the whole theory of scalar (A_p)-weights can be transferred to the matrix case except two results...the Peter Jones factorization and the Rubio-de-Francia extrapolation theory. But today (May 1, 1996) we do not know what the analogues of these two things are in high dimensions.*

Their assessment at that time was somewhat optimistic: for instance, there is still not a complete theory of reverse Hölder inequalities for matrix weights (though see [21]). Nevertheless, they did identify two fundamental problems in the study of matrix weights. While factorization and extrapolation are interesting in and of themselves, they gained greater importance with the study of the matrix $\mathcal{A}_2$ conjecture. As we described above, extrapolation played an important role in proving the scalar A_2 conjecture. Moreover, for a number of technical reasons matrix $\mathcal{A}_2$ weights are easier to work with than matrix $\mathcal{A}_p$ weights, $p \neq 2$. (E.g., the simpler definition (3.1) can be used.) Therefore, it seems natural to try to prove a sharp version of extrapolation in the matrix case.

Both results have been fully proved for matrix weights. Here we state the two main theorems from [7].

Theorem 4.1. *Given $1 < p < \infty$, then $W \in \mathcal{A}_p$ if and only if there exist commuting matrices W_0, $W_1 \in \mathcal{A}_1$ such that*

$$W = W_0 W_1^{1-p}.$$

Remark 4.2. For simplicity and ease of comparison to the scalar case, we state Theorem 4.1 assuming that the matrices W_0 and W_1 commute. We can remove this hypothesis, but to do so we must replace the product $W_0^{\frac{1}{p}} W_1^{\frac{1}{p'}}$ with a more complicated expression, the geometric mean of the two matrices: see [5].

Theorem 4.3. *Given p_0, $1 \le p_0 < \infty$, suppose that for some operator T and every $W_0 \in \mathcal{A}_{p_0}$,*

$$\bigg(\int_{\mathbb{R}^n} |W_0^{\frac{1}{p_0}} T\vec{f}|^{p_0}\,dx\bigg)^{\frac{1}{p_0}} \le N_{p_0}([W_0]_{p_0})\bigg(\int_{\mathbb{R}^n} |W_0^{\frac{1}{p_0}} \vec{f}|^{p_0}\,dx\bigg)^{\frac{1}{p_0}}.$$

Then for all p, $1 < p < \infty$, and $W \in \mathcal{A}_p$,

$$\bigg(\int_{\mathbb{R}^n} |W^{\frac{1}{p}} T\vec{f}|^{p}\,dx\bigg)^{\frac{1}{p}} \le N_{p}([W]_{p})\bigg(\int_{\mathbb{R}^n} |W^{\frac{1}{p}} \vec{f}|^{p}\,dx\bigg)^{\frac{1}{p}}.$$

Remark 4.4. We actually prove a more general version of Theorem 4.3, replacing the operator T by a family of pairs of functions (f, g). This more abstract approach to extrapolation was first suggested in [19] and systematically developed in [18]. As an immediate corollary to this approach we get via extrapolation vector-valued inequalities of the form

$$\bigg(\int_{\mathbb{R}^n} \bigg(\sum_{k=1}^{\infty} |W(x)Tf_k(x)|^q\bigg)^{\frac{p}{q}}\,dx\bigg)^{\frac{1}{p}} \le \bigg(\int_{\mathbb{R}^n} \bigg(\sum_{k=1}^{\infty} |W(x)f_k(x)|^q\bigg)^{\frac{p}{q}}\,dx\bigg)^{\frac{1}{p}},$$

$1 < q < \infty$. Such inequalities appear to be new in the matrix weight setting.

Remark 4.5. In Theorem 4.3 the function N_p depending on N_{p_0} has exactly the same form as the functions gotten in the sharp constant extrapolation theorem of Dragičević, *et al.* [22]. This is precisely what is needed to reduce the matrix $\mathcal{A}_2$ conjecture to the case $p = 2$.

