Bergman and Weighted Szegö Reproducing Kernels, and Fundamental Open Problems and Related Topics

Saburou Saitoh
Institute of Reproducing Kernels,
saburou.saitoh@gmail.com

February 28, 2023

Abstract: In this paper, as direct impacts of Q. Guan's results on the conjugate analytic Hardy H_2 norm and the recent results by Akira Yamada for the product of reproducing kernels in connection with operator theory we will introduce fundamental open problems for the Bergman and weighted Szegö kernels and the related topics. We will also refer to more recent surprising and profound results by Q. Guan and Z. Yuan on the weighted Bergman kernels and weighted Szegö kernels on the line of Oikawa-Sario's problems, Suita's conjecture and Yamada's conjecture. Furthermore, we will refer to more up-to-date results for the related norm inequalities and applications. The symbolic result is: quotient may be represented by product.

Since the RIMS meeting, the publication deadline was so long, another meeting:

The Research Meeting for the Potential Theory, Nagoya 2023.2.10-12

was held and the page restriction was 25 pages (the talk paper was 35 pages), this article was reasonably arranged with the addition of the new information. The published materials were reduced.

Key Words: Bergman kernel, Szegö kernel, Rudin kernel, Hardy reproducing kernel, weighted Szegö kernels, Dirichlet integral, weighted Bergman

kernel, conjugate analytic Hardy H_2 norm, Green function, isoperimetric inequality, general theory of reproducing kernels, operator theory, positive definiteness, Hejhal's theorem, norm inequalities, Suita conjecture, Yamada's conjecture, Sobolev space, quotient and product, division by zero, division by zero calculus.

2010 Mathematics Subject Classification: 30A10, 30H10, 30H20, 30C40.

1 Introduction

In order to state the contents in a self contained manner, first, we shall introduce the needed basic materials.

The Bergman kernel

Let D be a bounded regular domain on the complex $z = x + iy$ plane whoes boundary is composed of a finite number N of disjoint analytic Jordan curves. Let $AL_2(D)$ be a Hilbert space (Bergman space) comprising analytic functions $f(z)$ on D and with finite norms $\|f\|_{AL_2(D)} = \left\{\int\int_D |f(z)|^2 dxdy\right\}^{\frac{1}{2}}$. As we see simply, $f \to f(z)(z \in D)$ is a bounded linear functional on $AL_2(D)$. Therefore, there exists a reproducing kernel $K(z, u)$ such that for any $u \in D$ and for any function $f \in AL_2(D)$,

$$f(u) = \int\int_D f(z)\overline{K(z,u)}dxdy. \tag{1.1}$$

From $K(z, u) = \overline{K(u, z)}$, $K(z, u)$ is analytic in $\overline{u}$ (complex conjugate). So, we shall denote it as $K(z, \overline{u})$. This is the Bergman kernel of D or on D.

Let $G(z, u)$ be the Green's function for the Laplace equation on D with pole at $u \in D$. Then, we have the identity

$$K(z, \overline{u}) = -\frac{2}{\pi}\frac{\partial^2 G(z,u)}{\partial z \partial \overline{u}}. \tag{1.2}$$

Here we shall introduce the adjoint $L-$ kernel for $K(z, \overline{u})$ by

$$L(z, u) = -\frac{2}{\pi}\frac{\partial^2 G(z,u)}{\partial z \partial u}.$$

The adjoint L kernel $L(z,u)$ has one double pole at u on D such that

$$L(z,u) = \frac{1}{\pi(z-u)^2} + \text{regular terms}$$

of the second order. Furthermore, along the boundary ∂D we have the identity

$$\overline{K(z,\overline{u})dz} = -L(z,u)dz. \tag{1.3}$$

This important identity shows that the Bergman kernel is indeed an analytic differential $K(z,\overline{u})dz$ and it is continued analytically to the double - closed Riemann surface - of D, and at the symmetric point u of $\overline{u}$ it has the same double pole as in the above. So, the Bergman kernel is a fundamental differential on the closed Riemann surface.

The Szegö kernel

Let $H_2(D)$ be the Hardy 2 analytic function space on D. A member of the class has nontangential boundary values belonging to $L_2(\partial D)$. Let $AL_2(\partial D)$ denote its closed subspace in $L_2(\partial D)$, and we introduce the norm in $H_2(D)$ by

$$\left\{ \int_{\partial D} |f(z)|^2 |dz| \right\}^{\frac{1}{2}} < \infty.$$

For a function $f \in AL_2(\partial D)$, as we see from the Cauchy integral formula, since $f \to f(u), u \in D$ is a bounded linear functional, there exists a reproducing kernel $\hat{K}(z,u)$ satisfying

$$f(u) = \int_{\partial D} f(z)\overline{\hat{K}(z,u)}|dz|.$$

As in the Bergman kernel, $\hat{K}(z,u)$ is analytic in $\overline{u}$, and so we shall denote it by $\hat{K}(z,\overline{u})$. This is the Szegö kernel of D or on D.

The important concept is the adjoint L kernel $\hat{L}(z,u)$ for $\hat{K}(z,\overline{u})$. That is, it is a meromorphic function on $D \cup \partial D$ and only at the point u, it has a simple pole

$$\hat{L}(z,u) = \frac{1}{2\pi(z-u)} + \text{regular terms}, \tag{1.4}$$

and along the boundary ∂D it satisfies the relation

$$\overline{\hat{K}(z,\overline{u})}|dz| = \frac{1}{i}\hat{L}(z,u)dz. \tag{1.5}$$

This adjoint L kernel is a uniquley determined by these properties on $D \cup \partial D$. From (1.5) we have the identity

$$\hat{L}(z,u) = -\hat{L}(u,z) \quad \text{on} \quad D \times D. \tag{1.6}$$

The adjoint L kernel $\hat{L}(z,u)$ is also characterized by the minimum property:

$$\min \int_{\partial D} |h(z,u)|^2 |dz| = \int_{\partial D} |\hat{L}(z,u)|^2 |dz|.$$

Here the minimum is considered on the meromorphic functions $\{h(z,u)\}$ on D with the singularity at u as in (1.4) and satisfying $\left\{\int_{\partial D} |h(z,u)|^2 |dz|\right\}^{\frac{1}{2}} < \infty$.

