Bergman, Szegö and Sobolev kernels

Akira Yamada

Abstract

Two topics are outlined below:

Part I: Results concerning *Hejhal's theorem* about the relation between the Bergman kernel and the Szegö kernel on planar regular regions.

Part II: Determination of the *explicit formula* for the reproducing kernel of the Sobolev space on a bounded interval.

Part I. The Bergman and the Szegö kernel

1 Introduction

Let Ω be an n-ply connected planar regular region ($g = n-1$), where a planar regular region is a domain in the complex plane which is bounded by a finite number of disjoint analytic Jordan curves. In 1950 Schiffer [18] obtained the identity between the Bergman and the Szegö kernels:

$$K_B(z,w) = 4\pi K_S^2(z,w) + \sum_{i,j=1}^{g} h_{ij} v_i(z)\overline{v_j(w)},$$

where $H = (h_{ij})$ is a Hermitian matrix, and $\{v_i\}_{i=1}^{g}$ is a canonical homology basis of the Abelian differentials of the first kind on the Schottky double $\hat{\Omega}$ of Ω. In 1972 Hejhal [10] succeeded to prove the positive-definiteness of the matrix H (i.e. $H > 0$). Hejhal's theorem implied $4\pi K_S^2 \ll K_B$, one of the most important facts in the theory of kernel functions and complex analysis in the complex plane. His theorem was obtained by using Riemann's theta function, and, indeed, considered to be a very deep result. So, the next question naturally arises: Is there an elementary proof of $H > 0$?

Unfortunately, we could not obtain another proof, though we obtained [21]

- Some results concerning Schiffer's identity and Hejhal's theorem by introducing integral operator whose kernel is the product of Szegö kernels.
- $H > 0 \iff$ contractivity of the integral operator + equality condition for the norms.

This article is an expository report of our recent results, and the details and proofs will be given elsewhere.

Notation. We adopt the following notations:

- $B = \{f$: regular functions in Ω with $\iint_\Omega |f|^2 dxdy < \infty\}$: the Bergman space on Ω, K_B: the Bergman kernel on Ω.
- $B_E = \{f \in B$ in Ω with single-valued indefinite integral$\}$: the exact Bergman space, K_E: the exact Bergman kernel on Ω.
- $S = \{f$: regular functions in Ω with $\int_{\partial\Omega} |f|^2 |dz| < \infty\}$: the Szegö space on Ω, K_S: the Szegö kernel on Ω.
- $S^{\otimes 2}$: the tensor product RKHS $S \otimes S$ on Ω^2, $K_S^{\otimes 2} = K_S \otimes K_S$: its reproducing kernel.
- $S^{\times 2}$: the operator range of the pullback $f \in S^{\otimes 2} \mapsto f \circ \phi \in \mathbb{C}^\Omega$ with ϕ: $x \in \Omega \mapsto (x, x) \in \Omega^2$, whose reproducing kernel is K_S^2.

1.1 The operator range and the reproducing kernel

A Hilbert space $\mathcal{H}$ is called a *reproducing kernel Hilbert space (RKHS)* if $\mathcal{H}$ is a functional Hilbert space $\mathcal{H}$ on a set E such that the evaluation map is bounded for every point $x \in E$: $\forall f \in \mathcal{H}$, $\forall x \in E$, $\exists C_x > 0$ s.t. $|f(x)| \leq C_x \|f\|$. For the general theory of RKHSs the reader is referred to e.g. [3,4,15]. By the Riesz representation theorem, for any $x \in E$ there exists a unique function $k_x \in \mathcal{H}$ such that $f(x) = \langle f, k_x \rangle$, $\forall f \in \mathcal{H}$. The function $k_x \in \mathcal{H}$ is called the *reproducing kernel* for the point $x \in E$. $k(x, y) = k_y(x) = \langle k_y, k_x \rangle$ is the *reproducing kernel* for $\mathcal{H}$. If $k(x, y)$ is the reproducing kernel of a RKHS on E, then it is easy to see that $k \gg 0$ on E, i.e. $\forall n \in \mathbb{N}$, $\forall x_1, \ldots, x_n \in E$, $\forall c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{i,j=1}^n c_i \bar{c}_j k(x_i, x_j) \geq 0$. Most important is the fact that the converse holds [3], i.e. if $k \gg 0$ on E, then there exists a unique RKHS on E whose reproducing kernel is k.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space. For a continuous linear operator $A\colon \mathcal{H} \to \mathbb{C}^E$, we have a linear isomorphism:

$$(\ker A)^\perp \cong \operatorname{ran} A.$$

The *operator range* $\mathcal{M}(A)$ is the space $\operatorname{ran} A = A(\mathcal{H})$ equipped with the inner product induced by the above isomorphism. $\mathcal{M}(A)$ is a Hilbert space, and its norm $\|\cdot\|_{\mathcal{M}(A)}$ is called the *range norm*, which is given by

$$\|y\|_{\mathcal{M}(A)} = \inf\{\|x\|\colon Ax = y,\ x \in \mathcal{H}\}.$$

Consider a complex topological vector space V contained in the product space $\mathbb{C}^E$ such that the inclusion operator $V \hookrightarrow \mathbb{C}^E$ is continuous. The space $\mathbb{C}^E$ itself and RKHSs on E are examples of such spaces.

Theorem 1.1. *Let V be as above and let H be a Hilbert space. Then, the operator range $\mathcal{M}(T)$ of a continuous linear operator $T\colon H \to V$ is a RKHS on E, and the reproducing kernel k_y of $\mathcal{M}(T)$ for the point $y \in E$ is given by*

$$k_y = TT^*\delta_y = Tg_y,$$

where $\delta_y \in V^*$ is the point evaluation map of V at $y \in E$, and $g_y \in H$ is the Riesz representation of the continuous linear functional $f \in H \mapsto (Tf)(y) \in \mathbb{C}$, i.e. $(Tf)(y) = \langle f, g_y\rangle_H$, $\forall f \in H$.

In particular, if V is a RKHS on E with the reproducing kernel K_y for $y \in E$, then,

$$k_y = TT^*K_y, \quad k(x,y) = \langle T^*K_y, T^*K_x\rangle_H,$$

where $k(x,y)$ is the reproducing kernel of $\mathcal{M}(T)$.

1.2 Sum and product of reproducing kernels

We summarize here basic facts about the sum and the product of reproducing kernels.

- If k_j is the reproducing kernel of the RKHS $\mathcal{H}_{k_j}$ on E $(j = 1, 2)$,

$$k_1,\ k_2 \gg 0 \implies k_1 + k_2 \gg 0,\ k_1k_2 \gg 0.$$

- The RKHSs $\mathcal{H}_{k_1+k_2}$ and $\mathcal{H}_{k_1k_2}$ are, respectively, the operator range of the following maps:

(a) $f \oplus g \in \mathcal{H}_{k_1} \oplus \mathcal{H}_{k_2} \mapsto f + g \in \mathbb{C}^E$. The identity $(f+g)(y) = \langle f \oplus g, k_{1,y} \oplus k_{2,y} \rangle$ shows that $k_y = k_{1,y} + k_{2,y}$ by Theorem 1.1.

