Characterisation of Partially Isometric Toeplitz Operators

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This article provides a summary of a published paper [1] that addresses a unsolved problem related to Toeplitz operators. A Toeplitz operator is a product of two operators: a projection and a multiplication operator. Despite its apparent simplicity, there are still numerous unresolved issues associated with Toeplitz operators. The paper focuses on characterizing partially isometric Toeplitz operators. Let us strat with the formal definition of Toeplitz operators.

The classical *Laurent operator* L_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is the bounded linear operator on $L^2(\mathbb{T}^n)$ defined by $L_\varphi f = \varphi f$, $f \in L^2(\mathbb{T}^n)$. The *Toeplitz operator* T_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is the compression of L_φ to $H^2(\mathbb{T}^n)$, that is

$$T_\varphi f = P_{H^2(\mathbb{T}^n)}(\varphi f) \qquad (f \in H^2(\mathbb{T}^n)),$$

where $P_{H^2(\mathbb{T}^n)}$ denotes the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{T}^n)$. Therefore

$$T_\varphi = P_{H^2(\mathbb{T}^n)} L_\varphi|_{H^2(\mathbb{T}^n)}.$$

Toeplitz operators are one of the most useful and prevalent objects in matrix theory, operator theory, operator algebras, and its related fields. For instance, Toeplitz operators provide some of the most important links between index theory, C^*-algebras, function theory, and non-commutative geometry. See the monograph by Higson and Roe [8] for a thorough presentation of these connections, and consult the paper by Axler [7] for a rapid introduction to Toeplitz operators.

Evidently, a lot of work has been done in the development of one variable Toeplitz operators, and it is still a subject of very active research, with an ever-increasing list of connections and applications. But on the other hand, many questions remain to be settled in the several variables case, and more specifically in the open unit polydisc case (however, see [2, 3, 4, 5, 6]). The difficulty lies in the obvious fact that the standard (and classical) single variable tools are either unavailable or not well developed in the setting of polydisc. Evidently, advances in Toeplitz operators on the polydisc have frequently resulted in a number of new tools and techniques in operator theory, operator algebras, and related fields.

If $n = 1$, then the only nonzero Toeplitz operators that are partial isometries are those of the form T_φ and T_φ^*, where $\varphi \in H^\infty(\mathbb{D})$ is an inner function. This was proved by Brown and Douglas in [11].A key ingredient in the proof of the Brown and Douglas theorem is the classical Beurling theorem [10]. Recall that the Beurling theorem connects inner functions in $H^\infty(\mathbb{D})$ with shift invariant subspaces of $H^2(\mathbb{D})$. However, in the present case of higher dimensions, this approach does not work, as is well known, Beurling type classification does not hold for shift invariant subspaces of $H^2(\mathbb{D}^n)$, $n > 1$ .So here, we exploit more analytic and geometric structure of $H^2(\mathbb{D}^n)$ and $L^2(\mathbb{T}^n)$ to achieve the main goal. And successfully got the following theorem .

Theorem 0.1. *Let φ be a nonzero function in $L^\infty(\mathbb{T}^n)$. Then T_φ is a partial isometry if and only if there exist inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$ such that φ_1 and φ_2 depends on different variables and*

$$T_\varphi = T^*_{\varphi_1} T_{\varphi_2}.$$

The article presents lemmas, propositions and theorems in a linear order, and the next lemma to be proven is a multi-variable version of the Brother Riesz theorem. It is important to note that the paper will use $\mathbb{H}^2(\mathbb{T}^n)$ and $\mathbb{H}^2(\mathbb{D}^n)$ interchangeably for ease of computation. we can state the first lemma as follows,

Lemma 0.2. *If $f \in H^2(\mathbb{T}^n)$ is nonzero, then $\boldsymbol{m}(\mathcal{Z}(f)) = 0$.*

The proof of this lemma is based on induction, where $n-1$ coordinates are frozen and the n-torus $\mathbb{T}^n$ is considered as a disjoint union of these circles.

