TWO PRESERVATION THEOREMS OF STRONGLY PROPER FORCING NOTIONS

TERUYUKI YORIOKA

Abstract. It is proved that a strongly proper forcing notion preserves the maximality of $\mathbb{C}$-indestructible mad families and non-meager sets of reals.

1. Introduction

Asperó–Mota and Neeman developed forcing iteration theory by use of Todorčević's side condition method (e.g. [1, 8]). The side condition method consists of systems of models of some $H(\kappa)$, which is the set of sets of hereditary cardinality less than κ (e.g. [10, 11]). A basic side condition method is Todorčević's $\in$-collapse, which consists of finite chains of countable elementary submodels of $H(\kappa)$ for some fixed regular cardinal κ. Since the $\in$-collapse adds a Cohen real, for example, Asperó–Mota iteration may not force that $\mathrm{cov}(\mathcal{M}) = \aleph_1 < 2^{\aleph_0}$. And, since the $\in$-collapse preserves the countable chain condition of Suslin trees, it is possible that some Asperó–Mota iterations and Neeman iterations force some weak forcing axioms and the negation of Suslin Hypothesis simultaneously.

The $\in$-collapse has the strong properness, defined by Shelah. In this article, we prove two preservation theorems of strongly proper forcing notions. One is on the almost disjointness number $\mathfrak{a}$ and the other is on the uniformity $\mathrm{non}(\mathcal{M})$ of the meager ideal. So it is consistent relative to the existence of a supercompact cardinal that $\mathfrak{a} = \mathrm{non}(\mathcal{M}) = \aleph_1$ and the forcing axiom for strongly proper forcing notions holds. And, this suggests a possibility of Asperó–Mota iterations and Neeman iterations which force $\mathfrak{a} = \aleph_1$ and $\mathrm{non}(\mathcal{M}) = \aleph_1$ with some weak forcing axioms.

In §2, we introduce Shelah's strong properness and its examples, and demonstrate the proofs of some basic preservation theorems of strongly proper forcing notions. In §3, we prove a preservation theorem of strongly proper forcing notions about the almost disjointness number, and in §4, we prove a preservation theorem of strongly proper forcing notions about the uniformity of the meager ideal.

2. Strongly proper forcing notions

Definition 2.1 (Shelah, [9, Ch. IX, 2.6 Definition])**.** A forcing notion $\mathbb{P}$ is called strongly proper if, for any sufficiently large regular cardinal θ, any countable elementary submodel N of $H(\theta)$ with $\mathbb{P} \in N$, any countable sequence $\langle D_n; n \in \omega \rangle$ with $D_n \subseteq \mathbb{P} \cap N$ dense in $\mathbb{P} \cap N$ and any $p \in \mathbb{P} \cap N$, there exists $q \leq_{\mathbb{P}} p$ such that for all $n \in \omega$, D_n is predense below q in $\mathbb{P}$.

A strongly proper forcing notion is proper. The typical example of strongly proper forcing notion is Cohen forcing. The other one is the following.

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Definition 2.2 (Todorčević, the $\in$-collapse, e.g. [12, Ch.7])**.** Let κ be an uncountable regular cardinal. The $\in$-collapse (for the cardinal κ) $\mathbb{P}_\kappa$ consists of finite $\in$-chains of countable elementary submodels of $H(\kappa)$, ordered by the superset relation.

Proposition 2.3. *The $\in$-collapse $\mathbb{P}_\kappa$ is strongly proper.*

Proof. Let θ be a large enough regular cardinal for $\mathbb{P}_\kappa$, N a countable elementary submodel of $H(\theta)$ with $\{\mathbb{P}_\kappa, H(\kappa)\} \in N$, and $p \in \mathbb{P}_\kappa \cap N$. Define $p^+ := p \cup \{N \cap H(\kappa)\}$. Then $p^+ \in \mathbb{P}_\kappa$ and $p^+ \supseteq p$, hence $p^+ \leq_{\mathbb{P}_\kappa} p$. Let us show that p^+ is strong $(N, \mathbb{P}_\kappa)$-generic in the sense of Mitchell [6, Definition 2.3], that is, for any dense subset D of $\mathbb{P}_\kappa \cap N$ in $\mathbb{P}_\kappa \cap N$, D is predense below p^+ in $\mathbb{P}_\kappa$, which suffices to finish the proof.