(b) $f \otimes g \in \mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2} \mapsto fg \in \mathbb{C}^E$, which is given by the pullback of the map $\phi\colon x \in E \mapsto (x, x) \in E^2$, so $(fg)(y) = \langle f \otimes g, k_{1,y} \otimes k_{2,y} \rangle$ implies $k_y = k_{1,y} k_{2,y}$ by Theorem 1.1.

Proposition 1.1. *Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be RKHSs on E. Then, as vector spaces, $\mathcal{H}_{K_1+K_2} = \mathcal{H}_{K_1} + \mathcal{H}_{K_2}$, and for $f_1 \in \mathcal{H}_{K_1}$ and $f_2 \in \mathcal{H}_{K_2}$ we have the Pythagorean inequality*

$$\|f_1 + f_2\|^2_{\mathcal{H}_{K_1+K_2}} \le \|f_1\|^2_{\mathcal{H}_{K_1}} + \|f_2\|^2_{\mathcal{H}_{K_2}}.$$

Equality holds in the above inequality if and only if

$$\langle f_1, h \rangle_{\mathcal{H}_{K_1}} = \langle f_2, h \rangle_{\mathcal{H}_{K_2}}, \quad \text{for all } h \in \mathcal{H}_{K_1} \cap \mathcal{H}_{K_2}.$$

Proof. See [3, p. 352] and [16, p. 44]. □

2 Useful identities

The conjugate kernels of K_B, K_E and K_S are denoted, respectively, by L_B, L_E and L_S. Since the conjugate kernels are important for calculations involving kernel functions, we summarize known identities among them without proof (cf. Bergman [4, p. 50, 60, 80]).

Proposition 2.1. *The following identities hold for $z, w \in \Omega$.*

$$\begin{aligned} K_B(z,w) &= -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \overline{w}}, & L_B(z,w) &= -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial w}, \\ K_E(z,w) &= \frac{2}{\pi}\frac{\partial^2 N(z,w)}{\partial z \partial \overline{w}}, & L_E(z,w) &= -\frac{2}{\pi}\frac{\partial^2 N(z,w)}{\partial z \partial w}, \\ k_B(z,w) &= \frac{1}{2\pi}[N(z,w) - G(z,w)]. \end{aligned}$$

Let $\{v_j\}_{j=1}^g$ be the canonical basis of the holomorphic differentials on the double $\hat{\Omega}$.

Theorem 2.1 (Schiffer [18], 1950)**.** *There exist Hermitian $g \times g$ matrices $H = (h_{jk})$, $\tilde{H} = (\tilde{h}_{jk})$, $C = (c_{jk})$ such that the following identities hold.*

$$
\begin{aligned}
K_B(z,w) &= 4\pi K_S(z,w)^2 + \sum_{j,k=1}^{g} h_{jk} v_j(z)\overline{v_k(w)}, \qquad (1)\\
4\pi K_S(z,w)^2 &= K_E(z,w) + \sum_{j,k=1}^{g} \tilde{h}_{jk} v_j(z)\overline{v_k(w)},\\
K_B(z,w) &= K_E(z,w) + \sum_{j,k=1}^{g} c_{jk} v_j(z)\overline{v_k(w)},\\
C &= H + \tilde{H}.
\end{aligned}
$$

Hejhal has made a great breakthrough step forward by refining the result of Schiffer. If $(2\pi i I_g, \tau)$ is the period matrix of the compact Riemann surface $\hat{\Omega}$, then, from the symmetry of $\hat{\Omega}$, $\tau < 0$. The Riemann's theta function $\theta(z)$ is given by

$$\theta(z) = \sum_{m \in \mathbb{Z}^g} \exp\{\tfrac{1}{2} m\tau {}^t m + m^t z\}, \quad z \in \mathbb{C}^g.$$

Theorem 2.2 (Hejhal [10], 1972)**.**

$$K_B(z,w) = 4\pi K_S^2(z,w) + \frac{1}{\pi}\sum_{i,j=1}^{g} \frac{\partial^2 \log\theta(0)}{\partial z_i \partial z_j} v_i(z)\overline{v_j(w)}.$$

Note that Hejhal's identity is a special case of Fay's trisecant formula [9]: for $e \in \mathbb{C}^g$, $x, y, a, b \in R$,

$$
\begin{aligned}
&\frac{\theta(x-a-e)\theta(y-b-e)}{E(x,a)E(y,b)} - \frac{\theta(x-b-e)\theta(y-a-e)}{E(x,b)E(y,a)}\\
&\qquad = \frac{\theta(e)\theta(x+y-a-b-e)E(x,y)E(b,a)}{E(x,a)E(x,b)E(y,a)E(y,b)},
\end{aligned}
$$

where $E(x,y)$ is the prime form.

Corollary 2.1. $H = \pi^{-1}\Big(\dfrac{\partial^2 \log\theta(0)}{\partial z_i \partial z_j}\Big) > 0$ *and* $4\pi K_S^2 \ll K_B$*.*

Proof. Since $\theta(z)$ is an even function,

$$\frac{\partial^2 \log\theta}{\partial z_i \partial z_j}(0) = \frac{\theta(0)\theta_{ij}(0) - \theta_i(0)\theta_j(0)}{\theta^2(0)} = \frac{\theta_{ij}(0)}{\theta(0)}.$$

Clearly, $\theta(0) > 0$, and we have

$$\theta_{ij}(0) = \sum_{m\in\mathbb{Z}^g} m_i m_j \exp\{\tfrac{1}{2} m\tau{}^t m\}.$$

Thus, it is easy to see that $(\theta_{ij}(0)) > 0$, which implies $\left(\frac{\partial^2 \log\theta}{\partial z_j \partial z_k}(0)\right) > 0$. From Schiffer's identity and the fact $H > 0$ it follows easily that $4\pi K_S^2 \ll K_B$. $\square$

In what follows, we shall call the fact $H > 0$ as *"Hejhal's theorem"*.

2.1 Schiffer's identity

The following Schiffer's identity [18, p. 346] led us to begin the study of our paper.

Proposition 2.2 (Schiffer, 1950)**.** *If f is a regular function on Ω with finite Dirichlet integral, then, for $z, w \in \Omega$,*

$$\iint_\Omega K_S(z,\zeta)K_S(w,\zeta)f'(\zeta)\,d\xi d\eta = \frac{1}{2}(f(z) - f(w))L_S(z,w),$$

where L_S is the conjugate kernel of the Szegö kernel K_S.

Letting $w \to z$, for $f \in B_E$,

$$4\pi \iint_\Omega K_S(z,\zeta)^2 f(\zeta)\,d\xi d\eta = f(z).$$

Exploring the meaning of this formula was the motivation that led us to the research of this paper.