A deep and important property of the adjoint L kernel $\hat{L}(z,u)$ is that it does not have any zero point on $D \cup \partial D$. From this fact and (1.5),

$$f_0(z;u) = \frac{\hat{K}(z,\overline{u})}{\hat{L}(z,u)}$$

is the **Ahlfors function** of D with respect to u. That is, $f_0(z;u)$ is an analytic function on D, among the analytic functions satisfying the properties

$$f(u;u) = 0, f'(u;u) \geq 0 \quad \text{and} \quad |f(z;u)| \leq 1 \quad \text{on} \quad D,$$

it satisfies the extremal property $f_0'(u;u) = \max f'(u;u)$. Then, as we see from (1.5) and the principle of argument, the function $f_0(z;u)$ maps D onto the unit disc and the disc is covered N times. Here, we assume D is N-ply connected.

From (1.5) along ∂D we see

$$\overline{\hat{K}(z,\overline{u})^2 dz} = -\hat{L}(z,u)^2 dz. \tag{1.7}$$

From this relation, we see that the square of the Szegö kernel is an analyic differential, it is continued to the double of D analytically that is a closed Riemann surface as in the Bergman kernel. In this sense, the Szegö kernel is a half order differential on the closed Riemann surface and so its properties are very involved for the sake of multi-valuedness on the closed Riemann surface. The profound theory for these properties was done by D. A. Hejhal [22] and J. D. Fay [18] by using the **Riemann theta functions** and the **Klein prime forms**. As we see from (1.5), the Szegö kernel is a Cauchy kernel on the Riemann surface and so it is very important reproducing kernel.

The Hardy reproducing kernel

Let $H_2(D)(\hat{H}_2(D), \text{resp.})$ be the Hilbert space equipped the norm in the space $H_2(D)$:

$$\left\{ \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \frac{\partial G(z,t)}{\partial \nu_z} |dz| \right\}^{\frac{1}{2}} \tag{1.8}$$

$$\left(\left\{ \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \left(\frac{\partial G(z,t)}{\partial \nu_z} \right)^{-1} |dz| \right\}^{\frac{1}{2}}, \text{resp.} \right). \tag{1.9}$$

Here, $\partial/\partial\nu_z$ is the inner normal derivative with respect to D. $\partial G(z,t)/\partial\nu_z$ is a positive continuous function on ∂D and so as in the Szegö space, we can consider the reproducing kernel $K_t(z,\overline{u})(\hat{K}_t(z,\overline{u}), \text{resp.})$ satisfying

$$f(u) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{K_t(z,\overline{u})} \frac{\partial G(z,t)}{\partial \nu_z} |dz|,$$

$$\left(f(u) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{\hat{K}_t(z,\overline{u})} \left(\frac{\partial G(z,t)}{\partial \nu_z} \right)^{-1} |dz|, \text{resp.} \right).$$

We call $K_t(z,\overline{u})$ and $\hat{K}_t(z,\overline{u})$ the Hardy reproducing kernel and its conjugate reproducing kernel of D.

For an arbitrary open Riemann surface S, we can introduce the space $H_2(S)$ similarly and the Hardy reproducing kernel. The reproducing kernel $K_t(z,\overline{u})$ is introduced by W. Rudin in 1955.

Let $G^*(z,t)$ be the conjugate harmonic function of $G(z,t)$ and when we form the multi-valued meromorphic function $W(z,t) = G(z,t) + iG^*(z,t)$, $idW(z,t)$ is a single-valued meromorphic differential, and it satisfies along ∂D,

$$\frac{\partial G(z,t)}{\partial \nu_z} |dz| = idW(z,t).$$

Therefore, the integral (1.8) is represented by

$$\frac{1}{2\pi}\int_{\partial D}\frac{|f(z)dz|^2}{idW(z,t)}.$$

This will mean that $\hat{H}_2(D)$ is indeed comprised of analytic differentials $f(z)dz$. Therefore, the conjugate space may be considered on bordered Riemann surfaces with some good boundaries.

For $K_t(z,\overline{u})$(resp.$\hat{K}_t(z,\overline{u})$), the adjont L kernel $L_t(z,u)$(resp.$\hat{L}_t(z,u)$), that is a meromorphic function on $D\cup\partial D$ with only one simle pole at u with residue 1, is characterized by the identity, along ∂D

$$\overline{K_t(z,\overline{u})}idW(z,t)=\frac{1}{i}L_t(z,u)dz \tag{1.10}$$

$$\left(\overline{\hat{K}_t(z,\overline{u})dz}=\frac{1}{i}\hat{L}_t(z,u)idW(z,t)\quad \text{along}\quad \partial D, \text{resp.}\right). \tag{1.11}$$

From (1.10) and (1.11), we have a very important property:

$$L_t(z,u)=-\hat{L}_t(u,z)\quad \text{on}\quad D\times D. \tag{1.12}$$

The relations (1.10), (1.11) and (1.12) show that $L_t(z,u)dz$ is a meromorphic differential with respect to z, a meromorphic function in u, and along $z=u$ with one simple pole with residue 1; that is the Cauchy kernel of D. Therefore, these reproducing kernels are very important as in the Bergman and Szegö kernels as the third kind of reproducing kernels in one complex variable analysis ([45]). These reproducing kernels were examined deeply on Riemann surfaces by J. D. Fay [18] and A. Yamada [81] in terms of the Riemann theta functions and the Klein prime forms.