Let $q \leq_{\mathbb{P}_\kappa} p^+$. Then $q \cap N$ is in $\mathbb{P}_\kappa \cap N$, so there exists $r \in D$ such that $r \leq_{\mathbb{P}_\kappa} q \cap N$. Then $r \cup q$ is also in $\mathbb{P}_\kappa$ and an extension of r and q in $\mathbb{P}_\kappa$. ☐

The $\in$-collapse collapses κ to $\aleph_1$ and is Chodounský–Zapletal's Y-proper [3]. The $\in$-collapse has an $\aleph_2$-pic variation which does not collapse any cardinals over the Continuum Hypothesis (e.g. [11, §4]). This variation is also strongly proper. Strong properness is closed under countable support iterations [9, Ch. IX, 2.7A Remark]. So, if there exists a supercompact cardinal, there exists a strongly proper forcing notion which forces the forcing axiom for strongly proper forcing notions.

Remark 2.4. Sacks forcing and Silver forcing are strongly proper (see e.g. [13, Lemma 4.1.6, Corollary 4.1.9]).

In the rest of this section, we demonstrate three preservation results of strongly proper forcing notions.

Proposition 2.5. *A strongly proper forcing notion preserves the Aronszajn-ness of an ω_1-tree.*

Proof. Let $\mathbb{P}$ be a strongly proper forcing notion and T an (ω_1-)Aronszajn tree. For $\gamma \in \omega_1$, we denote by T_γ a set of all elements lying in the γ-th level of T, and define $T_{<\gamma} := \bigcup_{\alpha<\gamma} T_\alpha$. Assume that there are $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{X}$ such that

$$p \Vdash_{\mathbb{P}} \text{“ } \dot{X} \subseteq T \text{ is an uncountable chain ”}.$$

Let M be a countable elementary submodel of $H(\theta)$ such that $\left\{\mathbb{P}, T, p, \dot{X}\right\} \in M$, and let $\delta := \omega_1 \cap M$. For each $t \in T_\delta$, define

$$D_t := \left\{q \in \mathbb{P} \cap M : \exists s \in T_{<\delta}(s \not<_T t \ \& \ q \Vdash_{\mathbb{P}} \text{“ } s \in \dot{X} \text{ ”})\right\}.$$

We claim that each D_t is dense in $\mathbb{P} \cap M$. Given $r \in \mathbb{P} \cap M$, let $Y = \left\{s \in T : r \nVdash_{\mathbb{P}} \text{“ } s \notin \dot{X} \text{ ”}\right\}$, which is in M. If $Y \cap M = \{s \in T_{<\delta} : s <_T t\}$, then

$$M \models \text{“ } Y \text{ is an uncountable branch of } T \text{ ”},$$

which contradicts to the Aronszajn-ness of T, hence $Y \cap M \neq \{s \in T_{<\delta} : s <_T t\}$. Note that, by the elementarity of M,

$$M \models \text{“ } \forall \xi < \omega_1 \exists \eta \geq \xi (Y \cap T_\eta \neq \emptyset) \text{ ”}.$$

Thus we can find $s \in Y \cap M$ so that $s \not<_T t$. Since

$$M \models \text{“ } s \in Y \text{ ”},$$

there is $q \in \mathbb{P} \cap M$ such that $q \leq_{\mathbb{P}} r$ and $q \Vdash_{\mathbb{P}}$" $s \in \dot{X}$ ". Since

$$M \models \text{“ } T = \bigcup_{\xi<\omega_1} T_\xi \text{ ”},$$

$T \cap M = \bigcup_{\xi<\delta} T_\xi \cap M = T_{<\delta}$, so $s \in T_{<\delta}$.

