Classification of the matrix algebras over pseudo-solenoids via K_1 groups

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1 Introduction and main theorems

A *pseudo-solenoid* X is a compact connected metrizable space that is the limit of an inverse sequence of the circles $\mathbb{T}$

$$\mathbb{T} \xleftarrow{p_1} \mathbb{T} \xleftarrow{p_2} \mathbb{T} \xleftarrow{p_3} \cdots$$

where p_i is a "crooked" map (see [7]) that has a positive winding number: $\deg\, p_i \geq 1$ for each i. It has an exotic property, called *hereditary indecomposablility*: for each pair K, L of compact connected subsets of X with $K \cap L \neq \emptyset$, we have either $K \subset L$ or $L \subset K$. In particular every continuous map $[0,1] \to X$ is a constant map. In addition to this complexity, the class of the pseudo-solenoids exhibits a topological rigidity in that, two pseudo-solenoids X and Y are homeomorphic if and only if X and Y are shape equivalent (Here "shape equivalence" is an analogue of "homotopy equivalence" for spaces that do not have homotopy type of CW complexes. See [16] for topological shape theory). These conditions are also equivalent to the condition $\check{\mathrm{H}}^1(X;\mathbb{Z}) \cong \check{\mathrm{H}}^1(Y;\mathbb{Z})$, where $\check{\mathrm{H}}^1(\cdot;\mathbb{Z})$ denotes the first integral Čech cohomology ([10], [18]). Futhermore a continuous surjection $\alpha : Y \to X$ between pseudo-solenoids X and Y is approximated by homeomorphisms arbitrarily closely if and only if α induces an isomorphism $\alpha^* : \check{\mathrm{H}}^1(X;\mathbb{Z}) \to \check{\mathrm{H}}^1(Y;\mathbb{Z})$ (see Theorem 2.5).

This note reports a recent result of the author [12] on a non-commutative analogue of these results. The subject is the matrix algebra $M_n(C(X)) \cong C(X) \otimes M_n(\mathbb{C})$ over a pseudo-solenoid X: in view of characterization of $*$-homomorphisms of the algebra $M_n(C(X))$ for a compact Hausdorff space

X ([21]) together with the above results, we may expect that the algebras $M_n(C(X))$ are classified by their C^*-algebra-shape types in the sense of Blackadar [3]. Theorem 1.1, Theorem 1.2 and Corollary 1.3 below show that this is indeed the case. For each $*$-homomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$ of compact Hausdorff spaces X and Y, there exist a continuous map $\alpha : Y \to X$ and a (not necessarily continuous) map $u : Y \to U(n)$ of Y to the unitary group $U(n)$ such that

$$(\varphi f)(y) = u(y) \cdot (f(\alpha(y))) \cdot u(y)^*, \quad y \in Y$$

for each $f \in M_n(C(X))$. It turns out that the map α is uniquely determined by φ and is denoted by α_φ in the sequel. The K_1-group of a C^*-algebra A is denoted by $K_1(A)$ ([19]). For a metric d on a metrizable space X and for continuous maps $\alpha, \beta : Y \to X$, let $d(\alpha, \beta) = \sup_{y \in Y} d(\alpha(y), \beta(y))$.

Theorem 1.1. *Let X and Y be pseudo-solenoids and let $n \geq 1$. For a $*$-homomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$, the following conditions are equivalent.*

(a) There exists a sequence $\{\varphi_k : M_n(C(X)) \to M_n(C(Y)) \mid k \geq 1\}$ of $*$-isomorphisms such that $\lim_{k\to\infty} \|\varphi_k f - \varphi f\| = 0$ for each $f \in M_n(C(X))$.

(b) There exists a sequence $\{\alpha_k : Y \to X \mid k \geq 1\}$ of homeomorphisms such that $\lim_{k\to\infty} d(\alpha_k, \alpha_\varphi) = 0$.

(c) The homomorphism φ is a shape equivalence in the sense of [3].

(d) The induced homomorphism $K_1(\varphi) : K_1(M_n(C(X))) \to K_1(M_n(C(Y)))$ of the K_1-groups is an isomorphism.

It can be shown that the condidion (b) above does not depend on the choice of a metric on X. Also in [11], the condition (b) is shown to be equivalent to α_φ being a shape equivalence.