For the conjugate Hardy norm

$$\frac{1}{2\pi}\int_{\partial D}|f(z)|^2\left(\frac{\partial G(z,t)}{\partial \nu_z}\right)^{-1}|dz|=\frac{1}{2}\int_{\partial D}\frac{|f(z)dz|^2}{idW(z,t)},$$

we obtain the best possible norm inequality:

$$\int\int_D|f'(z)|^2dxdy\leq\frac{1}{2}\int_{\partial D}\frac{|f'(z)dz|^2}{idW(z,t)}, \tag{1.13}$$

and from this we see the naturality of the norm looked curriously. This inequality is not so simple to derive and for its proof we must examine deeply the relations among the Hardy reproducing kernel, its conjugate kernel and the Bergman kernel ([45]).

For the Bergman kernel and the Szegö kernel on D, we have the basic and deep relation

$$K(z,\overline{u}) >> 4\pi\hat{K}(z,\overline{u})^2$$

– the left minus the right is a positive definite quadratic form function – which was given by D. A. Hejhal [22]. This profound result was given on the long historical lines as in

G.F.B. Riemann (1826-1866); F. Klein (1849-1925); S. Bergman; G. Szegö; Z. Nehari; M.M. Schiffer; P.R. Garabedian (1949 published); D.A. Hejhal (1972 published).

It seems that any elementary proof is impossible, however, the result will, in particular, mean the fairly simple inequality:

For two functions φ and ψ of $H_2(D)$, we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\int\int_D |\varphi(z)\psi(z)|^2 dxdy \leq \frac{1}{2\pi}\int_{\partial D}|\varphi(z)|^2|dz|\frac{1}{2\pi}\int_{\partial D}|\psi(z)|^2|dz|, \qquad (1.14)$$

and we can determine completely the cases holding the equality here. In the thesis [42] of the author published in 1979 the result was given. The author realized the importance of the abstract and general theory of reproducing kernels by N. Aronszajn. In the paper, the core part was to determine the equality statement in the above inequality, surprisingly enough, some deep and general independing poof was appeared 26 years later in A. Yamada ([82]). A. Yamada was developed deeply equality problems for some general norm inequalities derived by the theory of reproducing kernels and it was published in the book appendix of [30]. Very recently his theory is developing much more in [41].

Of course, in the thesis we can find some fundamental idea for nonlinear transforms. In particular, for the special case $\varphi \equiv \psi \equiv 1$, for the plane measure $m(D)$ of D and the length ℓ of the boundary we have the isoperimetric inequality

$$4\pi m(D) \leq \ell^2.$$

Meanwhile, for the Hardy reproducing kernel we can give the conjecture in the converse direction, from the case of doubly connected domains

$$\pi K(z,\overline{u}) << K_t(z,\overline{u})\hat{K}_t(z,\overline{u}) \tag{1.15}$$

([45]). From the relation we can derive various results, the conjecture will be, however, very difficult to solve.

The author attacked to this problem several years, it was, however, impossible to solve it and in those days, the author derived the above result applying the result of Hejhal conversely to the general theory of reproducing kernels by Aronszajn. The author thinks for the conjecture we must use the deep theory of J. D. Fay, however, his profound theory seems to be too deep and great for the author. So, from the viewpoints of fundamental results and applicable analysis, the author turned his research interest for the general theory of reproducing kernels.

2 Basic open problems

For the space $H_2(D)$, by exchanging

$$\frac{1}{2\pi}\frac{\partial G(z,t)}{\partial \nu_z}$$

with a general positive continuous function ρ, we consider the weighed Szegö space $H_2^\rho(D)$ and the corresponding the weighted Szegö kernel $K_\rho(z,\overline{u})$ and the conjugate kernel $L_\rho(z,u)$; that is, for the space

$$\left\{\int_{\partial D}|f(z)|^2\rho(z)|dz|\right\}^{\frac{1}{2}} < \infty$$

with the weighted norm, we consider the reproducing kernel $K_\rho(z,\overline{u})$ satisfying

$$f(u) = \int_{\partial D} f(z)\overline{K_\rho(z,\overline{u})}\rho(z)|dz|.$$

and

$$\overline{K_\rho(z,\overline{u})}\rho(z)|dz| = \frac{1}{i}L_\rho(z,u)dz, \tag{2.1}$$

along ∂D. In particular, note that $L_\rho(z,u)$ has a similar singularity at $z=u$ as $\hat{L}(z,u)$.

For the weight ρ^{-1}, we see that the very important relations as in the Hardy case

$$L_\rho(z,u) = -L_{\rho^{-1}}(u,z)$$

and

$$\overline{K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u})dz} = -L_\rho(z,u)L_{\rho^{-1}}(u,z)dz. \tag{2.2}$$

From the boundary relations (2.1) and (2.2) on kernels, for any simply connected regular domain we have the identity

$$4\pi K(z,\overline{u}) \equiv K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u}). \tag{2.3}$$

We do not know whether the case (2.3) happens for $N > 1$ for some ρ. Note, in particular, that for this case we have the similar result in generalizations of the inequality (1.14) for the weighted Szegö norms.

Now, we can state the basic open problems:

Basic big open problems:

(A) Look for the condition of ρ such that

$$4\pi K(z,\overline{u}) >> K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u}) \tag{2.4}$$

is valid.

(B) Look for the condition of ρ such that

$$4\pi K(z,\overline{u}) << K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u}) \tag{2.5}$$

is valid.

By the deep result of D. A Hejhal and our example, this open problems are valid, in general, as very difficult problems. We can state the partial open problems as in

Partial big open problems:

(A′) Look for the condition of ρ such that

$$4\pi K(z,\overline{z}) > K_\rho(z,\overline{z})K_{\rho^{-1}}(z,\overline{z}) \tag{2.6}$$

is valid.

