Extremal eigenvalue statistics and spectrum of d-dimensional random Schrödinger operator

Fumihiko Nakano
Mathematical Institute, Tohoku University

Abstract

This is a review of joint works [5, 6] with K. Kawaai and Y. Maruyama(Tohoku University). We consider Schrödinger operator with random decaying potential on $\ell^2(\mathbf{Z}^d)$ and (i) we showed that IDS coincides with that of free Laplacian in general cases, (ii) we show some examples, with heavy-tailed single-site distribution, such that the set of rescaled extremal eigenvalues converges to a inhomogeneous Poisson process, and positive real axis belongs to the essential spectrum, (iii) we show the other examples with light-tailed single-site distribution such that the Hamiltonian is bounded almost surely, and the essential spectrum coincides with that of the free Laplacian.

Contents

1 Introduction

In this paper, we consider the d-dimensional Schrödinger operator with random decaying potential :

$$
\begin{aligned}
H &:= H_0 + V, \\
(H_0 u)(n) &= \sum_{|m-n|=1} u(m), \\
(Vu)(n) &:= \frac{\omega_n}{(n)^\alpha} u(n), \quad (n) := (1+|n|), \quad \alpha \ge 0, \quad u \in \ell^2(\mathbf{Z}^d).
\end{aligned}
$$

where $\{\omega_n\}_{n \in \mathbf{Z}^d}$: is i.i.d. with common distribution μ. When $\alpha = 0$, H is usually called "Anderson model" and many properties have been known : e.g., $\sigma(H) = [-2d, 2d] + supp\ \mu$, a.s., and there typically exists an interval $I_{loc} (\subset \sigma(H))$ in which the spectrum is composed

of densely distributed eigenvalues with exponentially decaying eigenfunctions(Anderson localization). On the other hand, when $\alpha > 0$, in many cases the spectrum $\sigma(H)$ of H has a form of $\sigma(H) = [-2d, 2d] \cup S$ (disjoint sum) and $\sigma_c(H) \subset [-2d, 2d]$. Moreover, if supp μ is unbounded, it is likely that $\lim_{|n|\to\infty} V(n) = 0$ for fixed ω, while $\sup_\omega V(n) = \infty$ for fixed n, so that something unusual can happen.
We remark that for the one-dimensional case ($d = 1$), $\alpha = 1/2$ is critical : (i) $\alpha > 1/2 \Longrightarrow \sigma(H) \cap [-2, 2]$ is ac, (ii) $\alpha < 1/2 \Longrightarrow \sigma(H) \cap [-2, 2]$ is pp, (iii) $\alpha = 1/2 \Longrightarrow \sigma(H) \cap [-2 + E_c, 2 - E_c]$ is sc and the compliment is pp, for some E_c.

2 IDS

Let $L \in \mathbf{N}$ and we set the box Λ_L of size $2L + 1$ and the finite-box Hamiltonian H_L which is the restriction of H on Λ_L :

$$\Lambda_L := \left\{ n = (n_1, \cdots, n_d) \in \mathbf{Z}^d \,\middle|\, |n_i| \le L,\ i = 1, 2, \cdots, d \right\}$$
$$H_L := 1_L H 1_L, \quad (1_L)(n) := 1(n \in \Lambda_L).$$

To set up the problem, let $E_j(L)$, $j = 1, 2, \cdots, |\Lambda_L|$ be the eigenvalues of H_L and let μ_L be the empirical measure for the eigenvalues of H_L : a random probability measure on $\mathbf{R}$ defined by

$$\mu_L := \frac{1}{|\Lambda_L|} \sum_j \delta_{E_j(L)}. \tag{2.1}$$

Among many known results, we recall (i) if $\alpha = 0$, there exists a deterministic measure μ_{DS}, s.t. $\mu_L \xrightarrow{w} \mu_{DS}$, a.s. (ii) in particular, let μ_L^0 be the empirical measure for the free Laplacian (that is, the Hamiltonian H_0). Then we have an ac probability measure μ_{DS}^0 with supp $\mu_{DS}^0 = [-2d, 2d]$ such that $\mu_L^0 \xrightarrow{w} \mu_{DS}^0$. In fact, μ_{DS}^0 is equal to the spectral measure of H_0 associated to δ_0. (iii) if $\alpha > 0$ and if $\mathbf{E}[\omega_0^2] < \infty$, we have $\mu_L \xrightarrow{w} \mu_{DS}^0$ ([4]). We first remark that the second moment condition in [4] is not necessary :