Theorem 1.2. *For each isomorphism $F : K_1(M_n(C(X))) \to K_1(M_n(C(Y)))$, there exists a $*$-isomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$ such that $F = K_1(\varphi)$.*

Combining Theorem 1.1, Theorem 1.2 and [11, Theorem 11], we obtain the following corollary.

Corollary 1.3. *Let X, Y be two pseudo-solenoids. Then the following conditions are equivalent.*

(a) $M_n(C(X)) \cong M_n(C(Y))$ *as C^*-algebras.*

(b) $M_n(C(X))$ *and* $M_n(C(Y))$ *are shape equivalent in the sense of [3].*

(c) $K_1(M_n(C(X))) \cong K_1(M_n(C(Y)))$.

(d) X *and* Y *are homeomorphic.*

(e) X *and* Y *are shape equivalent.*

(f) $\check{\mathrm{H}}^1(X;\mathbb{Z}) \cong \check{\mathrm{H}}^1(Y;\mathbb{Z})$.

Also one can show (see [12, Proposition 3.3]) that $M_m(C(X))$ and $M_n(C(X))$ are not isomorphic as C^*-algebras if $m \neq n$. This together with the above corollary shows that the matrix C^*-algebras $M_n(C(X))$ over pseudo-solenoids X, $n \geq 1$, form a (rather small) family of C^*-algebras of positive real rank that are classified by their K_1-groups (cf. [8]).

What is crucial in the argument is the characterization of $*$-homomorphisms of the matrix algebra $M_n(C(X))$ due to Thomsen (Theorem 2.6), from which it follows that the homotopy information of a $*$-homomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$ is carried by the continuous map $\alpha_\varphi : Y \to X$.

Throughout, $\mathbb{T}$ denotes the unit circle on the complex plane. For a continuous map $p : \mathbb{T} \to \mathbb{T}$, deg p denotes the winding number of p. For a compact Hausdorff space X, $M_n(C(X))$ denotes the C^*-algebra of the size n matrices over $C(X)$. All $*$-homomorphisms between unital C^*-algebras are assumed to be unital unless otherewise stated.

2 Preliminaries on pseudo-solenoids and $*$-homomorphisms on matrix algebras

First we recall some basics on pseudo-solenoids. For an inverse sequence

$$X_1 \xleftarrow{p_1} X_2 \xleftarrow{p_2} X_3 \longleftarrow \cdots$$

of compact metrizable spaces and for integers i, j with $i \leq j$, let $p_{ij} = p_i \circ p_{i+1} \circ \cdots \circ p_{j-1} : X_j \to X_i$. In particular $p_{i\ i+1} = p_i$. The above sequence is simply denoted by

$$(X_i, p_{ij} : X_j \to X_i). \tag{2.1}$$

Let X be the inverse limit $X = \varprojlim(X_i, p_{ij} : X_j \to X_i)$ and let $p_{i\infty} : X \to X_i$ be the projection to the factor space X_i. A *solenoid* Σ is the limit $\Sigma = \varprojlim(X_i, p_{ij} : X_j \to X_i)$ of an inverse sequence $(X_i, p_{ij} : X_j \to X_i)$, where $X_i = \mathbb{T}$ for each i and the map $p_{i\ i+1} : \mathbb{T} \to \mathbb{T}$ is given by

$$p_{i\ i+1}(z) = z^{d_i}, \quad z \in \mathbb{T}$$

for a positive integer d_i.

For two cotinuous maps $f, g : S \to T$ between topological spaces S and T, we write "$f \simeq g$" when they are homotopic.

Definition 2.1. *[11] An inverse sequence (2.1) is said to have Property* $(*)$ *if the following condition holds: for each i and for each $\epsilon > 0$ and for each continuous map $\lambda : X_j \to X_i$ with $j \geq i$ and $\lambda \simeq p_{ij}$, there exist an integer $k \geq j$ and a map $\mu : X_k \to X_j$ such that $\mu \simeq p_{jk}$ and $d(\lambda \circ \mu, p_{ik}) < \epsilon$.*

A pseudo-solenoid is originally defined as a compact connected hereditarily indecomposable metrizable space that is the limit of an inverse sequence of simple closed curves with essential bonding maps. The next definition is convenient for our purpose. As shown in [11, Theorem 7], it is equivalent to the original definition.