3 Integral operator and Hejhal's theorem

Let $\mathcal{H}$ be a RKHS on E and let $\phi\colon F \to E$ be a map. Then, the RKHS $\phi^*(\mathcal{H})$ on F, the pullback of $\mathcal{H}$ by ϕ, is defined as the operator range of $\mathcal{H}$ by the linear map given by

$$f \in \mathcal{H} \mapsto \phi^* f = f \circ \phi \in \mathbb{C}^F,$$

which induces a coisometry $T_\phi\colon \mathcal{H} \to \phi^*(\mathcal{H})$.

Lemma 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be RKHSs on a set E and F, respectively. Suppose $\phi^*(\mathcal{H}) \subset \mathcal{K}$ and let $\iota\colon \phi^*(\mathcal{H}) \hookrightarrow \mathcal{K}$ be the inclusion operator. Then,*

(i) *The operators ι and $T = \iota T_\phi \colon \mathcal{H} \to \mathcal{K}$ are bounded.*

(ii) *For $f \in \mathcal{K}$, $z \in E$ and $w \in F$, the adjoint T^* is given by*

$$(T^*f)(z) = \langle f, k_z \circ \phi \rangle_{\mathcal{K}},$$
$$T^*K_w = k_{\phi(w)},$$

where k_z and K_w are the reproducing kernels for $z \in E$ of $\mathcal{H}$ and for $w \in F$ of $\mathcal{K}$, respectively.

Proof. R_ϕ is bounded because by definition R_ϕ is a coisometry. Since $\phi^*(\mathcal{H})$ and $\mathcal{K}$ are both RKHSs on F, usual application of the closed graph theorem implies the boundedness of the inclusion ι, so that R is bounded. By the reproducing property, for $z \in E$ and $f \in \mathcal{K}$,

$$(R^*f)(z) = \langle R^*f, k_z \rangle_{\mathcal{H}} = \langle f, Rk_z \rangle_{\mathcal{K}} = \langle f, k_z \circ \phi \rangle_{\mathcal{K}}.$$

Putting $f = K_w$ in the above we have the second identity. □

Lemma 3.2. *$S^{\times 2} = B$ as vector spaces.*

Proof. From the identity (1) we easily conclude that there exists a constant $M > 0$ such that

$$4\pi K_S(z,w)^2 \ll K_B(z,w) + M \sum_{j=1}^{g} v_j(z)\overline{v_j(w)}.$$

Since K_S^2 is the reproducing kernel of the RKHS $S^{\times 2}$, by the general theory of the reproducing kernel we have $S^{\times 2} \subset B + \Gamma$. Since Γ is the space of the first kind of Abelian differentials on $\hat{\Omega}$, it is clear that $\Gamma \subset B$. Thus, $S^{\times 2} \subset B$. On the other hand, by a similar reasoning we see that $B \subset S^{\times 2} + \Gamma$. From $\Gamma \subset S$ and $1 \in S$, we conclude that $\Gamma \subset S^{\times 2}$. Thus, $B \subset S^{\times 2}$, and so $B = S^{\times 2}$ as vector spaces. □

Let Ω be a planar regular region, and let $\delta\colon z \in \Omega \mapsto (z,z) \in \Omega \times \Omega$ be the diagonal map. Since $S^{\times 2} = \delta^*(S^{\otimes 2})$, Lemma 3.1 and Lemma 3.2 imply that the inclusion operator $\iota\colon S^{\times 2} \hookrightarrow B$ and the operator $T = \iota R_\delta \colon S^{\otimes 2} \to B$ are bounded.

Theorem 3.1. *The following hold.*

(i) *If $f \in B$, $g \in S^{\times 2}$ and $(z, w) \in \Omega^2$, then*

$$\begin{aligned} T^* f(z,w) &= \langle f, K_{S,z} K_{S,w} \rangle_B = \iint_\Omega K_S(z,\zeta) K_S(w,\zeta) f(\zeta)\, dm(\zeta), \\ TT^* f(z) &= (\iota^* f)(z) = \iint_\Omega K_S(z,\zeta)^2 f(\zeta)\, dm(\zeta), \\ R_\delta^* g(z,w) &= \langle g, K_{S,z} K_{S,w} \rangle_{S^{\times 2}}. \end{aligned}$$

(ii) *If $f \in B_E$, then*

$$\begin{aligned} T^* f(z,w) &= 2^{-1} L_S(z,w) \int_w^z f(\zeta)\, d\zeta \in S^{\otimes 2}, \\ 4\pi\, TT^* f &= 4\pi\, \iota^* f = f. \end{aligned}$$

(iii) *The operator T is surjective, and its adjoint T^* is an injective left semi-Fredholm operator with $\operatorname{ran} T^* = (\ker T)^\perp$ closed in $S^{\otimes 2}$. Also, ι is a linear topological isomorphism.*

(iv) *If $f = f_E \oplus v \in B_E \oplus \Gamma$ is the orthogonal decomposition of $f \in B$, then*

$$4\pi TT^* f = f_E + \sum_{j,k=1}^{g} \overline{\tilde{h}_{jk}} \langle v, v_j \rangle v_k.$$

The operator TT^* is invertible with $TT^*(B_E) = B_E$ and $TT^*(\Gamma) = \Gamma$.

Proposition 3.1. *For planar regular regions, the matrices $\tilde{H}$ and C are positive definite, and we have*

$$\langle T^* v_j, T^* v_k \rangle_{S^{\otimes 2}} = \pi^2 \int_{B_k} \Big(\int_{B_j} K_S^2(z,w)\, \overline{dw} \Big)\, dz.$$

We next prepare a lemma concerning the relation between a Hermitian matrix and a finitely generated reproducing kernel.

Lemma 3.3. *Let $K(z, w)$ be the function*

$$K(z,w) = \sum_{j,k=1}^{n} a_{jk} f_j(z) \overline{f_k(w)},$$

where $A = (a_{jk})$ is an $n \times n$ Hermitian matrix, and $\{f_j\}_{j=1}^n$ is a linearly independent set of functions on E. If $f = {}^t(f_1, \ldots, f_n)$, then the following hold.

(i) $A \geq 0$ *if and only if* $K(z,w) \gg 0$ *on* $E \times E$.

(ii) *If* $A \geq 0$*, then the RKHS* $\mathcal{H}_K$ *on* E *with the reproducing kernel* K *is given by* $\operatorname{span}\{f(z)^*Af \colon z \in E\}$ *with the inner product satisfying*

$$\langle f(z)^*Af, f(w)^*Af\rangle_{\mathcal{H}} = \langle Af(w), f(z)\rangle_{\mathbb{C}^n} = \langle f(w), Af(z)\rangle_{\mathbb{C}^n}$$

Also, we have $\dim \mathcal{H}_K = \operatorname{rank} A$.

Proof. (i) If $A \geq 0$, then by setting $\beta_j = \sum_{k=1}^m \alpha_k f_j(z_k)$, $j = 1, \ldots, n$, for any $m \in \mathbb{N}$, $\alpha_j \in \mathbb{C}$ and $z_j \in E$, $j = 1, \ldots, m$, we have

$$\sum_{k,l=1}^{m} \alpha_k \overline{\alpha}_l K(z_k, z_l) = \sum_{i,j=1}^{n} a_{ij} \beta_i \overline{\beta}_j \geq 0. \tag{2}$$

Thus, $K \gg 0$.