The proofs of Theorems 4.3 and 4.1 are both long and extremely technical, and it is beyond the scope of this article to give many details. Instead we will provide a conceptual overview of the proofs and of the tools we developed for them. As part of the proof we have begun to lay the groundwork for studying harmonic analysis on convex set-valued functions. These results are of interest in their own right. Moreover, beyond their application to proving these two theorems, we believe that it will be useful for exploring more deeply the convex body sparse bounds of Nazarov, *et al.*, and so will be an important tool for proving the $\mathcal{A}_2$ conjecture. We will return to this point in the next section.

Underlying our proofs of Theorems 4.3 and 4.1 was the systematic philosophy of trying to replicate the proofs of extrapolation and factorization in the scalar case, particularly the elementary proofs of these results in [15, 18]. As we noted above, the proofs of extrapolation and factorization depended on the duality of scalar A_p and sharp bounds for weighted norm inequalities for the maximal operator; these in turn were used to build the Rubio de Francia iteration operator. Extrapolation further required the Jones factorization theorem, though as we saw in the proof sketched above, we only used the easier direction of this result.

The fundamental technical obstacle to the proof was lack of an appropriate definition of the maximal operator. While the Christ-Goldberg maximal operator M_W was sufficient to prove strong (p, p) bounds for singular integrals, it has the drawback that it maps a vector-valued function $\vec{f}$ to scalar-valued function $M_W\vec{f}$. Therefore, it cannot be iterated, and so cannot be used to construct a Rubio de Francia iteration operator. To overcome this problem, we passed from vector-valued functions to the larger category of convex set-valued functions, and defined a convex-set valued maximal operator.

We begin with some definitions related to convex sets. For complete details, see [7] and the references it contains. Let $\mathcal{K}$ denote the family of all convex sets $K \subset \mathbb{R}^d$ that are closed, bounded, and symmetric: i.e., if $x \in K$, then $-x \in K$. Sometimes it is necessary also to assume K is absorbing: that is, that $0 \in \operatorname{int}(K)$. However, we will not worry about this technical hypothesis. Given a set $K \subset \mathbb{R}^d$, let $|K| = \{|v| : v \in K\}$; given a matrix W, define $WK = \{Wv : v \in K\}$; note that WK is also a convex set. Given two convex sets K and L, their Minkowski sum is the set $K + L = \{u + v : u \in K, v \in L\}$.

As we noted above, every norm has associated to it the convex set which is its unit ball. The converse is also true: to every convex set K these is associated to it a unique norm, ρ_K on $\mathbb{R}^d$. Moreover, arguing as we did before, there exists a matrix W such that $\rho_K \approx \rho_W$. While we generally want to work with matrices $W \in \mathcal{A}_p$, at key points in the argument it is necessary to pass to working with more general norms and the underlying convex sets.

We now define convex-set valued functions F; there is actually a well-developed theory of such objects, but it does not appear to be well-known among harmonic analysts: see [3, 9]. Let $F : \mathbb{R}^n \to \mathcal{K}$ be such a map. There are several equivalent ways to define measurability of such functions F; for our purposes a useful and intuitive definition is that there exists a countable family of functions $\vec{f}_k : \mathbb{R}^n \to \mathbb{R}^d$ such that for almost every x,

$$F(x) = \overline{\{\vec{f}_k(x) : k \in \mathbb{N}\}}.$$

Such functions are referred to as selection functions.

Given such a convex set-valued function F, we can define for each x the associated John Ellipsoid $E(x)$. This ellipsoid-valued function is measurable. This fact has been used in the literature (see, for instance, Goldberg [27]) but we could not find a proof in the literature; a proof is given in [7].

The integral of a convex set-valued function can be defined using selection functions–this object is referred to as the Aumann integral [4] (see also [3]). Given $\Omega \subset \mathbb{R}^n$ and a function $F : \Omega \to \mathcal{K}$, define

$$S^1(\Omega, F) = \{\vec{f} \in L^1(\Omega, \mathbb{R}^d) : \vec{f}(x) \in F(x)\}.$$

Then the Aumann integral of F is defined to be the set

$$\int_\Omega F(x)\, dx = \left\{ \int_\Omega \vec{f}(x)\, dx : \vec{f} \in S^1(\Omega, F) \right\}.$$

It can be shown that since $F(x)$ is closed, bounded and convex, the Aumann integral is also a closed, convex set in $\mathbb{R}^d$.