Definition 2.2. *A pseudo-solenoid X is the limit of an inverse sequence*

$$X = \varprojlim(X_i, p_{ij} : X_j \to X_i)$$

with Property $(*)$ *such that* $X_i = \mathbb{T}$ *and* $\deg\ p_{i\ i+1} \geq 1$ *for each i. When* $\deg\ p_{i\ i+1} = 1$ *for each i, X is called the pseudo-circle.*

Every pseudo-solenoid has the same topological shape type of a solenoid. See [9], [10], [18] for basic information on pseudo-solenoids and see also [2], [20] for recent developements. The spaces have been studied also from dynamical system point of view (e.g. [5], [13]). The name "the" pseudo-circle above is justified by the next theorem.

Theorem 2.3. *[9],[10],[11, Corollary 8] Let X and Y be pseudo-solenoids. Then X and Y are homeomorphic if and only if X and Y are shape equivalent.*

In what follows, a metric on a metrizable space is always denoted by d. A specific choice of a metric is not important for our argument.

Definition 2.4. *A continuous map $\alpha : X \to Y$ between compact metrizable spaces is called a near-homeomorphism if for each $\epsilon > 0$, there exists a homeomorphism $\theta : X \to Y$ such that $d(\alpha, \theta) < \epsilon$.*

Every near-homeomorphism is a surjection.

Theorem 2.5. *[11, Theorem 11] Let X, Y be pseudo-solenoids and let $\alpha : Y \to X$ be a continuous surjection. The following conditions are equivalent.*

(a) The map α is a near-homeomorphism.

(b) The map α is a shape equivalence.

(c) The induced homomorphism $\alpha^* : \check{\mathrm{H}}^1(X;\mathbb{Z}) \to \check{\mathrm{H}}^1(Y;\mathbb{Z})$ of the first integral Čech cohomology is an isomorphism.

For later use, we outline the proof of the implication (b)⇒(a) above. Let $X = \varprojlim(X_i, p_{ij} : X_j \to X_i), Y = \varprojlim(Y_i, q_{ij} : Y_j \to Y_i)$. For the proof it is enough to assume that each of X_i and Y_i is a compact ANR and the inverse sequences $(X_i, p_{ij} : X_j \to X_i)$ and $(Y_i, q_{ij} : Y_j \to Y_i)$ have Property $(*)$.

Assume that $\alpha : Y \to X$ is a shape equivalence. Then we can take subsequences $\{m_i\}, \{n_i\}$ of positive integers, sequences of continuous maps $\{\alpha_i : Y_{n_i} \to X_{m_i} \mid i \geq 1\}$ and $\{\beta_i : X_{m_{i+1}} \to Y_{n_i} \mid i \geq 1\}$, and a sequence $\{\epsilon_i\}$ of positive numbers with $\lim_{i\to\infty} \epsilon_i = 0$ such that

$$d(p_{m_j m_i} \circ \alpha_i \circ q_{n_i\infty}, p_{m_j\infty} \circ \alpha) < \epsilon_i \quad \text{for each } j \leq i, \quad \text{and}$$
$$\alpha_i \circ \beta_i \simeq p_{m_i m_{i+1}}, \ \beta_i \circ \alpha_{i+1} \simeq q_{n_i n_{i+1}}.$$

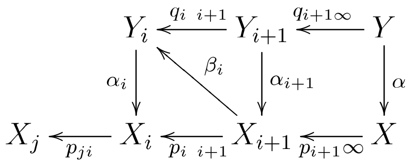
For notational simplicity we may assume without loss of genearity that $X_i = X_{m_i}, Y_i = Y_{n_i}$. Then we have

(1) $d(p_{ji} \circ \alpha_i \circ q_{i\infty}, p_{j\infty} \circ \alpha) < \epsilon_i$ for each $j \leq i$

(2) $\alpha_i \circ \beta_i \simeq p_{i\ i+1}, \ \beta_i \circ \alpha_{i+1} \simeq q_{i\ i+1}$, and in particular,

(3) $\alpha_i \circ q_{i\ i+1} \simeq p_{i\ i+1} \circ \alpha_{i+1}$

for each $i \geq 1$. See the diagram below.



In what follows we inductively define subsequences $\{i_k\}$, $\{j_k\}$ of positive integers and sequences of continuous maps $\{\theta_k : Y_{j_k} \to X_{i_k} \mid k \geq 1\}$ and $\{\tau_k : X_{i_{k+1}} \to Y_{j_k} \mid k \geq 1\}$ as follows. Take sequences $\{\xi_k\}$ and $\{\eta_k\}$ of positive numbers such that $\lim_{k\to\infty} \xi_k = \lim_{k\to\infty} \eta_k = 0$.