(B') Look for the condition of ρ such that

$$4\pi K(z,\overline{z}) < K_\rho(z,\overline{z})K_{\rho^{-1}}(z,\overline{z}) \tag{2.7}$$

is valid.

For doubly connected domains $N=2$, since the differentials

$$4\pi K(z,\overline{u})dz\overline{du} - K_\rho(z,\overline{u})K_{\rho^{-1}}(z,\overline{u})dz\overline{du}$$

is one dimensional first order differential that is complex Hermitian form of $dz\overline{du}$, the above basic big open problems and partial big open problems are the same problems.

We do not know whether they are always the same problems or not.

It is very interested in the result of S. R. Bell and B. Gustafsson ([2]) that for the Hejhal case, also for $N=3$ both problems are same.

3 Q. Guan's results

Surprisingly enough, since 40 years later after publication of [22], for some open question proposed there, an entirely unexpected partial solution was published in [12] that is an entirely new result.

For the conjecture (1.15), he proved that

$$\pi K(z,\overline{z}) \leq \hat{K}_z(z,\overline{z}) \tag{3.1}$$

and surprisingly enough, he completely determined (solved) the equality problem. Here note the important fact

$$K_t(z,\overline{t}) \equiv 1.$$

His interest is on the long and large topics on the Oikawa-Sario's problems and the Suita conjecture that determine the magnitudes of many conformal invariant quantities.

We recall the following solution of the conjecture posed by Suita [35].

Theorem: ([11]). *Let* $c_\beta(z_0) = \lim_{z\to z_0} \exp(G(z,z_0) - \log|z - z_0|)$*. Then* $(c_\beta(z_0))^2 \le \pi K(z_0,\overline{z_0})$ *and* $(c_\beta(z_0))^2 = \pi K(z_0,\overline{z_0})$ *holds for some* $z_0 \in D$ *if and only if* D *conformally equivalent to the unit disc, i.e.* $N = 1$*.*

Note that

$$(c_\beta(z_0))^2 \le \pi K(z_0,\overline{z_0})$$

that was proved by Blocki in [8] for planar domains D.

We can see the great impact of the Suita conjecture from many internet sites like Wikipedia; Suita conjecture, however, we see its frontier information in [20] and [36]. In particular, it seems that the impact of T. Ohsawa was great who introduced and connected to several complex analysis group and to several complex analysis. For a complete version of Yamada, see [37].

For some detail and global information on the Suita conjecture, see Ohsawa ([21]), the last 5 stories.

Then, Guan derived the inequalities

$$(c_\beta(z_0))^2 \le \pi K(z_0,\overline{z_0}) \le \hat{K}_{z_0}(z_0,\overline{z_0})$$

that mean that the values $(c_\beta(z_0))^2$ and $\hat{K}_{z_0}(z_0,\overline{z_0})$ are the extremals for the value $K(z_0,\overline{z_0})$ and he solved completely the equality problems for these inequalities by an elementary means.

For the various meanings and applications of the left hand side inequality see [21], and so, we are interested in some applications of the right hand side inequality.

Surprisingly enough, Guan derived there the following identity:

For any fixed $t \in D$ *and for any fixed analytic function* f *on* D *which is continuous on* $D \cup \partial D$*, the identity*

$$\lim_{r\to 1-0} \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)}\ge r\}} |f(z)|^2 dxdy \tag{3.2}$$

$$= \frac{1}{2}\int_{\partial D} |f(z)|^2 (\partial G(z,t)/\partial\nu)^{-1} |dz|$$

holds.

Main Result:

The following inequalities seem to be interesting on its own sense:

Theorem: *For any given* $\epsilon > 0$ *and for any fixed analytic function* $f(z)$ *on* $D \cup \partial D$*, there exists (a large)* $r : (0 < r < 1)$ *satisfying the inequality*

$$\int\int_D |f'(z)|^2 dxdy - \epsilon \leq \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)} \geq r\}} |f'(z)|^2 dxdy.$$

This inequality may be looked as an isoperimetric inequality, because the Dirichlet integral on a domain is estimated (restricted) by the Dirichlet integral on some small boundary neighborhood of the domain. Here, the neighborhood size and estimation are stated by the level curve of the Green function, precisely.

Even the case of the identity function $f(z) = z$, we can enjoy the senses of the estimation and the result.

Essence of the Proof of the Main Result

The integral of the conjugate analytic Hardy $H^2(D)$ norm seems to be not popular, however, the norm will have a beautiful structure and the norm is conformally invariant. Furthermore, we obtain the inequality:

For any analytic function $f(z)$ on $D \cup \partial D$, we have the inequality

$$\int\int_D |f'(z)|^2 dxdy \leq \frac{1}{2} \int_{\partial D} |f'(z)|^2 (\partial G(z,t)/\partial \nu)^{-1} |dz|. \tag{3.3}$$

This result was derived from some complicated theory of reproducing kernels in ([23]). The equality problem in the inequality was also established; that is, equality holds if and only if the domain is simply-connected and the function $f'(z)$ is expressible in the form $CK(z, \overline{t})$ for the Bergman reproducing kernel $K(z, \overline{u})$ on the domain D and for a constant C.

This surprising result was derived from some comparison of the magnitudes of the Bergman and the Rudin (Hardy H^2) reproducing kernels. See the original paper ([12]). Its source was given by [22] and then the topics were cited in the book [27] in details.

We can obtain the main result by combining of the Guan identity and this inequality.

