Convergence theorems for families of monotone Lipschitzian mappings in ordered Banach spaces

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Abstract

In this paper, we prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order. We also show weak convergence theorems for a finite family of monotone nonexpansive mappings.

1 Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of T, i.e.,

$$F(T) = \{z \in C : Tz = z\}.$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. The fixed point theory for such mappings is rich and varied. It finds many applications in nonlinear functional analysis. The existence of fixed points for nonexpansive mappings in Banach and metric spaces has been investigated since the early 1960s (For example, see [7, 8, 9, 11, 15]). Among other things, in 1975, Baillon [5] proved the following first nonlinear mean convergence theorem in a Hilbert space: Let C be a nonempty bounded

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closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$,

$$\{S_n x\} = \left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\}$$

converges weakly to a fixed point of T (see also [22]).

In recent years, a new direction has been very active essentially after the publication of Ran and Reurings results [19]. They proved an analogue of the classical Banach contraction principle [6] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [25, 26, 18, 14]). Bin Dehaish and Khamsi [13] proved a weak convergence theorem of Mann's type [17] for monotone nonexpansive mappings in Banach spaces endowed with a partial order (see also [17, 20]). Shukla and Wiśnicki [21] obtained a nonlinear mean convergence theorem for monotone nonexpansive mappings in such Banach spaces.

Takahashi and Tamura [24] proved some weak convergence theorems for a pair of nonexpansive mappings in a Banach space by using the iteration scheme considered by Das and Debata [12]. Their iteration scheme is as follows. Given any initial data x_1. Consider

$$X_{n+l} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$$

for all $n \geq 1$, where α_n, β_n are sequences in $[0,1)$. Takahashi and Shimoji [23] introduced an iteration scheme, given by finite nonexpansive mappings, which generalizes Das and Debata' s scheme [], and then proved weak convergence theorems for a finite commuting family of nonexpansive mappings in a Banach space.

In this paper, we prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order. We also show weak convergence theorems for finite monotone nonexpansive mappings.

2 Preliminaries and notations

Throughout this paper, we assume that

$$x \preceq y \text{ implies } x+z \preceq y+z,$$

$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x, \rightarrow] = \{z \in E : x \preceq z\}$ and $[\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x, \rightarrow]$ and $[\leftarrow, y]$ are closed. Recall that an order interval is any of the subsets

$$[a, \rightarrow] = \{x \in X; a \preceq x\} \quad \text{or} \quad [\leftarrow, a] = \{x \in X; x \preceq a\}.$$

for any $a \in E$. As a direct consequence of this, the subset

$$[a,b] = \{x \in X; a \preceq x \preceq b\} = [a, \rightarrow] \cap [\leftarrow, b]$$

is also closed and convex for each $a, b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E. Let C be a nonempty subset of E. A mapping $T : C \rightarrow C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \rightarrow C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x, y \in C$ such that $x \preceq y$. For a mapping $T : C \rightarrow C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. We write $x_n \rightarrow x$ (or $\lim\limits_{n\rightarrow\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\rightarrow\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x + y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x + y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Let $S_E = \{x \in E : \|x\| = 1\}$ be a unit sphere in a Banach space E.

3 Monotone and approximate fixed point sequences

In this section, we study approximate fixed point sequences and monotone sequences. Let C be a nonempty subset of E and let T be a mapping of C into E. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping T of C into itself if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(see also [16, 22]). A sequence $\{x_n\}$ in E is said to be *monotone* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [13]). The following result is crucial in this paper.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself. Let $\{x_n\}$ be a sequence in C which is a monotone, and approximate fixed point sequence of T and S, i.e.,*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Then, then the sequence $\{x_n\}$ converges weakly to a point of $F(S)\cap F(T)$.

4 Nonlinear mean convergence theorems

In this section, we show nonlinear mean convergence theorems for monotone nonexpansive mappings. The following lemma plays an important role in our results.