Theorem 1 *Let $\alpha > 0$. For any i.i.d. $\{\omega_n\}$, we have*

$$\mu_L^\omega \xrightarrow{w} \mu_{DS}^0, \quad a.s.$$

Remark
(1) It is well known that, if $\alpha = 0$, $\sigma(H) = supp\ \mu_{DS}$, a.s. However, Theorem 1 says it is not the case for $\alpha > 0$. In fact for $\alpha > 0$, Theorem 1 implies that μ_{DS}^0 is not supported on $\sigma(H_0)^c = [-2d, 2d]^c$, while, as we shall see later, there are examples in which $[-2d, 2d]^c \cap \sigma(H) \neq \emptyset$ and furthermore we have Anderson localization on that set. But the eigenvalue distribution are much thinner there than the usual cases.
(2) Although there are many cases in which $\pm 2d$ lies in the boundary of the spectrum, IDS does not have Lifschitz tail behavior near $\pm 2d$ so that usual tool to show the Anderson localization does not work there.

3 Extreme value statistics

We first discuss the eigenvalue statistics in the bulk. In order to do that, we usually pick up $E_0 \in \sigma(H)$ and consider

$$\xi_L := \sum_j \delta_{|\Lambda_L|(E_j(L)-E_0)}$$

to study the local eigenvalue statistics near E_0. For $\alpha = 0$, if E_0 lies in the localized region and $n(E_0) := \dfrac{d\mu_{DS}}{dE}(E_0) > 0$, then $\xi_L \overset{d}{\to} Poisson(n(E_0)dE)$ [11]. However, for $\alpha > 0$, it may not be the case, because $n(E_0) = 0$ if $E_0 \notin [-2d, 2d]$.

Instead, if μ has heavy tail at infinity, the first few eigenvalues of H_L presumably go to infinity as L goes to infinity, so that it may be reasonable to consider the scaling limit of those. In fact, Dolai [4] obtained the limit distribution of the maximal eigenvalue of H_L in a special case of μ. We begin by setting up some notations. We denote the tail of common distribution μ by

$$\mu[x, \infty) = \frac{1}{f(x)}, \quad x > 0,$$

for a function f. Let $\{E_j^H(L)\}_{j\geq 1} : E_1^H(L) \geq E_2^H(L) \geq \cdots$ be positive eigenvalues of H_L in decreasing order, and let

$$\widetilde{E}_j^H := \frac{f(E_j^H(L))}{\Gamma_L}, \quad j = 1, 2, \cdots,$$

be the scaling of those, where Γ_L will be chosen depending on f such that $\lim_{L\to\infty} \Gamma_L = \infty$. We set the point process with atoms being composed of the rescaled eigenvalues :

$$\xi_L := \sum_{j\geq 1} \delta_{\widetilde{E}_j^H(L)}.$$

We set the following two assumptions on f and Γ_L.

Assumption 1
$f : (0, \infty) \to (0, \infty)$ and Γ_L satisfy the following conditions :
(1) f is strictly increasing on $[R, \infty)$ for some $R > 0$, $\lim_{x\to\infty} f(x) = \infty$, $f \in C^1$, and $\lim_{L\to\infty} \Gamma_L = \infty$,
(2) $f'(x) = o(f(x))$, $x \to \infty$,
(3) $\sup_{|x-y|\leq 2d} |f(y)| \leq C|f(x)|$ for a positive constant C and sufficiently large x.

The condition (1) is natural, since $1/f$ gives the tail of a measure. Conditions (2), (3) are satisfied if f is of regular variation. On the other hand, the following one is essential for our problem and non-trivial :

Assumption 2

$$\lim_{L\to\infty} \sum_{n\in\Lambda_L} \mathbf{P}\left(\frac{f(V(n))}{\Gamma_L} \geq x\right) = \frac{1}{x}, \quad x > 0.$$

We note that, if $\alpha = 0$ and if Assumption 1 is satisfied, Assumption 2 is always valid with $\Gamma_L = |\Lambda_L|$. Let ν be a measure on $(0, \infty)$ defined by

$$d\nu := \frac{1}{x^2} 1_{(0,\infty)}(x) dx.$$

Under the two assumptions above, the rescaled extremal eigenvalues converge to a Poisson process :

Theorem 2 *Suppose f, Γ_L satisfy Assumption 1, 2. Then $\xi_L \stackrel{d}{\to}$ Poisson(ν).*

Here we consider the vague topology on the space of point processes on $\mathbf{R}$. As for the related results, the eigenvalue/eigenfunction statistics on the bulk for $d = 1$ is well studied [10, 8, 12, 9, 13, 15, 14] and the various limits such as clock, Sine$_\beta$ and Poisson appear. However, extremal eigenvalue statistics have not been studied even for $\alpha = 0$.