Remarks

We note the following interesting problems:

Problem 1 : *How will be some generalization of the Guan identity for a general weight for* $(\partial G(z,t)/\partial\nu)^{-1}$*?*

Problem 2 : *We can consider similar inequalities for various function spaces. For example, how will be the case for the harmonic Dirichlet integrals?*

Problem 3 : *The structure and proof of the equality (3.2) are very complicated (involved) and the Guan identity is very unique. So, we are interested in some direct proof of the theorem.*

Problem 4 : *The theorem seems to be valid for a general domain* D *and for general analytic functions with finite Dirichlet integrals on* D *apart from the proof in this paper. However, in the theorem* r *is depending on the function* f *and therefore the generalization of the theorem is not simple.*

For the equality (3.2), note the inequality

$$\left(\frac{1}{\pi}\int\int_D |f'(z)|^2 dxdy\right)^2 = \left(\frac{1}{2\pi i}\int_{\partial D} \overline{f(z)}f'(z)dz\right)^2$$
$$\leq \frac{1}{2\pi}\int_{\partial D} |f(z)|^2 \frac{\partial G(z,t)}{\partial \nu}|dz| \frac{1}{2\pi}\int_{\partial D} |f'(z)|^2 (\partial G(z,t)/\partial\nu)^{-1}|dz|.$$

Note that the relation of the Bergman norm and the weighted H^2 norm is very delicate. See [22, 23, 27].

4 Akira Yamada's results

A. Yamada ([39]) gave the various interpretations for the deep result by D. A. Hejhal ([18]) from the viewpoint of the operator theory. In particular, he introduces the operator that corresponds the tensor product of two Szegö spaces to the Bergman space by the restriction to the diagonal set of $D \times D$ and its adjoint operator. He examines the integral operator realizations of the operators and their interesting properties in detail.

5 Addition with the new paper [15] by Q. Guan and Z. Yuan

The result of Q. Guan was very surprised from the line of D. A. Hejhal and J. D. Fay based on the representation (3.2). On this line Q. Guan and Z. Yuan got very deep and surprising results [15]. This direction was suggested by the conjecture of A. Yamada [36] that is a generalization of the conjecture of Suita:

Let u be a harmonic function on D, and let $\rho = e^{2u}$. For the weighted Bergman kernel $K^B_\rho(z, \overline{u})$

Yamada's Conjecture:

$$\beta^2(z) \leq \pi\rho(z)K^B_\rho(z, \overline{z}).$$

In [11], Guan-Zhou proved the Yamada conjecture, and more general weighted versions were developed surprisingly and deeply in [13] and [14].

Following this line, Q. Guan and Z. Yuan is developing the corresponding result with the weighted Hardy H_2 reproducing kernels, and their results are too deep and complicated to state them simply. A typical case may be stated as follows:

Let λ be any positive continuous function on ∂D. By solving the Dirichlet problem, there exists the continuous harmonic function u on the closed domain of D satisfying that $u = -\frac{1}{2}\log\lambda$.

Then, they obtain surprisingly:

$$K_{\lambda\left(\frac{\partial G(\cdot,z)}{\partial\nu.}\right)^{-1}}(z, \overline{z}) \leq K^B_{e^{-2u}}(z, \overline{z}).$$

In particular, they are developing a detail and deep analysis for the identity (3.2).

Q. Guan and Z. Yuan gave a very delicate and profound property of

$$\frac{\partial G(z,t)}{\partial\nu}$$

for a very general situation for the Green function from the view point of the identity (3.2). For its complicated structure, it seems that the result is at this moment abstract in nature. However, as a typical case they were able to get the result for the general weighted Hardy H_2 norms.

This line was not an expected direction, however, we see that the results show more wide field for our mathematics. For example, how will be the values

$$K_{\left(\lambda\left(\frac{\partial G(\cdot,z)}{\partial \nu.}\right)^{-1}\right)^{-1}}(z,\overline{z})$$

and

$$K_{\lambda\left(\frac{\partial G(\cdot,z)}{\partial \nu.}\right)^{-1}}(z,\overline{z})K_{\left(\lambda\left(\frac{\partial G(\cdot,z)}{\partial \nu.}\right)^{-1}\right)^{-1}}(z,\overline{z})$$

?

Moore and surprising addition with the new paper [16] by Q. Guan and Z. Yuan

They furthermore consider the complete version for the product space $D \times D \times D \times \cdots \times D \subset C^n$; indeed, they surprisingly formulate the complete version and complete results containing the equality problems for the related estimates with the great paper.

See also [17] for a weighted version of Suita conjecture for higher derivatives.

In addition, S. R. Bell and B. Gustafsson [2] is discussing on the Hejhal's result and the Suita conjecture from a very interesting viewpoint.

6 One basic meaning of the norm inequalities

Now, we see an important meaning or application of the inequality (1.14); that is, when we fix any member ψ of $H_2(G)$, the multiplication operator

$$\varphi \longmapsto \varphi(z)\psi(z), \tag{6.1}$$

on $H_2(G)$ to the Bergman space is bounded. Therefore, by the general theory for general fractional functions, we can consider the generalized fractional functions: for any Bergman function $f(z)$ on the domain G

$$\frac{f(z)}{\psi(z)}, \tag{6.2}$$

at least in the sense of Tikhonov (we will refer to this later); that is, we can consider the best approximation problem for the functions $\psi(z)^{-1}f(z)$ by the functions $H_2(G)$. See [5, 7] for more detailed results.

As a very special fraction, we can consider the division by zero and division by zero calculus. See [31, 32] for the details.

As an important contribution of the theory of reproducing kernels is on the following fact:

For bounded linear operators on some reproducing kernel Hilbert spaces, we can give analytical and numerical solutions for the operator equations.

7 Quotient and product

We shall consider quotients g/f of general functions f and g in some deep and natural meanings, in a natural setting. The general theory of reproducing kernels will give the natural theory of the problem. In particular, we will consider the division by any functions containing the division by zero. We wish to know the meaning of the function g/f when the function f has zero points, with the definition of g/f.

First, what is a function $y = f(x)$ on (a, b)? For even the typical functions $L_2(a, b)$ having a good looking, we will not be able to get the functions as the corresponding from the points on (a, b) to some space on **R** or **C**, indeed the points are too many to consider them. This question will be more clear when we consider the inversions $1/f$ of the functions f of $L_2(a, b)$. Therefore, we will realize a function as a member of the function space, and the function space represents the functions as a global property over each point value.