There is a close connection between the Aumann integral and the convex averages $\langle\langle \vec{f} \rangle\rangle_Q$ used by Nazarov, *et al.* to define their convex body sparse operator. Given a vector-valued function $\vec{f}$, define $F_{\vec{f}}(x)$ to be the closed convex hull of the set $\{\vec{f}(x), -\vec{f}(x)\}$. Then $F_{\vec{f}}$ is a measurable, convex set-valued function, and

$$\langle\langle \vec{f} \rangle\rangle_Q = \fint_Q F_{\vec{f}}(x)\, dx.$$

Thus, the convex body sparse operator $T_\mathcal{S}$ is a convex set-valued function as defined above.

We use the Aumann integral to define a convex set-valued maximal operator. Given $F : \mathbb{R}^n \to \mathcal{K}$, let

$$MF(x) = \overline{\text{conv}} \left(\bigcup_Q \fint_Q F(y)\, dy \cdot \chi_Q(x) \right);$$

that is, MF is the closed, convex hull of the union of the Aumann integral averages of F over all cubes containing x. Then $MF(x)$ is a measurable, convex set-valued function. The intuition behind this definition is that the Hardy-Littlewood maximal operator finds the largest average in magnitude, and so uses the supremum. The convex set-valued maximal operator finds the largest average in magnitude in each direction; to preserve the information about direction we take the union of all averages.

It should be noted that this maximal operator does not preserve some natural subsets of $\mathcal{K}$. For instance, given a vector-valued function $\vec{f}$, then $MF_{\vec{f}}$ can be an absorbing convex set that properly contains the convex set that is the closure of $\{M\vec{f}, -M\vec{f}\}$. If F is ellipsoid-valued–that is, $F = W\mathbf{B}$, where W is a matrix valued function–then MF need not be ellipsoid valued.

The convex set-valued maximal operator has a number of properties that correspond to those of the Hardy-Littlewood maximal operator: for convex set-valued functions F and G, almost every $x \in \mathbb{R}^n$, and $\alpha \in [0, \infty)$,

(1) $F(x) \subset MF(x)$;
(2) $M(F+G)(x) \subset MF(x) + MG(x)$, where the sum is the Minkowski sum;
(3) $M(\alpha F)(x) = \alpha MF(x)$.

The maximal operator M also satisfies L^p norm inequalities. For $1 \leq p < \infty$, define $L^p_{\mathcal{K}}(\mathbb{R}^n, |\cdot|)$ to be the collection of all convex set-valued functions F such that

$$\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)} = \left(\int_{\mathbb{R}^n} |F(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Then for $1 < p < \infty$ we have that $\|MF\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)} \leq C\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,|\cdot|)}$.

Weighted norm inequalities for the convex maximal operator are governed by the matrix $\mathcal{A}_p$ weights. Define $L^p_{\mathcal{K}}(\mathbb{R}^n, W) = L^p_{\mathcal{K}}(W)$ to be all convex set-valued functions F such that

$$\|F\|_{L^p_{\mathcal{K}}(\mathbb{R}^n,W)} = \left(\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x)F(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Theorem 4.6. *Given p, $1 < p < \infty$, and $W \in \mathcal{A}_p$, for every $F \in L^p_{\mathcal{K}}(W)$,*

$$\|MF\|_{L^p_{\mathcal{K}}(W)} \leq C(n,d,p)[W]_{\mathcal{A}_p}^{p'-1}\|F\|_{L^p_{\mathcal{K}}(W)}.$$

The proof of Theorem 4.6 uses the L^p norm inequalities for the Christ-Goldberg maximal operator M_W; the sharp constant in terms of $[W]_{\mathcal{A}_p}$ was proved by Isralowitz and Moen [34]. It would be of interest to have a direct proof that did not require using the Christ-Goldberg maximal operator.