Conversely, suppose that $K \gg 0$. Let V be the subspace

$$\{(\beta_j) \in \mathbb{C}^n \colon \alpha_1, \ldots, \alpha_m \in \mathbb{C};\ z_1, \ldots, z_m \in E\ (m = 1, 2, \ldots)\}$$

of $\mathbb{C}^n$. In view of (2), to prove $A \geq 0$ it suffices to show that the orthogonal complement $V^\perp = \mathbb{C}^n \ominus V$ of V is $\{0\}$. So let us suppose that $(c_j) \in V^\perp$. Then,

$$0 = \sum_{j=1}^{n} \overline{c}_j \beta_j = \sum_{k=1}^{m} \alpha_k \sum_{j=1}^{n} \overline{c}_j f_j(z_k).$$

Since $\alpha_k \in \mathbb{C}$ and $z_k \in E$ are arbitrary, we have $\sum_{j=1}^n \overline{c}_j f_j = 0$. Since $\{f_j\}$ is linearly independent, $(c_j) = 0$, as desired.

(ii): First, define the inner product for elements of the form $f(z)^*Af$, $z \in E$, by

$$\langle f(w)^*Af, f(z)^*Af\rangle = \langle Af(z), f(w)\rangle_{\mathbb{C}^n} = \langle f(z), Af(w)\rangle_{\mathbb{C}^n}.$$

Next, expand the definition of the inner product to all the elements of $\mathcal{H}_K$ by linearity. This inner product for $\mathcal{H}_K$ is well-defined. Indeed, $\sum_j \alpha_j f(w_j)^*Af = 0$ on E if and only if $\langle f(z), A\sum_j \overline{\alpha}_j f(w_j)\rangle_{\mathbb{C}^n} = 0$ for all $z \in E$. Thus, the inner product is well-defined for the first variable of the inner product. Similarly, it is also well-defined for the second variable. The extended Schwarz inequality gives its positive definiteness.

By definition of the inner product, we have

$$\langle f(w)^*Af, f(z)^*Af\rangle = f(w)^*Af(z).$$

Thus, $\mathcal{H}_K$ is a finite dimensional RKHS on E with the reproducing kernel K, since $K_z = f(z)^*Af = K(\cdot, z)$. Now by diagonalizing the Hermitian matrix A, there exists a unitary matrix P and the eigenvalues $\{\lambda_j\}_{j=1}^n$ of A such that $\lambda_j > 0$ $(j = 1, \ldots, \sigma)$, $\lambda_j = 0$ $(j = \sigma + 1, \ldots, n)$, and

$$P^*AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

If $g = {}^t(g_1, \ldots, g_n) = P^*f$, then $f = Pg$ and

$$\begin{aligned} K(z, w) &= \langle Af(z), f(w)\rangle_{\mathbb{C}^n} = \langle P^*APg(z), g(w)\rangle_{\mathbb{C}^n} \\ &= \sum_{j=1}^{\sigma} \lambda_j g_j(z)\overline{g_j(w)}. \end{aligned}$$

Since $\{g_j\}$ is linearly independent, the form of K as above shows that $\{\lambda_j^{1/2} g_j\}_{j=1}^{\sigma}$ is a CONS for $\mathcal{H}_K$. Thus, $\dim \mathcal{H}_K = \sigma = \mathrm{rank}\, A$. □

Theorem 3.2. *The orthogonal decomposition $S^{\times 2} = B_E \oplus \Gamma_S$ holds, where Γ_S is the set Γ considered as a subspace of the RKHS $S^{\times 2}$. More precisely, for $f = f_1 \oplus f_2$ and $g = g_1 \oplus g_2$ in $B_E \oplus \Gamma_S$,*

$$\langle f, g\rangle_{S^{\times 2}} = 4\pi\langle f_1, g_1\rangle_B + \langle f_2, g_2\rangle_{S^{\times 2}}.$$

In particular, $2\sqrt{\pi}\,\|f\|_B = \|f\|_{S^{\times 2}}$ for all $f \in B_E$.

We next state Douglas's formula for the Dirichlet integral of harmonic functions on a planar regular region.

Theorem 3.3 ([6, 7, 11])**.** *The Dirichlet norm $\|f\|_D$ of a complex-valued harmonic function f with finite Dirichlet integral on a planar regular region Ω is given by:*

$$\|f\|_D^2 = \frac{1}{4\pi}\int_{\partial\Omega}\int_{\partial\Omega} |f(z) - f(w)|^2 \frac{\partial^2 G(z, w)}{\partial n_z \partial n_w}\, ds_z\, ds_w,$$

where $\partial/\partial n$ denotes the outer normal derivative and the value $f(z)$ is the nontangential boundary value of f at a.e. $z \in \partial\Omega$.

Along $\partial\Omega$ we have

$$\frac{\partial G(z,w)}{\partial n_z}\,ds = 2G_z(z,w)\,dn_z = 2iG_z\,dz,$$

so by Proposition 2.1, for $z, w \in \partial\Omega$, $(z \neq w)$,

$$\frac{\partial^2 G(z,w)}{\partial n_z \partial n_w}\,ds_z\,ds_w = -4\frac{\partial^2 G(z,w)}{\partial z \partial w}\,dzdw = 2\pi L_B(z,w)\,dzdw.$$

Hence, by using the conjugate Bergman kernel L_B, Douglas's formula is given by:

$$\|f\|_D^2 = \frac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} |f(z) - f(w)|^2 L_B(z,w)\,dzdw.$$

In 1981 Saitoh [14] obtained a similar formula which was an analog of the Douglas's formula for analytic functions. We give another proof of Saitoh's result by using Theorem 3.1.

Corollary 3.1 (Saitoh, 1981)**.** *If f is a Dirichlet-finite analytic function on a planar regular region Ω, the Dirichlet norm of f is given by*

$$\|f\|_D^2 = \pi\int_{\partial\Omega}\int_{\partial\Omega} |(f(z) - f(w))L_S(z,w)|^2 |dz||dw|,$$

where L_S is the conjugate Szegö kernel of Ω.

Proof. Since $df \in B_E$, by Theorem 3.1 (2) we have $4\pi TT^* df = df$ and $T^* df = 2^{-1} L_S(z,w)(f(z) - f(w))$. Thus,

$$\begin{aligned}\|f\|_D^2 &= \|df\|_B^2 = 4\pi\langle df, TT^* df\rangle_B = 4\pi\|T^* df\|^2_{S^{\otimes 2}} \\ &= \pi\int_{\partial\Omega}\int_{\partial\Omega} |(f(z) - f(w))L_S(z,w)|^2 |dz||dw|.\end{aligned}$$

□

3.1 Conditions equivalent to $H > 0$

Finally, we give several conditions equivalent to Hejhal's theorem as equality conditions for some contraction operators. To this end we need a lemma.

Lemma 3.4. *If an operator $W \colon H_1 \to H_2$ is a contraction between Hilbert spaces H_1 and H_2, then for $f \in H_1$ the following are equivalent.*

(i) $\|Wf\| = \|f\|$.