The general theory of reproducing kernels will give the natural theory of the problem for the quotients g/f of general functions f and g in some deep and natural meanings.

This content will represent a very interesting nature of the theory of reproducing kernels such that for **an arbitrary function** we considered some representation of its inversion [28] and as its great extensions we obtained the explicit representation [6] of the implicit functions in the theorem of implicit function existence.

Now, for some general two functions f, g on a set E, we will consider the quotient

$$\frac{g}{f}. \tag{7.1}$$

In order to consider such a function (7.1), we shall consider the related equation

$$f_1(p)f(p) = g(p) \quad \text{on} \quad E \tag{7.2}$$

for some function f_1 on the set E. If the solution f_1 in (7.2) on the set E exists, then the solution f_1 will give the meaning of the fractional (quotient) function (7.1). So, the problem may be transformed to the very general and popular equation (7.2).

Here, the serious problem is the case of the zero points of the function f; because we can not give the meaning of the function (7.1) there, intuitively. However, except the zero points of f, the solution f_1 gives the quotient (7.1) point wisely.

In this starting point, the function f is initially given. So, for analyzing the equation (7.2), we must introduce a suitable function space containing the function f_1 and then we find the induced function space containing the product $f_1 \cdot f$.

This idea means that the function g has a natural restriction, because it is the product of the function f and f_1 of some general function space. (*Children are intrinsically influenced by their mothers. g is children; the initial f is mothers.*)

Then, we will be able to consider the solution of the equation (7.2). Here, on this line, we will show that we can discuss the above problem in a very general setting. Indeed, this will be performed for an arbitrary function f on the set E that is non-identically zero on the set E by using the theory of reproducing kernels.

At first, we note that for an arbitrary function $f(p)$, there exist many reproducing kernel Hilbert spaces containing the function $f(p)$; the simplest reproducing kernel is given by $f(p) \times \overline{f(q)}$ on $E \times E$. In general, a reproducing kernel Hilbert space $H_K(E)$ on E admitting a reproducing kernel K on $E \times E$ is uniquley determined by a positive definite Hermitian form (kernel; now $f(p) \times \overline{f(q)}$) and the space is characterized by the very natural property that any point evaluation $f(p)$ is a bounded linear operator on $H_K(E)$ for any point $p \in E$. Therefore, secondary, we shall consider such a reproducing kernel Hilbert space $H_{K_1}(E)$ admitting a reproducing kernel K_1 containing the functions $f_1(p)$.

Then, we note the very interesting fact that the products $f_1 \cdot f$ determine a natural reproducing kernel Hilbert space that is induced by $H_{K_1}(E)$ and $H_K(E)$. In fact, the space in question is the reproducing kernel Hilbert space $H_{K_1K}(E)$ that is determined by the product $K_1 \cdot K$ and, furthermore, we obtain the fundamental norm inequality

$$\|f_1 f\|_{H_{K_1 \cdot K}(E)} \leq \|f_1\|_{H_{K_1}(E)} \|f\|_{H_K(E)}. \tag{7.3}$$

This important inequality (7.3) means that for the linear operator $\varphi_f(f_1)$ on $H_{K_1}(E)$ (for a fixed function f), defined by

$$\varphi_f(f_1)(p) \equiv f_1(p) f(p), \tag{7.4}$$

we obtain the inequality

$$\|\varphi_f(f_1)\|_{H_{K_1K}(E)} \leq \|f_1\|_{H_{K_1}(E)} \|f\|_{H_K(E)}. \tag{7.5}$$

This means that the mapping (multiplicative operator) φ_f is a bounded linear operator from $H_{K_1}(E)$ into $H_{K_1K}(E)$. See [30], Tensor Product of Reproducing Kernels; pages 105-109.

Moore-Penrose generalized solution

As the very natural solution of the operator equation (7.2), we will consider the best approximation, for any function g of the space $H_{K_1K}(E)$

$$\inf_{f_1 \in H_{K_1}(E)} \left\{ \|\varphi_f(f_1) - g\|_{H_{K_1K}(E)}{}^2 \right\}, \tag{7.6}$$

that leads to the Moore-Penrose generalized solution of (7.2).

So, simply we will recall the essential and general properties of the best approximation from [30], pages 166-169 and ([30], page178).

Now we shall apply the above general theory to our case. The situation will be essentially simple.

At first,

$$\varphi_f{}^*(g)(p) = (g, \varphi_f K_1(\cdot, p))_{H_{K_1K_2}(E)} = (g, f(\cdot)K_1(\cdot, p))_{H_{K_1K_2}(E)}$$

and the existence of the best approximations is that g is represented in the product for some function f_1

$$g = f_1 f$$

and the best approximation function is f_1 itself. In the present case,

$$H_k = \{\varphi_f{}^* \varphi_f; f_1 \in H_{K_1}(E)\} = \{f_1 f; f_1 \in H_{K_1}(E)\}.$$

For the multiplicative operator $\varphi_f f_1 = f_1 f$, **practically we can assume that it is injective** and furthermore, we assume that the operator

$$\varphi_f{}^* \varphi_f \varphi_f{}^* \varphi_f$$

is an identity on H_{K_1}. Then, the above theory is clear all and we can obtain the result:

For the product $g = f_1 f; f_1 \in H_{K_1}(E), f \in H_K(E)$, we obtain the representation

$$f_1(p) = \Big(f(p_2)\,(g(p_1), f(p_1) K_1(p_1, p_2))_{H_{K_1 K}(E)}\,, f(p_2) K_1(p_2, p) \Big)_{H_{K_1 K}(E)}. \tag{7.7}$$

In particular, *when* $g = 1 \cdot f \in H_{K_1 K_2}(E)$, we have the representation

$$\frac{1}{f(p)} = \Big(f(p_2)\,(1, f(p_1) K_1(p_1, p_2))_{H_{K_1 K}(E)}\,, f(p_2) K_1(p_2, p) \Big)_{H_{K_1 K}(E)}. \tag{7.8}$$

Quotient was represented by Product.

