Cardinal Collapsing and Product Forcing

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Abstract

Suppose κ is a singular strong limit cardinal of countable cofinality, and let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of regular cardinals cofinal in κ. In this short note, we show that if $\mathsf{cof}(2^\kappa) = \kappa^+$, then forcing with the full product $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ collapses 2^κ onto κ^+. This result gives a consistent positive answer to a question asked by Sy Friedman. We also provide a new proof of a result due to Shelah by showing that if, moreover, the sequence carries a scale of length κ^+, then forcing with $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ adds a generic filter for $\mathsf{Add}(\kappa^+, 1)$, and thus

$$\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)/\mathsf{fin} \simeq \mathsf{Add}(\kappa^+, 1).$$

Suppose $\langle \kappa_n : n < \omega \rangle$ is an increasing sequence of regular cardinals cofinal in κ. In [3], Sy Friedman and Radek Honzik observed that if $\prod_{n<\omega} \kappa_n$ carries a scale of length κ^+, then $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ collapses 2^κ onto κ^+. On the other hand, answering a question of Friedman and Rene David, Saharon Shelah [5] showed that if $\prod_{n<\omega} \kappa_n$ carries a scale of length κ^+, then forcing with $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ adds a generic for $\mathsf{Add}(\kappa^+, 1)$ over V. As the latter forcing collapses 2^κ onto κ^+, Friedman-Honzik's result follows from Shelah's theorem. In proofs of both results, the assumption that $\prod_{n<\omega} \kappa_n$ carries a scale of length κ^+ seems to be essential. In a personal communication, Sy Friedman asked the first author if one can remove the assumption of the existence of a scale from his result with Honzik. More precisely, he asked if the following is true.

Question 1. Suppose $\langle \kappa_n : n < \omega \rangle$ is an increasing sequence of inaccessible cardinals cofinal in κ. Does forcing with $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ collapse 2^κ onto κ^+?

We provide a consistent positive answer to Friedman's question:

Theorem 0.1. Assume κ is a singular strong limit cardinal of countable cofinality and that $\mathsf{cof}(2^\kappa) = \kappa^+$. Let $\langle \kappa_n : n < \omega \rangle$ be any increasing sequence of regular cardinals cofinal in κ and let $\langle \mathbb{P}_n : n < \omega \rangle$ be a sequence of non-trivial separative forcing notions, such that each $\mathbb{P}_n$ is κ_n-closed and of size $<\kappa$. Suppose that every decreasing sequence of $\mathbb{P}_n$-conditions of length $<\kappa_n$ has a greatest lower bound in $\mathbb{P}_n$. Then $\prod_{n<\omega} \mathbb{P}_n$ collapses 2^κ onto κ^+.

We also give a new proof of Shelah's theorem mentioned earlier, indeed we prove the following.

Theorem 0.2. Let κ be a singular strong limit cardinal of countable cofinality. Let $\langle \kappa_n : n < \omega \rangle$ be any increasing sequence of regular cardinals cofinal in κ which carries a scale of length κ^+. Then $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)/\mathsf{fin} \simeq \mathsf{Add}(\kappa^+, 1)$. In particular, forcing with $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ adds a generic filter for $\mathsf{Add}(\kappa^+, 1)$.

Proof of Theorem 0.1

We need two lemmas.

Lemma 0.3 ([2])**.** Assume $\mathsf{cof}(2^\kappa) = \kappa^+$ and that $\mathbb{Q}$ is a $(\kappa + 1)$-strategically closed forcing notion of size 2^κ such that Player II has a winning strategy where at limit stages he chooses the greatest lower bound of the previously chosen sequence. Then forcing with $\mathbb{Q}$ adds a new sequence of ordinals of length κ^+.

$\square$

Lemma 0.4 ([1])**.** Let $\mathbb{Q}$ be a $(\kappa + 1)$-strategically closed forcing notion of size 2^κ. Let $o(\mathbb{Q})$ be the least cardinal μ, such that forcing with $\mathbb{Q}$ adds a new μ-sequence of ordinals (or equivalently of elements of V). Then forcing with $\mathbb{Q}$ collapses 2^κ onto $o(\mathbb{Q})$.

$\square$

Remark 0.5. In [1], the lemma is not stated as above, but the proof and remarks after it shows that the above stronger result holds.

Now let $\mathbb{P} \coloneqq \prod_{n<\omega} \mathbb{P}_n$ and $\mathbb{Q} \coloneqq \prod_{n<\omega} \mathbb{P}_n/\mathsf{fin}$. Define $\pi : \mathbb{P} \to \mathbb{Q}$ by

$$\pi(\langle p_n : n < \omega \rangle) = [\langle p_n : n < \omega \rangle]/\mathsf{fin},$$

where $[\langle p_n : n < \omega \rangle]/\mathsf{fin}$ denotes the equivalence class of $\langle p_n : n < \omega \rangle$ in $\mathbb{Q}$.