(ii) $W^*Wf = f$.

(iii) $\|W^*Wf\| = \|f\|$.

Also, if this is the case, then $\|W^*Wf\| = \|Wf\|$.

Proof. If $\|Wf\| = \|f\|$, then by $\|W^*W\| \le 1$,

$$\begin{aligned}\|W^*Wf - f\|^2 &= \|W^*Wf\|^2 - 2\|Wf\|^2 + \|f\|^2 \\ &= \|W^*Wf\|^2 - \|f\|^2 \le 0.\end{aligned}$$

Thus, $W^*Wf = f$, and so (iii) holds. That (iii) implies (i) is clear from

$$\|f\| = \|W^*Wf\| \le \|Wf\| \le \|f\|.$$

If (iii) holds, then $\|W^*Wf\| \le \|Wf\| \le \|f\| = \|W^*Wf\|$. Thus, $\|W^*Wf\| = \|Wf\|$. ☐

Theorem 3.4. *Let* $K = K_B - 4\pi K_S^2$. *Then, the following are equivalent.*

(i) $H > 0$ *(Hejhal's Theorem).*

(ii) $H \ge 0$ *and* $\dim \mathcal{H}_K = g$, *where* $\mathcal{H}_K$ *is the RKHS on* Ω *with the reproducing kernel* K.

(iii) $2\sqrt{\pi}\,\iota\colon S^{\times 2} \hookrightarrow B$ *is a contraction, and* $f \in S^{\times 2}$ *is isometric with* $2\sqrt{\pi}\,\iota f$ *if and only if* $f \in B_E$.

(iv) $2\sqrt{\pi}\,T\colon S^{\otimes 2} \to B$ *is a contraction, and* $f \in S^{\otimes 2}$ *is isometric with* $2\sqrt{\pi}\,Tf$ *if and only if* $f \in T^*(B_E)$.

(v) $2\sqrt{\pi}\,T^*\colon B \to S^{\otimes 2}$ *is a contraction, and* $f \in B$ *is isometric with* $2\sqrt{\pi}\,T^*f$ *if and only if* $f \in B_E$.

(vi) $4\pi TT^*\colon B \to B$ *is a contraction, and* $f \in B$ *is isometric with* $4\pi TT^*f$ *if and only if* $f \in B_E$.

(vii) $4\pi\iota\iota^*\colon B \to B$ *is a contraction, and* $f \in B$ *is isometric with* $4\pi\iota\iota^*$ *if and only if* $f \in B_E$.

(viii) $2\sqrt{\pi}\,\iota^*\colon B \to S^{\times 2}$ *is a contraction, and* $f \in B$ *is isometric with* $2\sqrt{\pi}\,\iota^* f$ *if and only if* $f \in B_E$.

Part II. Reproducing kernel of Sobolev space on an interval

4 The Sobolev space on a bounded interval

Let $I = (a, b)$ be a bounded open interval on $\mathbb{R}$ and let $n \in \mathbb{N}$.

Definition 4.1. The Sobolev space $H^n(I)$ is the space of complex-valued functions f on I such that $f^{(j)} \in C^j(I)$ $(j = 0, \dots, n-1)$ and $f^{(n-1)}$ is absolutely continuous on I with finite norm [1], [5]:

$$\|f\|_S^2 = \langle f, f \rangle_S = \sum_{k=0}^{n} \int_I |f^{(k)}(x)|^2 \, dx. \tag{3}$$

- $H^n(I)$ is a RKHS on $\bar{I} = [a, b]$.
- In 2003 Watanabe [20] obtained an explicit form of the reproducing kernel for $n = 1, 2, 3$.
- Our result: explicit forms of the reproducing kernel $K(x, y)$ of $H^n(I)$ for general n [22].

4.1 Conditions for the reproducing kernel

Proposition 4.1. *Suppose that a function $K(x, y) \in C^\infty(\mathbb{R}^2)$ satisfies the following conditions (i)–(v): for all $x,\ y \in [a, b]$,*

(i) $\displaystyle\sum_{k=0}^{n}(-1)^k \frac{\partial^{2k} K}{\partial x^{2k}}(x, y) = \sum_{k=0}^{n}(-1)^k \frac{\partial^{2k} K}{\partial y^{2k}}(x, y) = 0,$ *(Euler-Lagrange equation)*

(ii) $\displaystyle\sum_{k=j+1}^{n} (-1)^k \frac{\partial^{2k-1-j} K}{\partial x^{2k-1-j}}(a, y) = 0,$ *$(j = 0, \dots, n-1)$,*

(iii) $\displaystyle\sum_{k=j+1}^{n} (-1)^k \frac{\partial^{2k-1-j} K}{\partial y^{2k-1-j}}(y, b) = 0,$ *$(j = 0, \dots, n-1)$,*

(iv) $\displaystyle\sum_{k=j+1}^{n}(-1)^k\Big\{\frac{\partial^{2k-1-j}K}{\partial y^{2k-1-j}}(y,y)-\frac{\partial^{2k-1-j}K}{\partial x^{2k-1-j}}(y,y)\Big\}=\delta_{j0}$, $(j=0,\dots,n-1)$.

(v) $\displaystyle\frac{\partial^j K}{\partial x^j}(y,y)=\frac{\partial^j K}{\partial y^j}(y,y)$, $(j=0,\dots,n-1)$.

Then, the function

$$k(x,y)=\begin{cases}K(x,y), & (x\le y),\\ K(y,x), & (y\le x),\end{cases}$$

is the reproducing kernel of $H^n(a,b)$.

This proposition is obtained by integration by parts. An outline of the proof is as follows. If $f\in H^n(a,b)$ and $g\in C^{2n}(\mathbb{R})$, $k=0,\dots,n$, then by integration by parts we obtain

$$\int_a^b f^{(k)}g^{(k)}\,dx=(-1)^k\int_a^b fg^{(2k)}\,dx-(-1)^k\sum_{j=0}^{k-1}(-1)^j[f^{(j)}g^{(2k-1-j)}]_a^b.$$

Summing these identities on k, we have

$$\int_a^b\sum_{k=0}^{n}f^{(k)}g^{(k)}\,dx=\int_a^b f\sum_{k=0}^{n}(-1)^k g^{(2k)}\,dx\\ -\sum_{j=0}^{n-1}(-1)^j\sum_{k=j+1}^{n}[(-1)^k f^{(j)}g^{(2k-1-j)}]_a^b.$$

Thus, from (i)–(iv), for $y\in[a,b]$,

$$\begin{aligned}\langle f,k(\cdot,y)\rangle_S=&\int_a^y f(x)\sum_{k=0}^{n}(-1)^k\frac{\partial^{2k}K}{\partial x^{2k}}(x,y)\,dx\\ &+\int_y^b f(x)\sum_{k=0}^{n}(-1)^k\frac{\partial^{2k}K}{\partial y^{2k}}(x,y)\,dx\\ &-\sum_{j=0}^{n-1}(-1)^j\sum_{k=j+1}^{n}\Big[(-1)^k f^{(j)}\frac{\partial^{2k-1-j}K}{\partial x^{2k-1-j}}(\cdot,y)\Big]_a^y\\ &-\sum_{j=0}^{n-1}(-1)^j\sum_{k=j+1}^{n}\Big[(-1)^k f^{(j)}\frac{\partial^{2k-1-j}K}{\partial y^{2k-1-j}}(y,\cdot)\Big]_y^b\quad=f(y).\end{aligned}$$

Condition (v) assures that $k(\cdot, y) \in H^n(a,b)$, and $k(\cdot, y)$ satisfies the reproducing property. Hence, $k(\cdot, y)$ is the reproducing kernel for the point y.