Since T_δ is countable by the ω_1-tree-ness of T, there exists $q \leq_{\mathbb{P}} p$ such that, for every $t \in T_\delta$, D_t is predense below q. Then

$$q \Vdash_{\mathbb{P}} \text{“ } \forall t \in T_\delta\ \exists s \in T_{<\delta} \cap \dot{X}(s \not<_T t) \text{ ”},$$

therefore

$$q \Vdash_{\mathbb{P}} \text{“ } \dot{X} \subseteq T_{<\delta} \text{ ”},$$

which is a contradiction. □

Proposition 2.6. *A strongly proper forcing notion preserves the gap-ness of a pregap in $\mathcal{P}(\omega)/\mathrm{fin}$.*

Proof. Recall the notions of (κ, λ)-pregaps and (κ, λ)-gaps in $\mathcal{P}(\omega)/\mathrm{fin}$. $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta : \alpha \in \kappa, \beta \in \lambda \rangle$ is called a (κ, λ)-pregap in $\mathcal{P}(\omega)/\mathrm{fin}$ if

- for any $\alpha \in \kappa$ and any $\beta \in \lambda$, a_α and b_β are infinite subsets of ω,
- for any $\alpha, \beta \in \kappa$, if $\alpha < \beta$, then $a_\alpha \subseteq^* a_\beta$, which means that $a_\alpha \setminus a_\beta$ is finite,
- for any $\alpha, \beta \in \lambda$, if $\alpha < \beta$, then $b_\alpha \subseteq^* b_\beta$, and
- for any $\alpha \in \kappa$ and any $\beta \in \lambda$, $a_\alpha \perp b_\beta$, which means that, $a_\alpha \cap b_\beta$ is finite.

An infinite subset c of ω separates a (κ, λ)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta : \alpha \in \kappa, \beta \in \lambda \rangle$ if, for any $\alpha \in \kappa$ and any $\beta \in \lambda$, $a_\alpha \subseteq^* c$ and $b_\beta \perp c$. A (κ, λ)-pregap $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta : \alpha \in \kappa, \beta \in \lambda \rangle$ is called a (κ, λ)-gap if there are no infinite subsets of ω which separate $(\mathcal{A}, \mathcal{B})$.

Let $\mathbb{P}$ be a strongly proper forcing notion and $(\mathcal{A}, \mathcal{B}) = \langle a_\alpha, b_\beta : \alpha \in \kappa, \beta \in \lambda \rangle$ a (κ, λ)-gap. Without loss, assume that κ is an uncountable regular cardinal. Let $p \in \mathbb{P}$ and $\dot{x}$ a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \text{“ } \forall b \in \mathcal{B}(b \perp \dot{x}) \text{ ”}.$$

Let us show that $p \not\Vdash_{\mathbb{P}}$" $\dot{x}$ separates $(\mathcal{A}, \mathcal{B})$ ".

Let M be a countable elementary submodel of $H(\theta)$ such that $\{\mathbb{P}, (\mathcal{A}, \mathcal{B}), p, \dot{x}, \kappa\} \in M$, and let $\delta := \sup(M \cap \kappa)$. Since κ is of uncountable cofinality and M is countable, $\delta < \kappa$. For each $n \in \omega$, define

$$D_n := \{q \in \mathbb{P} \cap M : \exists m \in a_\delta \setminus n(q \Vdash_{\mathbb{P}} \text{“ } m \notin \dot{x} \text{ ”})\}.$$

We claim that each D_n is dense in $\mathbb{P} \cap M$. Given $r \in \mathbb{P} \cap M$, let $c := \{k \in \omega : r \Vdash_{\mathbb{P}} \text{“ } k \in \dot{x} \text{ ”}\}$. (Note that $p \Vdash_{\mathbb{P}}$ " $c \subseteq \dot{x}$ ".) Then $c \in M$ and for all $b \in \mathcal{B}$, $b \perp c$. If $a_\delta \setminus n \subseteq c$, then

$$M \models \text{“ } c \text{ separates } (\mathcal{A}, \mathcal{B}) \text{ ”},$$

which is a contradiction because of the elementarity of M. Hence there exists $m \in a_\delta \setminus (n \cup c)$. Since

$$M \models \text{“ } m \notin c \text{ ”},$$

we can find $q \in \mathbb{P} \cap M$ such that $q \leq_{\mathbb{P}} r$ and $q \Vdash_{\mathbb{P}}$" $m \notin \dot{x}$ ".

By the strong properness of $\mathbb{P}$, there is $q \leq_{\mathbb{P}} p$ such that all D_n are predense below q in $\mathbb{P}$. Then

$$q \Vdash_{\mathbb{P}} \text{“ } \forall n \in \omega(a_\delta \setminus n \not\subseteq \dot{x}) \text{ ”},$$

that is $q \Vdash_{\mathbb{P}}$“ $a_\delta \not\subseteq^* \dot{x}$ ”, which finishes the proof. □

By the connection between unbounded families in $(\omega^\omega, <^*)$ and $(\mathfrak{b}, \omega)$-gaps, this proposition implies that strongly proper forcing notions add no dominating reals (c.f. [13, Corollary 4.1.7]).