Start with $i_1 = j_1 = 1$ and let $\theta_1 = \alpha_1 : Y_{j_1} \to X_{i_1}$. From (2) above we have $\theta_1 \circ \beta_1 \simeq p_{i_1 i_1+1}$. Apply Property $(*)$ to $\theta_1 \circ \beta_1$ in order to find $i_2 \geq i_1 + 1$ and a map $v_1 : X_{i_2} \to X_{i_1+1}$ such that

$$v_1 \simeq p_{i_1+1\ i_2} \text{ and } d(\theta_1 \circ \beta_1 \circ v_1, p_{i_1 i_2}) < \xi_1.$$

Let $\tau_1 = \beta_{i_1} \circ v_1 : X_{i_2} \to Y_{j_1}$. We obtain

(i) $d(\theta_1 \circ \tau_1, p_{i_1 i_2}) < \xi_1$ and

(ii) $\tau_1 \circ \alpha_{i_2} \simeq \beta_1 \circ p_{i_1+1\ i_2} \circ \alpha_{i_2} \simeq \beta_1 \circ \alpha_{i_1+1} \circ q_{i_1+1\ i_2} \simeq q_{i_1\ i_1+1} \circ q_{i_1+1\ i_2} = q_{i_1 i_2}$.

Apply Property $(*)$ to $\tau_1 \circ \alpha_{i_2}$ to find an integer $j_2 \geq i_2$ and a continuous map $w_1 : Y_{j_2} \to X_{i_2}$ such that

$$w_1 \simeq p_{i_2 j_2} \text{ and } d(\tau_1 \circ \alpha_{i_2} \circ w_1, q_{j_1 j_2}) < \eta_1.$$

Let $\theta_2 = \alpha_{i_2} \circ w_1 : Y_{j_2} \to X_{i_2}$. We have

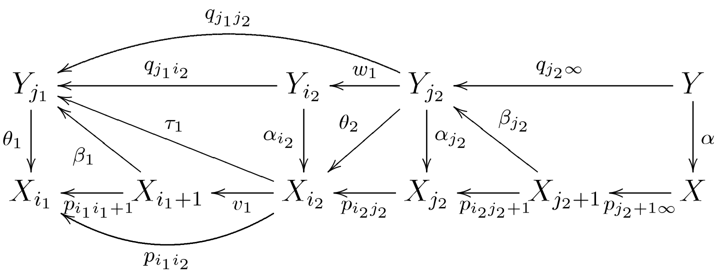
(iii) $d(\tau_1 \circ \theta_2, q_{j_1 j_2}) < \eta_1$,

(iv) $\theta_2 \circ q_{j_2\infty} \simeq \alpha_{i_2} \circ q_{i_2 j_2} \circ q_{j_2\infty} \simeq p_{i_2\infty} \circ \alpha$,

(v) $p_{i_2 j_2} \circ \alpha_{j_2} \simeq \alpha_{i_2} \circ q_{i_2 j_2} \simeq \theta_2$, and

(vi) $\theta_2 \circ \beta_{j_2} \simeq \alpha_{i_2} \circ q_{i_2 j_2} \circ \beta_{j_2} \simeq p_{i_2 j_2} \circ \alpha_{j_2} \circ \beta_{j_2} \simeq p_{i_2 j_2} \circ p_{j_2\ j_2+1} = p_{i_2\ j_2+1}$.

See the following diagram.



Repeating this process we have sequences $\{i_k\}, \{j_k\}$ of positive integers and $\{\theta_k : Y_{j_k} \to X_{i_k} \mid k \geq 1\}$ and $\{\tau_k : X_{i_{k+1}} \to Y_{j_k} \mid k \geq 1\}$ of continuous maps such that

$$d(\theta_k \circ \tau_k, p_{i_k i_{k+1}}) < \xi_k, \quad d(\tau_k \circ \theta_{k+1}, q_{j_k j_{k+1}}) < \eta_k.$$

By making ξ_k and η_k sufficiently small, we obtain a well-defined map $\theta : Y \to X$ defined by

$$p_{i_k\infty} \circ \theta := \lim_{\ell\to\infty} p_{i_k i_\ell} \circ \theta_\ell \circ q_{j_\ell\infty} \quad (k \geq 1)$$

which is a homeomorphism satisfying $d(\theta_1 \circ q_{j_1\infty}, p_{i_1\infty} \circ \theta) < \xi_1$ ([17]).