4 Examples

We show below two classes of functions satisfying Assumption 1, 2, and discuss a relation to spectral properties. For simplicity, we assume supp $\mu \subset (0, \infty)$.

4.1 Power functions

The first one is a family of power functions with some logarighmic corrections.

$$f(x) = f_{p,k}(x) := x^p (\log x)^{-k}, \quad p > 0, \quad k \in \mathbf{N} \cup \{0\}, \quad x > R$$

for some $R > 0$. We remark that Dolai [4] obtained the limiting distribution of $\widetilde{E}_1^H(L)$ when $p > 0$, $k = 0$.

Theorem 3
$f_{p,q}$, Γ_L satisfy Assumption 1, 2 in (1), (2) below.
(1) $\alpha p \le d$: $\xi_L \stackrel{d}{\to}$ Poisson(ν) with

$$\begin{cases} \Gamma_L = \gamma_{p,k} L^{d-\alpha p}, \quad \gamma_{p,k} := \frac{C_{d-1}}{d-\alpha p}\left(\frac{d}{d-\alpha p}\right)^k & (\alpha p < d) \\ \Gamma_L = h_k^{-1}\left(\gamma_k (\log L)^{k+1}\right), \quad h_k(x) := x(\log x)^k, \quad \gamma_k := \frac{C_{d-1}}{k+1} \cdot p^k & (\alpha p = d) \end{cases}$$

where $C_{d-1} := |S^{d-1}|$ is the surface area of the d-dimentional unit ball. Moreover, $\sigma(H) = \sigma_{ess}(H) = [-2d, \infty)$, we have Anderson localization on $(2d, \infty)$, and $\limsup_{|n|\to\infty} V(n) = \infty$, a.s.
(2) $\alpha p > d$: there exist positive constants C_1, C_2 such that the following estimate is valid for sufficiently large x.

$$\left(1 - \frac{C_1}{f(x)}\right) e^{-C_1 x^{-d/\alpha}} \le \mathbf{P}\left(\bigcap_{L \ge 1} \{E_1(L) \le x\}\right) \le \exp\left[-\frac{C_2}{f(x)}\right]. \tag{4.1}$$

Moreover, $\sigma_{ess}(H) = [-2d, 2d]$, a.s. and $\lim_{|n|\to\infty} V(n) = 0$, a.s.,

Remark
(i) We believe that for $\alpha p < d$ the result is true for any $k \in \mathbf{R}$.
(ii) Theorem 3(1) includes the case for the usual Anderson model where $\alpha = 0$.
(iii) It is natural to expect that the statement in Theorem 3 would be valid for general function f which is of regular variation of order p :

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^p, \quad \lambda > 0. \tag{4.2}$$

In fact, a formal computation indicates that f would satisfy Assumption 2 with $\Gamma_L = L^{d-\alpha p}/(d-\alpha p)$ (say, for the case of $\alpha p < d$). However, the constant $\gamma_{p,k}$ in Theorem 3 (1) implies that these observation is false in general and the quantity which vanishes in the limit in (4.2) has a non-zero contribution in the limiting behavior of ξ_L.
(iv) The leftmost inequality in (4.1) and Borel-Cantelli argument show that $\mathbf{P}(\limsup_{L\to\infty} E_1 < \infty) = 1$, while the rightmost one implies that there is no constant M such that $E_1^H(L) \le M$, a.s., having completely different behavior from that in Theorem 3(1). Moreover, since $\sigma_{ess}(H) = [-2d, 2d]$ a.s., to consider the limit of ξ_L would be meaningless in this case.