However, the situation is not so simple and Section 7 was published in the paper ([34]) in some complete form and the part was deleted.

8 More examples for the Bergman-Szegö spaces

By using complete orthonormal system expansions of the Bergman and Szegö kernels, we can represent the Moore Penrose generalized fractions in terms of a Taylor expansion; however, in the representation we have, in general, some complicated numerical solutions. From the paper [7], we shall see the typical results.

Theorem: *For any* $\mathbf{F}(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}_B$*, the Bergman space and for any fixed* $f_0(z) = a_m z^m$ *in* $\mathcal{H}_S$ *(with* $m \geq 0$ *and* $a_m \neq 0$*) and, the Szegö space, the Tikhonov fractional function* $(f_{\mathbf{F},\lambda})(z)$ *for* $H_K = \mathcal{H}_S$*,* $\mathcal{H} = \mathcal{H}_B$*, and* $Lf = f_0 \cdot f$ *is given by*

$$(f_{\mathbf{F},\lambda})(z) = \sum_{n=0}^{\infty} \frac{\bar{a_m} b_{n+m}}{\lambda(n+m+1) + |a_m|^2} z^n.$$

In particular, by tending $\lambda \to 0$, we obtain

Corollary: *For any* $\mathbf{F}(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}_B$*, the Bergman space and for any fixed* $f_0(z) = a_m z^m$ *in* $\mathcal{H}_S$ *(with* $m \geq 0$ *and* $a_m \neq 0$*), the Szegö space, the Moore-Penrose fractional function* $(f_{\mathbf{F},0})(z)$ *for* $H_K = \mathcal{H}_S$*,* $\mathcal{H} = \mathcal{H}_B$*, and* $Lf = f_0 \cdot f$ *is given by*

$$(f_{\mathbf{F},0})(z) = \sum_{n=0}^{\infty} b_{n+m} z^n.$$

Note that the terms

$$\sum_{n=0}^{m-1} b_n z^n$$

are negleted interestingly and the result is the same by the division by zero calculus.

By the Beurling's theorem (see [10]) on invariant subspaces (see page 84, Corollary 7.3), we obtain

Theorem: *For any fixed* $f_0(z) \in \mathcal{H}_S$*, the Szegö space, the adjoint* L^* *of the multiplication operator* $L : \mathcal{H}_S \longrightarrow \mathcal{H}_B$ $Lf = f_0(z)f(z)$ *is injective if and only if* f_0 *is an outer function.*

This section was published in the paper ([34]) and the part was deleted.

9 Norm inequalities in Sobolev spaces

We have similar results and theory for the Sobolev spaces as in the Bergman and Szegö spaces. For example, let ρ be a positive continuous function on (a, b) satisfying $\rho \in L_1(a, b)$. Let f_j be complex-valued functions on (a, b) satisfying $\lim_{x \to a-0} f_j(x) = 0$. Then, we have the inequality

$$\int_a^b |(f_1(x)f_2(x))'|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)\rho(x)}$$
$$\leq 2\int_a^b |f_1'(x)|^2 \frac{dx}{\rho(x)} \int_a^b |f_2'(x)|^2 \frac{dx}{\rho(x)},$$

when the integrals in the last part are finite. Equality holds here if and only if each f_j is expressible in the form $C_j K_\rho(x, x_2)$ for some constants C_j and for some point $x_2 \in [a, b]$ which is independent of j. Here, $K_\rho(x, \cdot)$ is the reproducing kernel of the Sobolev space with the norm

$$\sqrt{\int_a^b |f_1'(x)|^2 \frac{dx}{\rho(x)}} < \infty$$

([25, 33]).

We considered various norm inequalities and their applications, however, the materials were published in ([33]) and the part was deleted.

10 Division by zero

We note that the famous division by zero $1/0$, $0/0$ and any fractional $g/0$ for $f \equiv 0$ are 0 and the zero function, respectively, trivially in our sense in the both senses of any Tikhonov functionals with any parameter $\alpha > 0$ and $\alpha = 0$. See [31, 32] for the details.

This section was published in ([33]) and the part was deleted.

Acknowledgements

The author wishes to express his deep thanks Professors T. Ohsawa Y. Sawano, M. Seo and A. Yamada for their great contributions to the meeting and the paper.

This meeting and paper were supported by the RIMS.

References

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., **68** (1950), 337–404.

[2] S. R. Bell and B. Gustafsson, *Ruminations on Hejhal's theorem about the Bergman and Szego kernels*, arXiv:2112.06877 (2021).

[3] S. Bergman, *The kernel function and conformal mapping,* Amer. Math. Soc., Providence, R. I. 1950, 1970.

[4] S. Bergman and M. M. Schiffer, *Kernel functions and differential equations*, Academic Press. New York, 1953.

[5] L. P. Castro and S. Saitoh, *Fractional functions and their representations,* Complex Anal. Oper. Theory **7** (2013), no. 4: 1049-1063.

[6] L. P. Castro, K. Murata, S. Saitoh and M. Yamada, *Explicit integral representations of implicit functions*, Carpath. J. Math., **29** (2013), no. 2, 141-148.

[7] L. P. Castro, S. Saitoh and A. Yamada, *Solutions of Tikhonov Functional Equations and Applications to Multiplication Operators on Szegö Spaces*, Complex Analysis and Operator Theory **10** (2016), 1705–1723.

[8] Z. Blocki, *Suita conjecture and the Ohsawa–Takegoshi extension theorem*, Invent. Math., **193** (2013), 149–158.

[9] J. D. Fay, *Theta functions on Riemann surfaces,* Lecture Notes in Math., Vol. **352**, Springer-Verlag 1973.