For an arbitrary $\epsilon > 0$, take a large integer N and a small $\delta > 0$ such that

$$\mathrm{diam}(S) < \delta,\ S \subset X_{i_N} \ \Rightarrow \mathrm{diam}(p_{i_N\infty}^{-1}(S)) < \epsilon,$$

where $\mathrm{diam}(S)$ denotes the diameter of S (with an appropriate metric). We may assume that $\epsilon_N, \xi_N < \delta/2$. We may repeat the above construction of θ starting with $i_1 = j_1 = N$ and $\theta_1 = \alpha_N$. Then the resulting homeomorphism $\theta : Y \to X$ satisfies $d(p_{i_N\infty} \circ \theta, \alpha_N \circ q_{j_N\infty}) < \xi_N < \delta/2$. Also we have $d(\alpha_N \circ q_{j_N\infty}, p_{i_N\infty} \circ \alpha) < \epsilon_N < \delta/2$. Then we see $d(p_{i_N\infty} \circ \alpha, p_{i_N\infty} \circ \theta) < \delta$ and by the choice of δ, we have $d(\theta, \alpha) < \epsilon$. Therefore α is a near-homeomorphism.

Next we recall some results on $*$-homomorphisms of matrix algebras $M_n(C(X))$ over $C(X)$ with X being a compact metrizable space. The following theorem due to Thomsen plays a fundamental role (see [6] for an analogue in Lipschitz algebras). For a continuous map $\alpha : Y \to X$, let $\alpha^\sharp : M_n(C(X)) \to M_n(C(Y))$ be the homomorphism defined by $(\alpha^\sharp f)_{ij} = f_{ij} \circ \alpha$ $(1 \leq i, j \leq n)$ for $f = (f_{ij}) \in M_n(C(X))$. The unitary group of size n is denoted by $U(n)$.

Theorem 2.6. *[21],[22] Let* $\varphi : M_n(C(X)) \to M_n(C(Y))$ *be a* $*$*-homomorphism. There exist a continuous map* $\alpha : Y \to X$ *and a (not necessarily continuous) function* $u : Y \to U(n)$ *such that*

$$(\varphi f)(y) = u(y) \cdot f(\alpha(y)) \cdot u(y)^*, \quad y \in Y \tag{2.2}$$

for each $f \in M_n(C(X))$.

Remark 2.7. *The above continuous map* $\alpha : Y \to X$ *is uniquely deteremied by* φ*, while the unitary element* $u(y), y \in Y$*, is unique up to multiplication of unimodular scalar matrices.*

For a $*$-homomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$, let $\alpha_\varphi : Y \to X$ be the continuous map of (2.2). Remark 2.7 shows that α_φ is well-defined. Also let $U(n)/Z(U(n))$ be the quotient Lie group of $U(n)$ by the center $Z(U(n)) = \{z1_n \mid z \in \mathbb{T}\}$ with the projection $\pi : U(n) \to U(n)/Z(U(n))$. Remark 2.7 again shows that the map $\bar{u}_\varphi : Y \to U(n)/Z(U(n))$, given by $\bar{u}_\varphi(y) := \pi(u(y))$ for $y \in Y$, is well-defined and is uniquely determined by φ. Also Lemma 2.8 (1) below shows that the map $\mathrm{Ad}(\bar{u}_\varphi) : M_n(C(Y)) \to M_n(C(Y))$ given by

$$(\mathrm{Ad}(\bar{u}_\varphi)f)(y) = u_y \cdot f(y) \cdot u_y^*, \quad y \in Y$$

where u_y is an aribrary unitary matrix such that $\pi(u_y) = \bar{u}_\varphi(y)$, is well-defined as well. Under the notation, every homomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$ is written as

$$\varphi = \mathrm{Ad}(\bar{u}_\varphi) \circ \alpha_\varphi^\sharp \tag{2.3}$$

for continuous maps $\alpha_\varphi : Y \to X$ and $\bar{u}_\varphi : Y \to U(n)/Z(U(n))$.