We now define the analog of the A_1 condition for convex set-valued functions. Given $F : \mathbb{R}^n \to \mathcal{K}$, we say that $F \in \mathcal{A}_1^{\mathcal{K}}$ if $MF(x) \subset CF(x)$. Denote the infimum of all such constants C by $[F]_{\mathcal{A}_1^{\mathcal{K}}}$. There is a very close relationship between convex set-valued $\mathcal{A}_1^{\mathcal{K}}$ and matrix $\mathcal{A}_1$.

Theorem 4.7. *Given a matrix weight W, $W \in \mathcal{A}_1$ if and only if $F = W\mathbf{B} \in \mathcal{A}_1^{\mathcal{K}}$, and $[W]_{\mathcal{A}_1} \approx [F]_{\mathcal{A}_1^{\mathcal{K}}}$.*

The proof of this result required a careful development of the properties of norm functions and their duals on the one hand, and the properties of convex sets and their polar bodies on the other.

With the machinery we have developed, we define the convex set-valued analog of the Rubio de Francia iteration operator. Given $F : \mathbb{R}^n \to \mathcal{K}$ and $W \in \mathcal{A}_p$, $1 < p < \infty$, define

$$\mathcal{R}F(x) = \sum_{k=0}^{\infty} \frac{M^k F(x)}{2^k \|M\|^k_{L^p_{\mathcal{K}}(W)}}.$$

Then $\mathcal{R}F$ has the following properties which are the exact analogs of the properties of the scalar iteration operators:

(1) $F(x) \subset \mathcal{R}F(x)$;
(2) $\|\mathcal{R}F\|_{L^p_{\mathcal{K}}(W)} \leq 2\|F\|_{L^p_{\mathcal{K}}(W)}$;
(3) $\mathcal{R}F \in \mathcal{A}_1^{\mathcal{K}}$ and $[\mathcal{R}F]_{\mathcal{A}_1^{\mathcal{K}}} \leq 2\|M\|_{L^p_{\mathcal{K}}(W)}$.

We now give an overview of the proofs of Theorems 4.1 and 4.3. The proof of factorization in the Jones factorization theorem follows the scalar proof in [15] closely. There is one major technical obstacle: the scalar proof uses a variant of the Hardy-Littlewood maximal operator, $M_s f(x) = M(|f|^s)^{\frac{1}{s}}$, $s > 1$, which is a sublinear operator. To define $M_s F(x)$, requires replacing F by an appropriate ellipsoid (so that the power F^s is defined) and then proving that the resulting operator is sublinear. The converse, proving reverse factorization, is very delicate and much more difficult than in the scalar case. We were not able to prove it using the definitions of matrix $\mathcal{A}_p$ and $\mathcal{A}_1$ given by Roudenko and Frazier. For this proof we were required to work with the definition of matrix $\mathcal{A}_p$ in terms of norms as originally given by Treil and Volberg. Intuitively, the proof can be thought of as an interpolation argument between finite dimensional Banach spaces.

The proof of extrapolation follows the proof of sharp constant extrapolation in the scalar case given in [18]. Because of the way in which we define the matrix $\mathcal{A}_p$ condition, we are also able to include naturally in the proof a result for extrapolation from the endpoint $p = \infty$; this gives a quantitative version of an extrapolation result first proved by Harboure, Macías and Segovia [29]. Recently we learned that this quantitative version was proved earlier by Nieraeth [46, 47]. The proof uses reverse factorization and also requires passing between convex set-valued functions and closely associated ellipsoid valued ones.

5. Final remarks on the matrix A_2 conjecture

In this final section we return to the matrix $\mathcal{A}_2$ conjecture. As noted above, given Theorem 4.3, it is enough to prove the $\mathcal{A}_2$ conjecture in the case $p = 2$, as our version of matrix extrapolation yields the sharp constants for the other values of p. Moreover, the machinery of harmonic analysis on convex set-valued functions provides a way to work with the convex body sparse operator directly, avoiding the loss of information in the proofs in [16, 43] that yielded the best known estimates. We believe that this approach will yield the proof of the conjecture.