Lemma 4.1 ([1])**.** *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S)\cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Let $x \in C$. For each $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$, let*

$$U_n^{(m)}x = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^{k+m}T^{l+m}x.$$

Then, the sequence $\{U_n^{(m)}x\}_{n=1}^{\infty}$ in C is an approximate fixed point sequence of S and T uniformly in $m \in \mathbb{Z}^+$, i.e.,

$$\lim_{n\to\infty} \sup_{m\in\mathbb{Z}^+} \|U_n^{(m)}x - TU_n^{(m)}x\| = \lim_{n\to\infty} \sup_{m\in\mathbb{Z}^+} \|U_n^{(m)}x - SU_n^{(m)}x\| = 0.$$

Lemma 4.2 ([1]). *Let C be a nonempty closed convex subset of an ordered Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Let $x \in C$. For each $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$, let*

$$U_n^{(m)}x = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^{k+m}T^{l+m}x.$$

Then, for each $m \in \mathbb{Z}^+$, the sequence $\{U_n^{(m)}x\}_{n=1}^{\infty}$ in C is monotone.

We can prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings.

Theorem 4.3 ([1]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. For each $n \in \mathbb{N}$, let*

$$U_n x = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^k T^l x.$$

Then, $\{U_n x\}_{n=1}^{\infty}$ converges weakly to a point of $F(S) \cap F(T)$.

Using Theorem 4.3, we get some convergence theorems for monotone nonexpansive mappings in ordered uniformly convex Banach spaces (see [21]).

Theorem 4.4 ([21]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{S_n x\} = \{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\}$ converges weakly to a point of $F(T)$.*

Theorem 4.5 ([21]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{T^n x\}$ converges weakly a point of $F(T)$.*

5 Weak convergence theorems for finite mappings

In this section, we show weak convergence theorems for finite monotone nonexpansive mappings (see [2]). Let C be a nonempty convex subset of a Banach space E. Let $T_I, T_2, \ldots, T_r$ be finite mappings of C into itself and let $\alpha_1 \ldots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i = 1, 2, \ldots r$. We define a mapping W of C into itself as follows:

$$\begin{aligned} U_1 &= \alpha_1 T_1 + (1-\alpha_1)I, \\ U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)l, \\ &\vdots \\ U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I \\ W = U_r &= \alpha_r T_r U_{r-1} + (1-\alpha_r)I. \end{aligned}$$

Such a mapping W is called the W-*mapping* generated by $T_I, T_2, \ldots, T_r$ and $\alpha_1 \ldots, \alpha_r$ (see [23, 3]).

The following lemma plays an important role in our results.

Lemma 5.1 ([2]). *Let E be a ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_I, T_2, \ldots, T_r$ be finite nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{i=r} F(T_i)$ is nonempty. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r}$ be real numbers sch that $a \leq \alpha_{n,i} \leq b$ for every $i = 1, 2, \ldots, r$ and $n = 1, 2, \ldots$, and let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_I, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$,*

$$\begin{aligned} U_{n,1} &= \alpha_{n,1} T_1 + (1-\alpha_{n,1})I, \\ U_{n,2} &= \alpha_{n,2} T_2 U_{n,1} + (1-\alpha_{n,2})l, \\ &\vdots \\ U_{n,r-1} &= \alpha_{n,r-1} T_{r-1} U_{n,r-2} + (1-\alpha_{n,r-1})I \\ W = U_{n,r} &= \alpha_{n,r} T_r U_{n,r-1} + (1-\alpha_{n,r})I \end{aligned}$$

Suppose $x_1 \in C$, and x_n is given by

$$x_{n+1} = W_n x_n$$

for each $n = 1, 2, \ldots$. Then, the sequence $\{x_n\}_{n=1}^{\infty}$ in C is an approximate fixed point sequence of T_k for each $k = 1, 2, \ldots, r$, i.e.,

$$\lim_{n \to \infty} \|x_n - T_k x_n\| = 0$$

for each $k = 1, 2, \ldots, r$.