For a sequence $\{\varphi_k : M_n(C(X)) \to M_n(C(Y)) \mid k \geq 1\}$ of $*$-homomorphisms and a homomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$, we write "$\lim_{k\to\infty} \varphi_k = \varphi$" if $\lim_{k\to\infty} \|\varphi_k(f) - \varphi(f)\| = 0$ for each $f \in M_n(C(X))$.

The proofs of the next two lemmas are omitted (see [12]).

Lemma 2.8. *Let* $\varphi = \mathrm{Ad}(\bar{u}_\varphi) \circ \alpha_\varphi^\sharp : M_n(C(X) \to M_n(C(Y))$ *be a* $*$*-homomorphism.*

(1) The map $\bar{u}_\varphi : Y \to U(n)/Z(U(n))$ *is continuous.*

(2) Let $\{\varphi_k : M_n(C(X)) \to M_n(C(Y)) \mid k \geq 1\}$ *be a sequence of* $*$*-homomorphisms such that* $\lim_{k\to\infty} \varphi_k = \varphi$*. Then* $\lim_{k\to\infty} d(\alpha_{\varphi_k}, \alpha_\varphi) = 0$.

(3) *Let* $\{\alpha_k : Y \to X \mid k \geq 1\}$ *be a sequence of continuous maps such that* $\lim_{k\to\infty} d(\alpha_k, \alpha_\varphi) = 0$ *for a continuous map* $\alpha : Y \to X$*. Then there exists a sequence* $\{\varphi_k : M_n(C(X)) \to M_n(C(Y)) \mid k \geq 1\}$ *of* $*$*-homomorphisms such that* $\lim_{k\to\infty} \varphi_k = \varphi$*.*

Two $*$-homomorphisms $\varphi, \psi : A \to B$ of C^*-algebras are said to be *homotopic*, written as $\varphi \simeq \psi$, if there exists a family $(\Phi_t : A \to B)_{0\leq t\leq 1}$ of $*$-homomorphisms such that $\Phi_0 = \varphi, \Phi_1 = \psi$, and for each $a \in A$, the map $e_a : [0,1] \to B$ defined by $e_a(t) = \Phi_t(a), t \in [0,1]$, is continuous.

Lemma 2.9. *Let* X, Y *be compact metrizable spaces.*

(1) *Let* $\varphi, \psi : M_n(C(X)) \to M_n(C(Y))$ *be two* $*$*-homomorphisms. If* φ *and* ψ *are homotopic, then* $\alpha_\varphi \simeq \alpha_\psi : Y \to X$*.*

(2) *Let* $(\alpha_t : Y \to X)_{0\leq t\leq 1}$ *and* $(\bar{u}_t : Y \to U(n)/Z(U(n)))_{0\leq t\leq 1}$ *be homotopies. Let* $\varphi_t = \mathrm{Ad}(\bar{u}_t) \circ \alpha_t^\sharp : M_n(C(X)) \to M_n(C(Y))$ *for* $t \in [0,1]$*. Then the family* $(\varphi_t : M_n(C(X)) \to M_n(C(Y)))_{0\leq t\leq 1}$ *is a homotopy.*

By Künneth formula [4], we have

$$K_1(M_n(C(\mathbb{T}))) \cong K_1(C(\mathbb{T})) \otimes K_0(M_n(\mathbb{C})) \cong \mathbb{Z}.$$

Using the above isomorphism and the continuity of K groups, one can show the following lemmas. Details are again omitted ([12]).

Lemma 2.10. (1) *Let* $\varphi, \psi : M_n(C(\mathbb{T})) \to M_n(C(\mathbb{T}))$ *be two* $*$*-homomorphisms. If* $K_1(\varphi) = K_1(\psi)$*, then* $\alpha_\varphi \simeq \alpha_\psi : \mathbb{T} \to \mathbb{T}$*.*

(2) *For each homomorphism* $F : K_1(M_n(C(\mathbb{T}))) \to K_1(M_n(C(\mathbb{T})))$*, there exists a map* $\tau : \mathbb{T} \to \mathbb{T}$ *such that* $F = K_1(\tau^\sharp)$*.*

Lemma 2.11. *Let* $X = \varprojlim(X_i, p_{ij} : X_j \to X_i)$ *be the limit of an inverse sequence* $(X_i, p_{ij} : X_j \to X_i)$*, where* $X_i = \mathbb{T}$ *for each* i*. Let* $d_i = \deg\, p_{i\ i+1}$*. Then we have an isomorphism*

$$K_1(M_n(C(X))) \cong \{\frac{k}{d_1 \cdots d_r} \mid r \geq 1, k \in \mathbb{Z}\}.$$

3 Proof outline of main theorems

(1) Proof of Theorem 1.1.