[10] J. B. Garnett, *Bounded Analytic Functions*, Pure and Applied Mathematics, Vol. **96**, Academic Press Inc., New York, (1981).

[11] Q. A. Guan and X. Y. Zhou, *A solution of an L_2 extension problem with an optimal estimate and applications*, Ann. of Math. (2), **181** (2015), 1139–1208.

[12] Q. Guan, *A proof of Saitoh's conjecture for conjugate Hardy H_2 kernels,* J. Math. Soc. Japan, **71**, No. 4 (2019), 1173–1179.

[13] Q.A. Guan and Z.T. Mi, *Concavity of minimal L_2 integrals related to multiplier ideal sheaves*, arXiv:2106.05089v2.

[14] Q. A. Guan and Z. Yuan, *Concavity property of minimal L_2 integrals with Lebesgue measurable gain*, preprint. https://www.researchgate.net/publication/353794984.

[15] Q. Guan and Z. Yuan, *The weighted version of Saitoh's conjecture,* arXiv:2207.10976 [math.CV] (or arXiv:2207.10976v1 [math.CV] for this version), https://doi.org/10.48550/arXiv.2207.10976, [v1] Fri, 22 Jul 2022 09:47:24 UTC (12 KB)

[16] Q. Guan and Z. Yuan, *Hardy space, kernel function and Saitoh's conjecture on products of planar domains,* arXiv:2210.14579. (2022). 84 pages.

[17] Q. Guan, X. Sun and Z. Yuan, *A remark on a weighted version of Suita conjecture for higher derivatives*, arXiv:2212.02713 2022 , 22 pages.

[18] D. A. Hejhal, *Theta functions, kernel functions and Abel integrals*, Memoirs of Amer. Math. Soc., Providence, R. I. **129** (1972).

[19] Z. Nehari, *Conformal mapping,* McGraw-Hill Book Company Inc., New York (1952).

[20] T. Ohsawa, *The Bergman kernel and the Suita conjecture on Riemann surfaces (Japanese)*, The theory of reproducing kernels and their applications (Kyoto, 1998), Surikaisekikenkyu-sho Kokyuroku, **1067** (1998), 89-95.

[21] T. Ohsawa, *Function Theory Gaiden - 100 years of Bergman Kernel - Book –* , Gendai Suugakusya (2022), in Japanese.

[22] S. Saitoh, *The Bergman norm and the Szegö norm*, Trans. Amer. Math. Soc., **249** (1979)(2), 261–279.

[23] S. Saitoh, *The Dirichlet norm and the norm of Szegö type*, Trans. Amer. Math. Soc., **254** (1979), 355– 364.

[24] S. Saitoh, *Positive Definite Hermitian Matrics and Reproducing Kernels*, Linear Algebra and its Applications, **48** (1982), 119-130.

[25] S. Saitoh, *Hilbert spaces admitting reproducing kernels on the real line and related fundamental inequalities*, Riazi, J. Kar. Math. Assoc., **6** (1984), 25-31.

[26] S. Saitoh, *Quadratic Inequalities Deduced from the Theory of Reproducing Kernels*, Linear Algebra and its Applications, **93** (1987), 171-178.

[27] S. Saitoh, *Theory of reproducing kernels and its applications.* Pitman Research Notes in Mathematics Series, **189**. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1988. x+157 pp. ISBN: 0-582-03564-3.

[28] S. Saitoh, *Representations of inverse functions*, Proc. Amer. Math. Soc., **125** (1997), 3633–3639.

[29] S. Saitoh, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Res. Notes in Math. Series **369** (1997). Addison Wesley Longman Ltd, UK, CRC Press, Taylor & Francis Group, Boca Raton London, New York (in hard cover).

[30] S. Saitoh and Y. Sawano, *Theory of Reproducing Kernels and Applications,* Developments in Mathematics **44**, 2016 (Springer).

[31] S. Saitoh, *Introduction to the Division by Zero Calculus*, Scientific Research Publishing (2021).

[32] S. Saitoh, *Division by Zero Calculus - History and Development*, Scientific Research Publishing (2021).

[33] S. Saitoh, *Norm Inequalities for One Dimensional Sobolev Hilbert Spaces (An Extension)*, viXra:2209.0077 submitted on 2022-09-13 01:01:53.

[34] S. Saitoh, *REPRESENTATIONS OF QUOTIENTS* $g/f = h$ *IN TERMS OF* $g = fh$, International Journal of Reproducing Kernels, Vol. **2**, No. 1 (2023).

[35] N. Suita, *Capacities and kernels on Riemann surfaces*, Arch. Ration. Mech. Anal., **46** (1972), 212–217.

[36] A. Yamada, *Topics related to reproducing kernels, theta functions and the Suita conjecture (Japanese)*, The theory of reproducing kernels and their applications (Kyoto, 1998), Surikaisekikenkyu-sho Kokyuroku, **1067** (1998), 39-47.

[37] A. Yamada, *Fay's trisecant formula and Hardy H^2 reproducing kernels,* Reproducing Kernels and their Applications (1999), Kluwer Academic Publishers, 165–188.

[38] A. Yamada, *Equality conditions for general norm inequalities in reproducing kernel Hilbert spaces,* Advances in Analysis, World Scientific, 2005, 447–455.

[39] A. Yamada, *Bergman's kernel, Szegö's kernel and Dirichlet's integral,* (manuscript in Japanese on December 3, 2021). This paper was accepted for publication in the Complex Analysis and Operator theory.

[40] A. Yamada, *THE REPRODUCING KERNEL OF THE SOBOLEV SPACE ON AN INTERVAL*, to appear in the Proc. AMS.

[41] A. Yamada, *Equality problems in norm inequalities in reproducing kernel Hilbert spaces and completion,* (manuscript in Japanese, June 10, 2022).