Theorem 5.2 ([2]). *Let E be a ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_I, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{i=r} F(T_i)$ is nonempty. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r}$ be real numbers such that $a \leq \alpha_{n,i} \leq b$ for every $i = 1, 2, ..., r$ and $n = 1, 2, \ldots$, and for each $n = 1, 2, \ldots$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by*

$T_I, T_2, \ldots, T_r$ *and* $\alpha_{n,1}, \ldots, \alpha_{n,r}$,

$$\begin{aligned} U_{n,1} &= \alpha_{n,1}T_1 + (1-\alpha_{n,1})I, \\ U_{n,2} &= \alpha_{n,2}T_2U_{n,1} + (1-\alpha_{n,2})l, \\ &\vdots \\ U_{n,r-1} &= \alpha_{n,r-1}T_{r-1}U_{n,r-2} + (1-\alpha_{n,r-1})I \\ W = U_{n,r} &= \alpha_{n,r}T_rU_{n,r-1} + (1-\alpha_{n,r})I. \end{aligned}$$

Suppose $x_1 \in C$, *and* x_n *is given by*

$$x_{n+1} = W_n x_n$$

for each $n = 1, 2, \ldots$. *Then, the sequence* $\{x_n\}_{n=1}^{\infty}$ *in* C *converges weakly to a common fixed point of* $T_I, T_2, \ldots, T_r$.

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References

[1] S. Atsusiba, *A nonlinear mean convergence theorem for two monotone nonexpansive mappings in Banach space*, Linear and Nonlinear Anal., **8** (2022), 155-162.

[2] S. Atsusiba, *Weak convergence theorem for a finite family of monotone nonexpansive mappings in Banach spaces*, submitted.

[3] S. Atsusiba and W. Takahashi, *Strong convergence theorems for a finite family of non-expansive mappings and applications*, Indian J. Math. **41** (1999), 435-453.

[4] S. Atsusiba and W. Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domains*, Math. Japon. **52** (2000), no. 2, 183–195.

[5] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sei. Paris Ser. A-B, **280** (1975), 1511-1514.

[6] S.Banach, *Sur les opérations dans les ensembles abstraits et leurs applications*, Fundam. Math. **3** (1922), 133-181.

[7] L. P. Belluce and W. A. Kirk, *Fixed-point theorems for families of contraction mappings*, Pac. J. Math. **18** (1966), 213-217.

[8] L. P. Belluce and W. A. Kirk, *Nonexpansive mappings and fixed-points in Banach spaces*, Illinois Journal of Mathematics, **11** (1967), 474-479.

[9] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Natl. Acad. Sci. USA **54** (1965), 1041-1044.

[10] F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces. Nonlinear functional analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968), Amer. Math. Soc., Providence, R.I., 1976.

[11] R. E. Bruck, *Properties of fixed point sets of non-expansive mappings in Banach spaces*, Trans. Am. Math. Soc. **179** (1973), 251-262.

[12] G. Das and J.P. Debata, *Fixed points of quasi-nonexpansive mappings*, Indian J. Pure Appl. Math. **17** (1986), 1263-1269.

[13] B.A. Bin Dehaish and M.A. Khamsi, *Mann iteration process for monotone nonexpansive mappings*, Fixed Point Theory Appl. **2015** (2015), 177, 7pp.

[14] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc. **136** (2007), 1359-1373.

[15] T.C. Lim, *A fixed point theorem for families of nonexpansive mappings*, Pac. J. Math. **53** (1974), 487-493.

[16] E. Matoušková and S. Reich, *Reflexivity and approximate fixed points*, Studia Math. **159** (2003), 403–415.

[17] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[18] J.J. Nieto, Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223-239.

[19] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435-1443.

[20] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[21] R. Shukla and A. Wiśnicki, *Iterative methods for monotone nonexpansive mappings in uniformly convex Banach spaces*, Adv. Nonlinear Anal. **10** (2021), 1061–1070

[22] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[23] W. Takahashi and K. Shimoji, *Convergence Theorems for Nonexpansive Mappings and Feasibility Problems*, Mathematical and Computer Modelling **32** (2000), 1463-1471.

[24] W. Takahashi and T. Tamura, *Limit theorems of operators by convex combinations for nonexpansive retractions in Banach spaces*, J. Approximation Theory **91** (1997), 386-397.

[25] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl. **117** (1986), 100-127.

[26] M. Turinici, *Fixed points for monotone iteratively local contractions*, Demonstr. Math. **19** (1986), 171-180.