Lemma 0.6. π is a projection, i.e.,

1. π is order-preserving and $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$.
2. If $[p]/\mathsf{fin} \leq_{\mathbb{Q}} [\mathsf{q}]/\mathsf{fin}$, then there exists $r \leq_{\mathbb{P}} q$ such that $[r]/\mathsf{fin} \leq_{\mathbb{Q}} [\mathsf{p}]/\mathsf{fin}$.

Proof. Easy! □

Observe that our forcing $\mathbb{Q}$ is $(\kappa+1)$-strategically closed and there exists a winning strategy for Player II where at limit stages, he chooses the greatest lower bound of the previously chosen sequence. It follows from Lemma 0.3 that forcing with $\mathbb{Q}$ adds a new κ^+-sequence of ordinals. Now Lemma 0.4 implies that $\mathbb{Q}$ collapses 2^κ to κ^+, and by Lemma 0.6, forcing with $\mathbb{P}$ collapses 2^κ to κ^+ as well.

Proof of Theorem 0.2

The proof is given in two stages. At the first stage we show that forcing with $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ collapses 2^κ onto κ^+. In the next stage, we analyse the forcing notion $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)/\mathsf{fin}$ and use our results to conclude the theorem.

Stage 1: We show that forcing with $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ collapses 2^κ onto κ^+. Fix a scale $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ in $\prod_{n<\omega} \kappa_n$. Let

$$\mathcal{F} = \{ f \in \prod_{n<\omega} \kappa_n : f =^* f_\alpha, \text{ for some } \alpha < \kappa^+ \}.$$

Then $|\mathcal{F}| = \kappa^+$, and it is cofinal in $(\prod_{n<\omega} \kappa_n, \leq)$. Let $G_n : \kappa_n \to 2$ be the Cohen generic function, added by $\mathsf{Add}(\kappa_n, 1)$. For each $f \in \mathcal{F}$, define $g_f : \kappa \to 2$, so that for each $n < \omega$,

$$g_f(\kappa_{n-1} + \xi) = G_{n+1}(f(n+1) + \xi),$$

where $\kappa_{n-1} \leq \xi < \kappa_n$ and $\kappa_{-1} = 0$. We demonstrate that for each $g : \kappa \to 2$ in V, there is $f \in \mathcal{F}$ with $g = g_f$. But it is enough to show that the following set is dense in $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$.

$$D_g = \{ p \in \prod_{n<\omega} \mathsf{Add}(\kappa_n, 1) : \exists f \in \mathcal{F}, \forall n, \forall \xi < \kappa_n [g(\xi) = p(n+1)(f(n+1) + \xi)] \}.$$

Then D_g is dense in $\prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$. To see this, let $p \in \prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$. By extending p, we may assume that for each $n < \omega, p(n) : \zeta_n \to 2$, for some $\zeta_n < \kappa_n$. It then follows that

$$\langle \zeta_n : n < \omega \rangle \in \prod_{n<\omega} \kappa_n.$$

Pick $\alpha < \kappa^+$ such that $g <^* f_\alpha$. It follows that $g < f$ for some $f \in \mathcal{F}$. Define the condition $q \in \prod_{n<\omega} \mathsf{Add}(\kappa_n, 1)$ by $q(n) : f(n) \to 2$, $q(n) \supseteq p(n)$ and for all $\xi < \kappa_n$,

$$q(n+1)(f(n+1) + \xi) = g(\xi).$$

Then q is well-defined, extends p, and belongs to D_g. It follows that for some $f \in \mathcal{F}$, $g = g_f$.

Stage 2: We complete the proof of Theorem 0.2. For each $n < \omega$, set $\mathbb{P}_n = \mathsf{Add}(\kappa_n, 1)$. Let $\mathbb{P} = \prod_{n<\omega} \mathbb{P}_n$ and $\mathbb{Q} = \prod_{n<\omega} \mathbb{P}_n/\mathsf{fin}$. Let also $\pi : \mathbb{P} \to \mathbb{Q}$ be defined as before. The next claim can be proved easily.

Claim 0.7. (a) $\mathbb{Q}$ is $<\kappa^+$-strategically closed.

(b) The quotient forcing $\mathbb{P}/\dot{G}_{\mathbb{Q}}$ is κ^+-c.c.

As forcing with $\mathbb{P}$ preserves cardinals $\leq \kappa^+$ and that by Stage 1, it collapses 2^κ onto κ^+. It follows from Claim 0.7(b) that it is the forcing $\mathbb{Q}$ that collapse 2^κ onto κ^+. Now we need the following well-known fact:

Fact 0.8 (see [4])**.** Suppose $\kappa < \lambda$ are infinite cardinals and $\lambda^\kappa = \lambda$. Suppose $\mathbb{Q}$ is a $<\kappa^+$-strategically closed forcing notion of size λ and suppose that forcing with $\mathbb{Q}$ collapses λ onto κ^+. Then $\mathbb{Q} \simeq \mathsf{Col}(\kappa^+, \lambda)$.

By Claim 0.7(a) and Fact 0.8, we have $\mathbb{Q} \simeq \mathsf{Add}(\kappa^+, 1)$, and that by Lemma 0.6, forcing with $\mathbb{P}$ adds a generic for $\mathbb{Q}$, which completes the proof of Theorem 0.2.

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