Theorem 2.7 (Miyamoto, [7]). *A strongly proper forcing notion preserves the countable chain condition of a Suslin tree.*

Proof. Let $\mathbb{P}$ be a strongly proper forcing notion and T a Suslin tree. Assume that $p \in \mathbb{P}$ and $\dot{A}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \text{“ } \dot{A} \text{ is a maximal antichain in } T \text{ ”}.$$

Let M be a countable elementary submodel of $H(\theta)$ such that $\left\{T, \mathbb{P}, p, \dot{A}\right\} \in M$ and $\delta := \omega_1 \cap M$.

For $t \in T_\delta$, define

$$D_t := \left\{q \in \mathbb{P} \cap M : \exists s \in T_{<\delta} \left(s <_T t \ \&\ q \Vdash_{\mathbb{P}} \text{“ } s \in \dot{A} \text{ ”}\right)\right\}.$$

Each D_t may *not* be in M. We claim that each D_t is dense below p in $\mathbb{P} \cap M$. Let $r \in \mathbb{P} \cap M$ be a stronger condition of p in $\mathbb{P}$. Then (inside M) $\left\{s \in T : r \not\Vdash_{\mathbb{P}} \text{“ } s \notin \dot{A} \text{ ”}\right\}$ is predense in T, so we can find a maximal antichain A' in this set. By the elementarity of M, we may assume that $A' \in M$. Since T is a Suslin tree, A' is countable, hence $A' \subseteq M$. Then (outside M) there exists $s \in A'$ compatible with t in T. Since $M \models$“ $s \in A'$ ”, there exists $q \leq_{\mathbb{P}} r$ in M such that

$$q \Vdash_{\mathbb{P}} \text{“ } s \in \dot{A} \text{ ”}.$$

Since $T \cap M = \bigcup_{\alpha<\delta} T_\alpha$, $s \in T_{<\delta}$ and so $s <_T t$ holds, hence $q \in D_t$.

By the strong properness of $\mathbb{P}$, there exists $q \leq_{\mathbb{P}} p$ such that D_t is predense below q for every $t \in T_\delta$. Then

$$q \Vdash_{\mathbb{P}} \text{“ } \forall t \in T_\delta\, \exists s \in \dot{A}(s <_T t) \text{ ”},$$

therefore

$$q \Vdash_{\mathbb{P}} \text{“ } \dot{A} \subseteq T_{<\delta}, \text{ which is countable ”}.$$

□

It is not known whether a strongly proper forcing notion preserves the destructibility of a destructible gap like a Suslin tree.

3. $\mathbb{C}$-indestructible madfamilies

A subset $\mathcal{A}$ of $[\omega]^{\aleph_0}$ is called almost disjoint if any two elements of $\mathcal{A}$ is pairwise disjoint, and an almost disjoint family $\mathcal{A}$ on ω is called a mad family if $\mathcal{A}$ is infinite and is maximal with respect to almost disjoint families, that is, any infinite subset of ω has an infinite intersection with some element of $\mathcal{A}$. For a forcing notion $\mathbb{P}$, a mad family is called $\mathbb{P}$-indestructible if $\mathbb{P}$ forces that $\mathcal{A}$ is still a mad family. A Cohen forcing is denoted by $\mathbb{C}$ in this article.

Theorem 3.1 (Brendle–Yatabe [2, Theorem 2.4.8], Hrušák [4, Theorem 5], Kurilić [5, Theorem 2]). *A mad family $\mathcal{A}$ is $\mathbb{C}$-indestructible iff, for any function f from $\mathbb{C}$ into ω, there exists $a \in \mathcal{A}$ such that $f^{-1}[a]$ is somewhere dense in $\mathbb{C}$.*

Theorem 3.2 (Hrušák, [4, Proposition 6 (2)])**.** *If* $\mathfrak{b} = 2^{\aleph_0}$*, then there exists a* $\mathbb{C}$*-indestructible mad family.*