Proposition 0.3. *$\|T_\varphi\| = \|\varphi\|_\infty$ for all $\varphi \in L^\infty(\mathbb{T}^n)$.*

Proposition is a well-known fact that and a short proof is provided using kernel functions. By combining Lemmas 0.2 and proposition 0.3 and expressing the norm in integration form, it can be shown that if a Toeplitz operator $T\varphi$ with norm one attains its norm, then it has to be a unimodular function (i.e., $|\varphi| = 1$ on $\mathbb{T}^n$), which is the precise statement of the next corollary,

Corollary 0.4. *Suppose φ is a nonzero function in $L^\infty(\mathbb{T}^n)$. If $\|T_\varphi f\| = \|\varphi\|_\infty \|f\|$ for some nonzero $f \in H^2(\mathbb{D}^n)$, then $\frac{1}{\|\varphi\|_\infty}\varphi$ is unimodular in $L^\infty(\mathbb{T}^n)$.*

We know that T_φ is a partial isometry. As a result of Corollary 0.4, φ is a unimodular function. This allows us to conclude that both the range of T_φ and the range of $T_\varphi^* = T_{\overline{\varphi}}$ are closed, M_{z_i}-invariant subspaces. Since the adjoint of T_φ is also a partial isometry the same result applies to $T_{\overline{\varphi}}$, we only need to state it for T_φ. Furthermore, the range of T_φ can be written as $kernel T_{\overline{\varphi}}^{\perp}$, and vice versa.

Lemma 0.5. *$\mathcal{R}(T_\varphi)$ is invariant under M_{z_i}, $i = 1, \ldots, n$.*

,which leads us to Lemma 0.6.

Lemma 0.6. *For each $i = 1, \ldots, n$, the function φ cannot depend on both z_i and $\bar{z}_i$ variables at a time.*

In other words, if we consider the power series expansion $\varphi = \sum_{k=1}^{\infty} \bar{z_i}^k \varphi_{-k} \oplus \sum_{k=0}^{\infty} z_i^k \varphi_k$, where $\varphi_k \in L^2(\mathbb{T}^{n-1})$ (i.e., φ_k is independent of z_i), $k \in \mathbb{Z}$, then either the sum on the left or the sum on the right of the "$\oplus$" must be zero. We can prove this using Lemma 0.5 and the defining property of Toeplitz operators, which is that $M_{z_i}^* T_\varphi M_{z_i} = T_\varphi$. Using this, we can find an element $f \in rangeT_\varphi$ that is independent of z_i and say that φ is analytic or co analytic in ith variable (z_i) depending on whether such a non-zero f exists or not.

Using lemma0.6 we can partition the variables into two disjoint sets A and C, where A contains the variables on which φ is analytic and C contains the variables on which φ is co-analytic. The cardinality of A plus the cardinality of C is equal to n. We can then write φ as $\sum_{k \in \mathbb{Z}_+^q} \bar{z}_C^k \varphi_{A,k}$, where $k \in \mathbb{Z}_+^q$, q is the cardinality of C, $\varphi_{A,k}$ are analytic functions that only depend on variables from A, and z_C^k denote the analytic monomials that are generated by C.

Using induction and Lemma 0.5, we can show that all $\varphi_{A,k}$ are in the range of T_φ. Consequently, we have that both $\overline{\varphi}_{A,k} \varphi_{A,l}$ and $\overline{\varphi}_{A,l} \varphi_{A,k}$ are analytic, which implies that $\varphi_{A,k} = \alpha_l \psi$ for some inner function ψ and constant α_l. We can then take out this ψ and combine the remaining terms to obtain Theorem 0.1.

Two immediate consequences of Theorem 0.1 are that hyponormal partially isometric Toeplitz operators (i.e., those satisfying $T^*T - TT^* \geq 0$) are just multiplication by inner functions, and partially isometric Toeplitz operators are power partial isometries. To connect this with the main result in Halmos and Wallen [9], we need a small lemma that says multiplication by an inner function is a shift (pure isometry). The final result is

Theorem 0.7. *Up to unitary equivalence, a partially isometric Toeplitz operator is either a shift, or a co-shift, or a direct sum of truncated shifts.*

If you set $\mathcal{I}$ as kernel of T_φ ,k-th block in a truncated shift will look like,

$$\begin{array}{c|cccccc|}
 & \varphi_1^k\mathcal{I} & \varphi_1^{k-1}\varphi_2^1\mathcal{I} & \varphi_1^{k-2}\varphi_2^2\mathcal{I} & \cdots & \varphi_1^1\varphi_2^{k-1}\mathcal{I} & \varphi_2^k\mathcal{I} \\
\varphi_1^k\mathcal{I} & 0 & 0 & 0 & \cdots & 0 & 0 \\
\varphi_1^{k-1}\varphi_2^1\mathcal{I} & \mathbf{I} & 0 & 0 & \cdots & 0 & 0 \\
\varphi_1^{k-2}\varphi_2^2\mathcal{I} & 0 & \mathbf{I} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & 0 & 0 \\
\varphi_1^1\varphi_2^{k-1}\mathcal{I} & 0 & 0 & 0 & \ddots & 0 & 0 \\
\varphi_2^k\mathcal{I} & 0 & 0 & 0 & \cdots & \mathbf{I} & 0
\end{array}$$

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