The implications of $(a) \Leftrightarrow (b) \Rightarrow (c)$ of Theorem 1.1 are consequences of Theorem 2.5 and the next two proposition whose proofs are omitted (see [12]). For a $*$-homomorphism $\varphi : M_n(C(X)) \to M_n(C(Y))$, we keep the notation (2.3).

Proposition 3.1. *Let X, Y be compact metrizable spaces and $n \geq 1$. Let $\varphi : M_n(C(X)) \to M_n(C(Y))$ be a $*$-homomorphism. The following conditions are equivalent.*

- *(a) There exists a sequence $\{\varphi_k : M_n(C(X)) \to M_n(C(Y)) \mid k \geq 1\}$ of $*$-isomorphisms such that $\lim_{k\to\infty} \varphi_k = \varphi$.*
- *(b) $\alpha_\varphi : Y \to X$ is a near-homeomorphism.*

Proposition 3.2. *Let $\varphi = \mathrm{Ad}(\bar{u}) \circ \alpha^\sharp : M_n(C(X)) \to M_n(C(Y))$ be a $*$-homomorphism, where X and Y are compact metrizable spaces and $\alpha : Y \to X$ and $\bar{u} : Y \to U(n)/Z(U(n))$ are continuous maps. Then the map α is a shape equivalence if and only if φ is a shape equivalence in the sense of [3].*

The implication $(c) \Rightarrow (d)$ of Theorem 1.1 follows from [3]. In order to complete the proof of Theorem 1.1, it thus remains to prove the implication (d)⇒(b).

Proof of (d) ⇒ (b) of Theorem 1.1.

Let $X = \varprojlim(X_i, p_{ij} : X_j \to X_i)$ and $Y = \varprojlim(Y_i, q_{ij} : Y_j \to Y_i)$, where $X_i = Y_i = \mathbb{T}$ for each i and assume that (X_i, p_{ij}) and (Y_i, q_{ij}) have Property $(*)$. Let $\varphi = \mathrm{Ad}(\bar{u}) \circ \alpha^\sharp$ and assume that $K_1(\varphi) : K_1(M_n(C(X))) \to K_1(M_n(C(Y)))$ is an isomorphism. Take subsequences $\{m_i\}, \{n_i\}$ of positive integers and a sequence $\{\varphi_i : M_n(C(X_{m_i})) \to M_n(C(Y_{n_i})) \mid i \geq 1\}$ of $*$-homomorphisms such that

$$\begin{aligned} q^\sharp_{n_i n_{i+1}} \circ \varphi_i &\simeq \varphi_{i+1} \circ p^\sharp_{m_i m_{i+1}}, \\ \varphi \circ p_{m_i\infty} &\simeq q_{n_i\infty} \circ \varphi_i \end{aligned}$$

for each $i \geq 1$. For notational simplicity, let $X_i := X_{m_i}$ and $Y_i = Y_{n_i}$. Since $K_1(\varphi)$ is an isomorphism, by taking a subsequence if necessary, we may assume that there exist sequences $\{i_k\}, \{j_k\}$ of positive integers and a sequence

$\{\eta_k : K_1(M_n(C(Y_{j_k}))) \to K_1(C(X_{i_{k+1}})) \mid k \geq 1\}$ of $*$-homomorphisms which form the commutative diagram below:

$$\begin{array}{ccc} K_1(M_n(C(X_{i_k})) & \xrightarrow{p^\sharp_{i_k i_{k+1}}} & K_1(M_n(C(X_{i_{k+1}})) \\ {\scriptstyle K_1(\varphi_k)}\downarrow & \nearrow^{\eta_k} & \downarrow{\scriptstyle K(\varphi_{k+1})} \\ K_1(M_n(C(Y_{j_k})) & \xrightarrow[q^\sharp_{j_k j_{k+1}}]{} & K_1(M_n(C(Y_{j_{k+1}})) \end{array}$$