Theorem 3.3. *A strongly proper forcing notion preserves the maximality of a* $\mathbb{C}$*-indestructible mad family.*

Proof. Let $\mathbb{P}$ be a strongly proper forcing notion, $\mathcal{A}$ a $\mathbb{C}$-indestructible mad family, $p \in \mathbb{P}$, and $\dot{x}$ a $\mathbb{P}$-name for an infinite subset of ω. Let us find $q \leq_{\mathbb{P}} p$ and $a \in \mathcal{A}$ such that

$$q \Vdash_{\mathbb{P}} \text{“ } \dot{x} \cap a \text{ is infinite ”}.$$

Denote $\theta := \left(2^{|\mathbb{P}|}\right)^+$. Take a countable elementary submodel M of $H(\theta)$ such that $\{\mathbb{P}, \mathcal{A}, p, \dot{x}\} \in M$. If there exists $q \leq_{\mathbb{P}} p$ such that

$$b_q := \{k \in \omega : q \Vdash_{\mathbb{P}} \text{“ } k \in \dot{x} \text{ ”}\}$$

is infinite, then the maximality of $\mathcal{A}$ follows the existence of our desired $a \in \mathcal{A}$. So we assume that any extension q of p in $\mathbb{P}$ satisfies that b_q is finite.

For each $r \leq_{\mathbb{P}} p$, define $k_r := \max(b_r \cup \{0\})$. Let C be a subset of $\mathbb{P} \cap M$ which is dense below p in $\mathbb{P} \cap M$. Since C is a dense subset of the countable forcing notion $\mathbb{P} \cap M$, by shrinking C if necessary, we may assume that there exists an order-isomorphism h of (a dense subset of) $\mathbb{C}$ onto C. Define the function f from $\mathbb{C}$ into ω such that, for each $\sigma \in \mathbb{C}$, $f(\sigma) = k_{h(\sigma)}$. Since $\mathcal{A}$ is $\mathbb{C}$-indestructible, there are $a \in \mathcal{A}$ and $\sigma \in \mathbb{C}$ such that $f^{-1}[a]$ is dense below σ in $\mathbb{C}$. Then $h(\sigma) \leq_{\mathbb{P}} p$ and $h(\sigma) \in M$.

Let us show that, for each $n \in \omega$,

$$D_n := \{q \in \mathbb{P} \cap M : n \leq k_q \text{ and } k_q \in a\}$$

is dense below $h(\sigma)$ in $\mathbb{P} \cap M$. To show this, let $n \in \omega$ and $s \in \mathbb{P} \cap M$ be such that $s \leq_{\mathbb{P}} h(\sigma)$. Since $\dot{x}$ is a $\mathbb{P}$-name for an infinite subset of ω and belongs to M, by the elementarity of M, there exists $r \in C$ such that $r \leq_{\mathbb{P}} s$ and $k_r \geq n$. Since $h^{-1}(r) \leq_{\mathbb{C}} \sigma$, there is $\tau \in f^{-1}[a]$ such that $\tau \leq_{\mathbb{C}} h^{-1}(r)$. Then $f(\tau) \in a$, $h(\tau) \leq_{\mathbb{P}} r$, $h(\tau) \leq_{\mathbb{P}} h(\sigma)$, and

$$n \leq k_r \leq k_{h(\tau)} = f(\tau),$$

hence $h(\tau) \in D_n$.

By the strong properness of $\mathbb{P}$, there exists $q \leq_{\mathbb{P}} h(\sigma)$ such that, for all $n \in \omega$, D_n is predense below q in $\mathbb{P}$. Then

$$q \Vdash_{\mathbb{P}} \text{“ } \dot{x} \cap a \text{ is infinite ”}.$$

□

4. Non-meager sets of reals

Theorem 4.1. *A strongly proper forcing notion preserves non-meager sets of reals.*

Proof. Let $\mathbb{P}$ be a forcing notion. For each $\sigma \in \omega^{<\omega}$, denote $[\sigma] := \{f \in \omega^\omega : \sigma \subseteq f\}$. For $r \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{F}$ for a nowhere dense subset of ω^ω, define

$$G(r, \dot{F}) := \left\{ f \in \omega^\omega : \forall k \in \omega, r \Vdash_{\mathbb{P}} \text{“ } \dot{F} \cap [f \restriction k] \neq \emptyset \text{ ”} \right\}.$$