The characteristic equation of the Euler-Lagrange equation $y^{(2n)} - y^{(2n-2)} + \cdots + (-1)^n y = 0$ for the Sobolev norm (3) is given by

$$\omega^{2n} - \omega^{2n-2} + \cdots + (-1)^n = 0. \tag{4}$$

The set of solutions of (4) is denoted by

$$\Omega = \{\pm\omega_1, \pm\omega_2, \dots, \pm\omega_n\}$$

and the subset of Ω whose real part is positive is denoted by Ω_+:

$$\begin{aligned} \Omega_+ &= \{\omega \in \Omega \colon \operatorname{Re} \omega > 0\} = \{\omega_1, \dots, \omega_n\}, \\ \omega_k &= \sin \frac{k\pi}{n+1} - i \cos \frac{k\pi}{n+1} \quad (= s_k - i c_k). \end{aligned} \tag{5}$$

4.2 Orthogonality of trigonometric vectors

Using complex numbers we obtain the probably well-known trigonometric identities.

Proposition 4.2 (cf. [12])**.** *The following identities hold for $j, l \in \mathbb{Z}$.*

(i)

$$\sum_{k=1}^{n} \sin \frac{jk\pi}{n+1} \sin \frac{lk\pi}{n+1} = \begin{cases} \frac{n+1}{2}, & (j \equiv l,\ 2j \not\equiv 0 \pmod{2n+2}), \\ -\frac{n+1}{2}, & (j \equiv -l,\ 2j \not\equiv 0 \pmod{2n+2}), \\ 0, & (\textit{otherwise}). \end{cases}$$

(ii)

$$\sum_{k=1}^{n} \sin \frac{jk\pi}{n+1} \cos \frac{lk\pi}{n+1} = \begin{cases} \dfrac{\sin \frac{j\pi}{n+1}}{\cos \frac{l\pi}{n+1} - \cos \frac{j\pi}{n+1}}, & (j + l\text{: } \textit{odd}), \\ 0, & (\textit{otherwise}). \end{cases}$$

Next Lemma and Proposition are used to show the identities in the conditions in Proposition 4.1.

Lemma 4.1. *If $z, w, x, y \in \mathbb{C}$ and $j, k = 0, 1, \dots,$ then*

$$\begin{aligned}
&\mathrm{Re}\,(z \cosh^{(j)} x)\,\mathrm{Re}\,(w \cosh^{(k)} y) + \mathrm{Im}\,(z \sinh^{(j)} x)\,\mathrm{Im}\,(w \sinh^{(k)} y) \\
&\quad = \frac{1}{2}\cos(\mathrm{Im}\,(x-y))\big\{\mathrm{Re}\,(z\bar{w})\cosh^{(j+k)}(\mathrm{Re}\,(x+y)) \\
&\qquad\qquad + (-1)^k\,\mathrm{Re}\,(zw)\cosh^{(j+k)}(\mathrm{Re}\,(x-y))\big\} \\
&\qquad - \frac{1}{2}\sin(\mathrm{Im}\,(x-y))\big\{\mathrm{Im}\,(z\bar{w})\sinh^{(j+k)}(\mathrm{Re}\,(x+y)) \\
&\qquad\qquad + (-1)^k\,\mathrm{Im}\,(zw)\sinh^{(j+k)}(\mathrm{Re}\,(x-y))\big\}.
\end{aligned}$$

Proposition 4.3. *If $f(x)$ is either $\mathrm{Re}\,[\omega\cosh(\omega(x-a))]$ or $\mathrm{Im}\,[\omega\sinh(\omega(x-a))]$ with $\omega \in \Omega$, then $f(x)$ satisfies the following:*

(i) $\sum_{k=0}^{n}(-1)^k f^{(2k)}(x) = 0$ *for all* $x \in \mathbb{R}$.

(ii) $\sum_{k=j+1}^{n}(-1)^k f^{(2k-1-j)}(a) = 0$ *for* $j = 0, \dots, n-1$.

5 Main result: Sobolev kernel for general n

By Propositions and Lemmas stated above we are able to establish the main result of Part II, the concrete form of the reproducing kernel of $H^n(a,b)$ for $n = 1, 2, \dots$.

Theorem 5.1. *The reproducing kernel $k(x,y)$ of the one-dimensional Sobolev space $H^n(a,b)$ on a bounded interval (a,b) belongs to $C^{2n-2}(a,b)$, and is given by*

$$\begin{aligned}
k(x,y) &= \frac{2}{n+1}\sum_{\omega\in\Omega_+}\frac{\mathrm{Re}\ \omega}{\sinh(\mathrm{Re}\ \omega(b-a))}k_\omega(x,y) \\
&= \frac{1}{n+1}\sum_{k=1}^{n}\frac{s_k}{\sinh(s_k(b-a))}\Big[\cos(c_k(x-y))\cosh(s_k(x+y-a-b)) \\
&\qquad - c_{2k}\cos(c_k|x-y|)\cosh(s_k(b-a-|x-y|)) \\
&\qquad + s_{2k}\sin(c_k|x-y|)\sinh(s_k(b-a-|x-y|))\Big],
\end{aligned}$$

where s_k and c_k are the numbers defined in (5), *$k_\omega(x,y)$ is the function*

$$\begin{aligned}
&\mathrm{Re}\,\{\omega\cosh(\omega(x\wedge y-a))\}\,\mathrm{Re}\,\{\omega\cosh(\omega(x\vee y-a)-\mathrm{Re}\ \omega(b-a))\} \\
&\quad + \mathrm{Im}\,\{\omega\sinh(\omega(x\wedge y-a))\}\,\mathrm{Im}\,\{\omega\sinh(\omega(x\vee y-a)-\mathrm{Re}\ \omega(b-a))\},
\end{aligned}$$

and $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$.