By Lemma 2.10 (2), there exists a continuous map $\beta_k : X_{m_{k+1}} \to Y_{n_k}$ such that $K_1(\beta_k^\sharp) = \eta_k$. Then we have

$$K_1(\varphi_{k+1} \circ \beta_k^\sharp) = K_1(q^\sharp_{n_k n_{k+1}}),\ K_1(\beta_k^\sharp \circ \varphi_k) = K_1(p^\sharp_{m_k m_{k+1}}).$$

By Lemma 2.10 (1), we see

$$\beta_k \circ \alpha_{\varphi_{k+1}} \simeq q_{n_k n_{k+1}}, \quad \alpha_{\varphi_{j_k}} \circ \beta_k \simeq p_{m_k m_{k+1}}.$$

Thus α is a shape equivalence. Theorem 2.5 implies that α is a near-homeomorpism.

This completes the proof of Theorem 1.1.

$\square$

(2) Proof of Theorem 1.2. Let $X = \varprojlim(X_i, p_{ij} : X_j \to X_i)$ and $Y = \varprojlim(Y_i, q_{ij} : Y_j \to Y_i)$, where each of X_i and Y_i is homeomorphic to $\mathbb{T}$ and the sequences (X_i, p_{ij}) and (Y_i, q_{ij}) have Property $(*)$. Let $d_i = \deg p_{i\ i+1}$. By Lemma 2.11, we have an isomorphism

$$K_1(C(X)) \cong K_1(C(Y)) \cong \{\frac{k}{d_1 \cdots d_r} \mid r \geq 1, k \in \mathbb{Z}\}.$$

Let D be the right-most group of the above. It follows from the above isomorphism that every homomorphism $F : K_1(M_n(C(X))) \to K_1(M_n(C(Y)))$ is uniquely determined by the element $F(1)$. Identify the group $K_1(M_n(C(X_1)))$ with $K_1(M_n(C(\mathbb{T})))$ that is generated by the element $[\iota]_1$ of Lemma 2.10. Observe then that $K_1(p^\sharp_{1\infty})([\iota]_1) = 1 \in D$ under the identification of $K_1(C(X))$ with D. Take an integer j and an element $a \in K_1(M_n(C(Y_j)))$ such that $K_1(q^\sharp_{j\infty})(a) = F(1)$. This defines a homomorphism $F_1 : K_1(M_n(C(X_1))) \to$

$K_1(M_n(C(Y_j)))$ such that $F \circ K_1(p^\sharp_{1\infty}) = K_1(q^*_{j\infty}) \circ F_1$. Since F is an isomorphism, we can find subsequences $\{m_i\}, \{n_i\}$ of positive integers and sequences of $*$-homomorphisms $\{F_i : K_1(M_n(C(X_{m_i}))) \to K_1(M_n(C(Y_{n_i}))) \mid i \geq 1\}$, $\{G_i : K_1(M_n(C(Y_{n_i}))) \to K_1(M_n(C(X_{m_{i+1}}))) \mid i \geq 1\}$ satisfying

$$\begin{aligned} &m_1 = 1,\ n_1 = j, \\ &G_i \circ F_i = K_1(p^\sharp_{m_i m_{i+1}}), \quad F_{i+1} \circ G_i = K_1(q^\sharp_{n_i n_{i+1}}) \ \text{ for each } i \geq 1. \end{aligned}$$

Apply Lemma 2.10 (2) to obtain a map $\alpha_1 : Y_{n_1} \to X_{m_1}$ such that $K_1(\alpha_1^\sharp) = F_1$.

By repeating the proof of Theorem 2.5 (b) $\Rightarrow$ (c) (see Section 2), we obtain a homeomorphism $\theta : Y \to X$ such that $p_{1\infty} \circ \theta \simeq \alpha_1 \circ q_{j\infty}$. Then we have $K_1(\theta^\sharp) \circ K_1(p^\sharp_{1\infty}) = K_1(q^\sharp_{j\infty}) \circ K_1(\alpha^\sharp_1) = K_1(q^\sharp_{j\infty}) \circ F_1$. Then we see

$$\begin{aligned} K_1(\theta^\sharp)(1) &= K_1(\theta^\sharp) \circ K_1(p^\sharp_{1\infty})([\iota]_1) = K_1(q^\sharp_{j\infty}) \circ F_1([\iota]_1) \\ &= F \circ K_1(p^\sharp_{1\infty})([\iota]_1) = F(1), \end{aligned}$$

and thus $K_1(\theta^\sharp) = F$.

□

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