4.2 Exponential functions

We next consider a family of exponential functions :

$$f(x) = f_\delta(x) := e^{x^\delta}, \quad 0 < \delta \le 1.$$

In this case, the tail of ω_n is smaller than the previous one, so that we expect that the behavior of eigenvalues become more gentle.
Theorem 4
(1) $0 < \delta < 1$, $\alpha = 0$: $\xi_L \stackrel{d}{\to}$ *Poisson*(ν) *with* $\Gamma_L = |\Lambda_L|$. *Moreover,* $\sigma(H) = \sigma_{ess}(H) = [-2d, \infty)$, *we have Anderson localization on* $(2d, \infty)$, *and* $\limsup_{|n|\to\infty} V(n) = \infty$, *a.s.*
(2) $0 < \delta \le 1$, $\alpha > 0$: *we can find positive constants* C_j, $j = 1, 2$ *such that for sufficiently large* x, *we have*

$$1 - C_1 e^{-x^\delta} \le \mathbf{P}\left(\bigcap_{L\ge 1} \{E_1^H(L) \le x\} \right) \le \exp\left[-C_2 x^{-d/\alpha} e^{-2D_{\alpha,\delta} x^\delta} \right]$$

where $D_{\alpha,\delta} = \max\{1, 2^{\alpha\delta - 1}\}$. *Moreover* $\sigma_{ess}(H) = [-2d, 2d]$, *a.s. and* $\lim_{|n|\to\infty} V(n) = 0$, *a.s.*

Theorems 3,4 imply that we have a phase transition : there exists α_c such that
(1) $\alpha \le \alpha_c$: $\xi_L \stackrel{d}{\to}$ *Poisson*(ν), $\sigma(H) = \sigma_{ess}(H) = [-2d, \infty)$, we have Anderson localization on $(2d, \infty)$, and $\limsup_{|n|\to\infty} V(n) = \infty$, a.s.
(2) $\alpha > \alpha_c$: $\limsup_{L\to\infty} E_1 < \infty$, a.s., $\sup_{\omega, L} E_1 = \infty$, $\sigma_{ess}(H) = [-2d, 2d]$, a.s. and $\lim_{|n|\to\infty} V(n) = 0$, a.s.
It would be interesting if we could prove above statements for more general cases.

This work is partially supported by JSPS KAKENHI Grant Number 20K03659(F.N.), and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center Located in Kyoto University.

References

[1] Bai, Z., D., Methodologies in spectral analysis of large dimensional random matrices, a review, Statistica Sinica **9**(1999), 611-677.

[2] Breuer, J., Grinshpon, Y., and White M.J., Spectral fluctuations for Schrödinger operators with a random decaying potential, Ann. Henri Poincaré **22**(2021), 3763-3794.

[3] Delyon, F., Kunz, H., and Souillard, B., From power pure point to continuous spectrum in disordered systems, Ann. Inst. H. Poincaré **42**(1985), 283-309.

[4] Dolai, D. R., The IDS and asymptotic of the largest eigenvalue of random Schrödinger operators with decaying random potential, Rev. Math. Phys. **33**(2021), no.8, 2150026.

[5] Kawaai, K., Maruyama, Y., and Nakano, F., Limiting distribution of extremal eigenvalues of d-dimensional random Schrödinger operator, Reviews in Mathematical Physics, **35** No.2(2023), 2250041.

[6] Kawaai, K., Maruyama, Y., and Nakano, F., A relation between the extremal eigenvalue statistics and the spectrum of d-dimensional random Schrödinger operator, in preparation.

[7] Kirsch, W., Krishna, M., and Obermeit, J., Anderson model with decaying randomness : mobility edge, Math. Z. **235**(2000), 421-433.

[8] Kotani, S., Nakano, F., Level statistics for the one-dimensional Schroedinger operators with random decaying potential, Interdisciplinary Mathematical Sciences Vol. 17 (2014) p.343-373.

[9] Kotani, S., Nakano, F., Poisson statistics for 1d Schrödinger operators with random decaying potentials, Electronic Journal of Probability **22**(2017), no.69, 1-31.

[10] Kritchevski, E., Valkó, B., Virág, B., The scaling limit of the critical one-dimensional random Sdhrödinger operators, Commun. Math. Phys. **314**(2012), 775-806.

[11] Minami, N., Local fluctuation of the spectrum of a multidimensional Anderson tight binding model, Commun. Math. Phys. **177**(1996), no.3, 709 - 725.

[12] Nakano, F., Level statistics for one-dimensional Schrödinger operators and Gaussian beta ensemble, Journal of Statisical Physics **156**(2014), 66-93.

[13] Nakano, F., Fluctuation of density of states for 1d Schrödinger operators, Journal of Statisical Physics **166**(2017):1393-1404.

[14] Nakano, F., Shape of eigenvectors for the decaying potential model, Annales Henri Poincaré. **24**(2023), 871-893.

[15] Rifkind, B., Virág, B, Eigenvectors of the 1-dimensional critical random Schrödinger operator, Geom. Funct. Anal. **28** (2018), 1394-1419.