Corollary 5.1 (cf. [13, 19]). *If $y \in [a,b]$, then, for $j = 0, \dots, n-1$,*

$$\frac{\partial^{2j} k}{\partial x^j \partial y^j}(y,y) = \frac{1}{n+1} \sum_{\omega \in \Omega_+} \frac{\mathrm{Re}\ \omega}{\sinh(\mathrm{Re}\ \omega(b-a))} \Big[\cosh(\mathrm{Re}\ \omega(2y-a-b)) - \mathrm{Re}\,(i\omega)^{2(j+1)} \cosh(\mathrm{Re}\ \omega(b-a))\Big].$$

In particular, if $y \in [a,b]$, then the maximum of $\frac{\partial^{2j}}{\partial x^j \partial y^j} k(y,y)$ is taken when $y = a$ and $y = b$, and the following identity holds.

$$\max_{y \in [a,b]} \frac{\partial^{2j}}{\partial x^j \partial y^j} k(y,y) = \frac{2}{n+1} \sum_{k=1}^{n} s_k s_{(j+1)k}^2 \coth(s_k(b-a)).$$

Remark 5.1. By differentiation under the integral sign, $k_y^{(j)} = \frac{\partial^j k}{\partial y^j}(\cdot, y) \in H^n(a,b)$ reproduces the derivative $f^{(j)}(y)$ of $f \in H^n(a,b)$ for $j = 0, \dots, n-1$:

$$f^{(j)}(y) = \Big\langle f, \frac{\partial^j k}{\partial y^j}(\cdot, y) \Big\rangle_S.$$

By Schwarz's inequality,

$$|f^{(j)}(y)|^2 \le \|f\|_S^2 \|k_y^{(j)}\|_S^2 = \|f\|_S^2 \frac{\partial^{2j} k}{\partial x^j \partial y^j}(y,y).$$

Thus,

$$\|f^{(j)}\|_\infty^2 \le \|f\|_S^2 \Big\| \frac{\partial^{2j} k}{\partial x^j \partial y^j}(y,y) \Big\|_\infty.$$

Hence, Corollary 5.1 gives the best constant of Sobolev inequality for $f^{(j)}$ (cf. [19]).

5.1 Sobolev kernel on infinite intervals

The reproducing kernel of the Sobolev space on infinite intervals is obtained from those kernels on bounded intervals by limiting procedure. The theorem we need is

Theorem 5.2 (Aronszajn [3, p. 362]). *Let $\{E_n\}$ be a monotone increasing sequence of sets with $E = \bigcup_n E_n$, and let $\mathcal{H}_n$ be a RKHS on E_n with its reproducing kernel k_n. If, for all $m \le n$, the restriction operator $\rho_m^n \colon f \in \mathcal{H}_n \mapsto f|_{E_m} \in \mathcal{H}_m$ is a well-defined contraction, then the following hold:*

(i) *For all* $x, y \in E$, *there exists a limit* k_∞: $\lim_{n\to\infty} k_n(x,y) = k_\infty(x,y)$.

(ii) k_∞ *is a positive definite kernel on* E. *If* $\mathcal{H}_\infty$ *is a RKHS on* E *with the reproducing kernel* k_∞, *then* $\mathcal{H}_\infty$ *consists of the functions* f *on* E *such that the restriction* $f|_{E_n} \in \mathcal{H}_n$ *for* $n = 1, 2, \dots$ *with* $\sup_n \|f|_{E_n}\| < \infty$. *The inner product of* f *and* $g \in \mathcal{H}_\infty$ *is given by*

$$\langle f, g\rangle_{\mathcal{H}_\infty} = \lim_{n\to\infty} \langle f|_{E_n}, g|_{E_n}\rangle_{\mathcal{H}_n}.$$

Corollary 5.2. *For* $a \in \mathbb{R}$, *the reproducing kernel* $k(x,y)$ *of the one-dimensional Sobolev space* $H^n(a,\infty)$ *belongs to* $C^{2n-2}(a,\infty)$ *and the following hold:*

(i)

$$\begin{aligned} k(x,y) &= \frac{2}{n+1} \sum_{\omega\in\Omega_+} \operatorname{Re}\omega \Big[\operatorname{Re}\{\omega \cosh(\omega(x\wedge y - a))\} \operatorname{Re}\{\omega \exp(\omega(a - x\vee y))\} \\ &\qquad\qquad - \operatorname{Im}\{\omega \sinh(\omega(x\wedge y - a))\} \operatorname{Im}\{\omega \exp(\omega(a - x\vee y))\}\Big] \\ &= \frac{1}{n+1} \sum_{k=1}^{n} s_k \Big[\cos(c_k(x-y)) \exp(s_k(2a - x - y)) \\ &\qquad\qquad - \cos(c_k|x-y| + \tfrac{2k\pi}{n+1}) \exp(-s_k|x-y|)\Big], \end{aligned}$$

(ii) *For* $y \in [a,\infty)$, $j = 0, \dots, n-1$,

$$\frac{\partial^{2j} k}{\partial x^j \partial y^j}(y,y) = \frac{1}{n+1} \sum_{\omega\in\Omega_+} \operatorname{Re}\omega \Big[\exp\{2 \operatorname{Re}\omega(a-y)\} - \operatorname{Re}(i\omega)^{2(j+1)}\Big].$$

In particular, if $y \in [a,\infty)$, *the maximum of* $\frac{\partial^{2j}}{\partial x^j \partial y^j} k(y,y)$ *is taken at* $y = a$, *and the following identity holds.*

$$\begin{aligned} \max_{y\in[a,\infty)} \frac{\partial^{2j}}{\partial x^j \partial y^j} k(y,y) &= \frac{1}{n+1} \sum_{k=1}^{n} s_k (1 - c_{2(j+1)k}) \\ &= \frac{1}{n+1} \left(\frac{\sin\frac{\pi}{n+1}}{1 - \cos\frac{\pi}{n+1}} + \frac{\sin\frac{\pi}{n+1}}{\cos\frac{\pi}{n+1} - \cos\frac{2(j+1)\pi}{n+1}} \right). \end{aligned}$$

Corollary 5.3. *The reproducing kernel $k(x, y)$ of the one-dimensional Sobolev space $H^n(\mathbb{R})$ belongs to $C^{2n-2}(\mathbb{R})$ and is given by*

$$k(x, y) = \frac{-1}{n+1} \sum_{k=1}^{n} s_k \cos(c_k|x-y| + \tfrac{2k\pi}{n+1}) \exp(-s_k|x-y|).$$

In particular, for $j = 0, \ldots, n-1$,

$$\frac{\partial^{2j}}{\partial x^j \partial y^j} k(x, x) = \frac{-1}{n+1} \sum_{k=1}^{n} s_k c_{2(j+1)k} = \frac{\sin \frac{\pi}{n+1}}{(n+1)\bigl(\cos \frac{\pi}{n+1} - \cos \frac{2(j+1)\pi}{n+1}\bigr)}.$$

6 Examples and norm inequalities

Example 6.1. The reproducing kernel of the Sobolev space $H^n(a, b)$ for small n is given as follows:

(i) $H^1(a, b)$ (cf. [8, p. 105], [2, p. 344]):

$$\begin{aligned} k(x, y) &= \frac{1}{2} \operatorname{csch}(b-a)\{\cosh(x+y-a-b) + \cosh(b-a-|x-y|)\} \\ &= \operatorname{csch}(b-a) \cosh(x \wedge y - a) \cosh(x \vee y - b), \end{aligned}$$