We claim that $G(p, \dot{F})$ is nowhere dense. To show this, the strong properness of $\mathbb{P}$ is not necessary. Let $\sigma \in \omega^{<\omega}$. Then there are $s \leq_{\mathbb{P}} r$ and $\tau \in \omega^{<\omega}$ such that $\sigma \subseteq \tau$

and $s \Vdash_{\mathbb{P}}$" $\dot{F} \cap [\tau] = \emptyset$ ". Then $G(r, \dot{F}) \cap [\tau]$ is empty. Because, if $f \in G(r, \dot{F}) \cap [\tau]$, then there is $k \in \omega$ such that $\tau \subseteq f \restriction k$, and then

$$s \Vdash_{\mathbb{P}} \text{“ } \dot{F} \cap [f \restriction k] \neq \emptyset \text{ and } \dot{F} \cap [\tau] = \emptyset \text{ ”},$$

which contradicts to the fact that $[f \restriction k] \subseteq [\tau]$.

Suppose that $\mathbb{P}$ is strongly proper, X is a non-meager subset of ω^ω, $p \in \mathbb{P}$, and $\left\{\dot{F}_n : n \in \omega\right\}$ is a set of $\mathbb{P}$-names for nowhere dense subsets of ω^ω. Let us show that

$$p \nVdash_{\mathbb{P}} \text{“ } X \subseteq \bigcup_{n \in \omega} \dot{F}_n \text{ ”}.$$

Suppose not. Denote $\theta := \left(2^{|\mathbb{P}|}\right)^+$, and take a countable elementary submodel M of $H(\theta)$ such that $\left\{\mathbb{P}, X, p, \left\{\dot{F}_n : n \in \omega\right\}\right\} \in M$, and take f in the set

$$X \setminus \left(\bigcup_{n \in \omega} \bigcup_{r \in \mathbb{P} \cap M} G(r, \dot{F}_n) \right).$$

For each $n \in \omega$, define

$$D_n := \left\{ s \in \mathbb{P} \cap M : \exists k \in \omega \left(s \Vdash_{\mathbb{P}} \text{“ } \dot{F}_n \cap [f \restriction k] = \emptyset \text{ ”} \right) \right\}.$$

Each D_n may not be in M. We claim that D_n is dense in $\mathbb{P} \cap M$. Let $r \in \mathbb{P} \cap M$. Then $f \notin G(r, \dot{F}_n)$, which means that there exists $k \in \omega$ such that

$$r \nVdash_{\mathbb{P}} \text{“ } \dot{F}_n \cap [f \restriction k] \neq \emptyset \text{ ”}.$$

This implies that there exists $s \in \mathbb{P} \cap M$ such that $s \leq_{\mathbb{P}} r$ and

$$s \Vdash_{\mathbb{P}} \text{“ } \dot{F}_n \cap [f \restriction k] = \emptyset \text{ ”},$$

because of the fact that $\left\{\mathbb{P}, r, \dot{F}_n, f \restriction k\right\} \in M$ and the elementarity of M. Then $s \in D_n$.

By the strong properness of $\mathbb{P}$, there exists $q \leq_{\mathbb{P}} p$ be such that all D_n are predense below q in $\mathbb{P}$. Then $q \Vdash_{\mathbb{P}}$" $f \in X \subseteq \bigcup_{n \in \omega} \dot{F}_n$ ", so there are $s \leq_{\mathbb{P}} q$ and $n \in \omega$ such that $s \Vdash_{\mathbb{P}}$" $f \in \dot{F}_n$ ". Since D_n is predense below p in $\mathbb{P}$, there are $r \in D_n$, $k \in \omega$ and $t \leq_{\mathbb{P}} s$ such that $r \Vdash_{\mathbb{P}}$" $\dot{F}_n \cap [f \restriction k] = \emptyset$ " and $t \leq_{\mathbb{P}} r$. But then

$$t \Vdash_{\mathbb{P}} \text{“ } f \in \dot{F}_n \text{ and } f \notin \dot{F}_n \text{ ”},$$

which is a contradiction. □

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Faculty of Science, Shizuoka University, Ohya 836, Shizuoka, 422-8529, JAPAN.
Email address: yorioka@shizuoka.ac.jp