We illustrate this approach by sketching a proof of weighted norm inequalities for the convex set-valued sparse operator that is quite elementary, but yields a suboptimal constant. By inequality (3.4) this yields an elementary proof of a matrix weighted norm inequality for singular integrals. Given $F \in L^2_{\mathcal{K}}(W)$, by a duality argument we can show that there exists $G \in L^2_{\mathcal{K}}(W^{-1})$, $\|G\|_{L^2_{\mathcal{K}}(W^{-1})} \leq \sqrt{d}$, such that

$$\|T_{\mathcal{S}}F\|_{L^2_{\mathcal{K}}(W)} \leq \int_{\mathbb{R}^n} \langle T_{\mathcal{S}}F(x), G(x)\rangle \, dx.$$

If we apply the definition of $T_{\mathcal{S}}$, linearity, and Cauchy's inequality, we get

$$\begin{aligned}
&\leq 2\sum_{Q\in\mathcal{S}} \left\langle \fint_Q F(y)\,dy, \fint_Q G(y)\,dy \right\rangle |E(Q)| \\
&= \sum_{Q\in\mathcal{S}} \int_{E(Q)} \left\langle \fint_Q W^{\frac{1}{2}}(x)W^{-\frac{1}{2}}(y)W^{\frac{1}{2}}(y)F(y)\,dy, \right. \\
&\qquad \left. \fint_Q W^{-\frac{1}{2}}(x)W^{\frac{1}{2}}(y)W^{-\frac{1}{2}}(y)G(y)\,dy \right\rangle dx \\
&\leq \sum_{Q\in\mathcal{S}} \int_{E_Q} M_W(W^{\frac{1}{2}}F) M_{W^{-1}}(W^{-\frac{1}{2}}G)\,dx \\
&\leq \|M_W(W^{\frac{1}{2}}F)\|_{L^2} \|M_{W^{-1}}(W^{-\frac{1}{2}}G)\|_{L^2},
\end{aligned}$$

where M_W is the Christ-Goldberg maximal operator. By the sharp constant estimate for M_W,

$$\begin{aligned}
&\leq C[W]^2_{A_2} \|F\|_{L^2_{\mathcal{K}}(W)} \|G\|_{L^2_{\mathcal{K}}(W^{-1})} \\
&\leq C[W]^2_{A_2} \|F\|_{L^2_{\mathcal{K}}(W)}.
\end{aligned}$$

In order to prove the sharp constant result using this approach, we would like to follow the scalar proof sketched above, which used the universal dyadic maximal operator $\tilde{M}^d_w$. We have shown that the above argument can be modified to replace the Christ-Goldberg maximal operators with an operator $\tilde{M}^d_W$ that is a matrix weighted version of the universal dyadic maximal operator. If this operator were bounded on $L^2(W)$ with a constant independent of the $\mathcal{A}_2$ characteristic of W, then we could prove the matrix $\mathcal{A}_2$ conjecture. However, Nazarov, Petermichl, Škreb, and Treil [44] recently showed that there exist matrices W such that $\tilde{M}^d_W$ is not bounded on $L^2(W)$. It remains an open question whether the above approach can be further adapted to overcome this obstacle, if a new approach to this problem is required, or if the matrix $\mathcal{A}_2$ conjecture is false and an exponent larger than 1 is required.

As in the scalar case, the currently best known exponent $\frac{3}{2}$ seems unnatural. However, it is worth noting that this exponent has recently appeared as a lower bound in the search for the sharp constant for rough singular integrals [30]. If one could prove a bound of the form $[W]_{\mathcal{A}_2}^{s(d)}$, where $s(d)$ was some exponent depending in the dimension d, $1 \leq s(d) \leq \frac{3}{2}$ and $s(1) = 1$, then a bound larger than 1 might be seen more reasonable.

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