(ii) $H^2(a, b)$ (cf. [20, p. 813]):

$$\begin{aligned} k(x, y) = \tfrac{1}{\sqrt{3}} \operatorname{csch}(\tfrac{\sqrt{3}}{2}(b-a)) \Bigl[&\tfrac{\sqrt{3}}{2} \sin(\tfrac{1}{2}|x-y|) \sinh(\tfrac{\sqrt{3}}{2}(b-a-|x-y|)) \\ &+ \cos(\tfrac{1}{2}(x-y)) \bigl\{ \cosh(\tfrac{\sqrt{3}}{2}(x+y-a-b)) \\ &+ \tfrac{1}{2} \cosh(\tfrac{\sqrt{3}}{2}(b-a-|x-y|)) \bigr\} \Bigr], \end{aligned}$$

(iii) $H^3(a, b)$ (cf. [20, p. 813]):

$$\begin{aligned} k(x, y) = \tfrac{1}{2\sqrt{2}} \operatorname{csch}(\tfrac{1}{\sqrt{2}}(b-a)) \Bigl[&\cos(\tfrac{1}{\sqrt{2}}(x-y)) \cosh(\tfrac{1}{\sqrt{2}}(x+y-a-b)) \\ &+ \sin(\tfrac{1}{\sqrt{2}}|x-y|) \sinh(\tfrac{1}{\sqrt{2}}(b-a-|x-y|)) \Bigr] \\ &+ \tfrac{1}{4} \operatorname{csch}(b-a)(\cosh(x+y-a-b) + \cosh(b-a-|x-y|)). \end{aligned}$$

Let $H_0^1(a,b)$ be the subspace of $H^1(a,b)$ consisting of functions f with $f(a)=f(b)=0$. $H_0^1(a,b)$ is a RKHS with the reproducing kernel $k0$ given by (cf. [20, p. 811]):

$$k0(x,y)=\operatorname{csch}(b-a)\sinh(x\wedge y-a)\sinh(b-x\vee y).$$

Note that the function $\sinh(x\wedge y-a)$ (resp. $\sinh(b-x\vee y)$) is the reproducing kernel of the RKHS H_a^1 (resp. H_b^1), where H_a^1 (resp. H_b^1) is the Hilbert space of functions $f\in H^1(a,b)$ with $f(a)=0$ (resp. $f(b)=0$) equipped with the inner product given respectively by

$$\int_a^b \frac{f'(x)\overline{g'(x)}}{\cosh(x-a)}\,dx,\qquad \int_a^b \frac{f'(x)\overline{g'(x)}}{\cosh(b-x)}\,dx.$$

Let H_j be a RKHS on a set E with the reproducing kernel k_j $(j=1,2,3)$. It is well-known (e.g. [17]) that, if $k_1=k_2k_3$, then, for $f\in H_2$ and $g\in H_3$, $fg\in H_1$, and the following norm inequality holds:

$$\|fg\|_{H_1}\le \|f\|_{H_2}\|g\|_{H_3}.$$

Equality holds in the above inequality if $f=C_1k_1(\cdot,y)$ and $g=C_2k_2(\cdot,y)$ for any constants $C_1,C_2\in\mathbb{C}$ and $y\in E$. Thus, we have

Proposition 6.1. *If $f,g\in H^1(a,b)$ with $f(a)=g(b)=0$, then*

$$\int_a^b (|fg|^2+|(fg)'|^2)\,dx\le \sinh(b-a)\int_a^b \frac{|f'|^2}{\cosh(x-a)}\,dx\int_a^b \frac{|g'|^2}{\cosh(b-x)}\,dx.$$

Equality holds if $f(x)=C_1\sinh(x\wedge y-a)$ and $g(x)=C_2\sinh(b-x\vee y)$ for any $C_1,\ C_2\in\mathbb{C}$ and $y\in(a,b)$.

Remark 6.1. In fact, the above condition for equality is necessary, which will be shown in our subsequent paper.

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Second, Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.

[2] J. Agler, *Nevanlinna-Pick interpolation on Sobolev space*, Proc. Amer. Math. Soc. **108** (1990), no. 2, 341–351.

[3] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[4] S. Bergman, *The kernel function and conformal mapping*, revised, American Mathematical Society, Providence, R.I., 1970. Mathematical Surveys, No. V.

[5] A. Berlinet and C. Thomas-Agnan, *Reproducing kernel Hilbert spaces in probability and statistics*, Kluwer Academic Publishers, Boston, MA, 2004. With a preface by Persi Diaconis.

[6] J. Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931), no. 1, 263–321.

[7] ______, *Green's function and the problem of Plateau*, Amer. J. Math. **61** (1939), 545–589.

[8] M. Duc-Jacquet, *Approximation des fonctionnelles par des fonctionnelles d'évaluation sur des espaces de Hilbert de fonctions*, J. Approximation Theory **11** (1974), 97–109.

[9] J. D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, Vol. 352, Springer-Verlag, Berlin-New York, 1973.

[10] D. A. Hejhal, *Theta functions, kernel functions, and Abelian integrals*, Memoirs of the American Mathematical Society, No. 129, American Mathematical Society, Providence, R.I., 1972.

[11] H. Osborn, *The Dirichlet functional. I*, J. Math. Anal. Appl. **1** (1960), 61–112.

[12] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and series. Vol. 1*, Gordon & Breach Science Publishers, New York, 1986. Elementary functions, Translated from the Russian and with a preface by N. M. Queen.

[13] W. Richardson, *Steepest descent and the least C for Sobolev's inequality*, Bull. London Math. Soc. **18** (1986), no. 5, 478–484.

[14] S. Saitoh, *A characterization of the adjoint L-kernel of Szegö type*, Pacific J. Math. **96** (1981), no. 2, 489–493.

[15] ______, *Theory of reproducing kernels and its applications*, Pitman Research Notes in Mathematics Series, vol. 189, Longman Scientific & Technical, Harlow, 1988.

[16] ______, *Integral transforms, reproducing kernels and their applications*, Pitman Research Notes in Mathematics Series, vol. 369, Longman, Harlow, 1997.

[17] ______, *Natural norm inequalities in nonlinear transforms*, General inequalities, 7 (Oberwolfach, 1995), 1997, pp. 39–52.

[18] M. Schiffer, *Various types of orthogonalization*, Duke Math. J. **17** (1950), 329–366.

[19] K. Watanabe, Y. Kametaka, A. Nagai, K. Takemura, and H. Yamagishi, *The best constant of Sobolev inequality on a bounded interval*, J. Math. Anal. Appl. **340** (2008), no. 1, 699–706.

[20] K. Watanabe, T. Yamada, and W. Takahashi, *Reproducing kernels of* $H^m(a,b)$ $(m = 1,2,3)$ *and least constants in Sobolev's inequalities*, Appl. Anal. **82** (2003), no. 8, 809–820.

[21] A. Yamada, *Bergman kernel, Szegö kernel and Dirichlet integral*, To appear in Complex Anal. Oper. Theory.

[22] ______, *The reproducing kernel of the Sobolev space on an interval*, To appear in Proc. Amer. Math. Soc.

Tokyo Gakugei University
Current address: 7–69, Kuwamizuhon-machi, Chuo-ku, Kumamoto-shi, Kumamoto, JAPAN
E-mail address: yamada@u-gakugei.ac.jp