NOTE ON LAYEREDNESS OF IDEALS OVER SUCCESSORS OF SINGULAR CARDINALS

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Abstract. We show that there is no layered ideal over $\mathcal{P}_\kappa\lambda$ for all regular cardinals $\kappa \leq \lambda$ if $\kappa = \mu^+$ for some singular μ. We also give a model in which $[\lambda]^{\aleph_{\omega+1}}$ carries a S-layered ideal for some stationary subset $S \subseteq \lambda$.

1. Introduction

In [10], Kunen established

Theorem 1.1 (Kunen [10] for $\mu = \aleph_0$, Laver). *Suppose that j is a huge embedding with critical point κ and $\mu < \kappa$ is regular. Then there is a poset P such that $P * \dot{S}(\kappa, j(\kappa))$ forces that μ^+ carries a saturated ideal.*

Here, $S(\kappa, \lambda)$ denotes a Silver collapse. In [4], Foreman–Magidor–Shelah [4] additionally proved that $P * \dot{S}(\kappa, j(\kappa))$ forces μ^+ carries a layered ideal if j is an almost-huge cardinal with critical point κ and $j(\kappa)$ is Mahlo. This P is a poset of Theorem 1.1.

For the definition of layeredness, see Section 2. Layeredness is one of the strengthenings of usual saturation. Indeed, layeredness lies between saturation and denseness. The denseness of ideals is the strongest saturation property that we can consider. Eskew pointed out

Theorem 1.2 (Eskew [3]). *There is no dense ideal over μ^+ if μ is a singular cardinal.*

If μ is a measurable cardinal and I is a saturated ideal over μ^+ then, by using Prikry-type forcings, we can force an ideal $\overline{I}$ generated by I is saturated and μ is singular. In [15], we proved that such ideals cannot satisfy the layeredness by showing

Theorem 1.3 (Tsukuura [15]). *Suppose that $2^\mu = \mu^+$, μ is measurable, and U is a normal ultrafilter over μ. For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$),*

1. *If $Z \subseteq \mathcal{P}_\kappa(X)$ and $\lambda^{<\kappa} = \lambda$ then $\mathcal{P}_U \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda^+}_{\geq\mu^+}$.*
2. *If $Z \subseteq [X]^\kappa$, I is λ-dense, and λ is a successor cardinal then $\mathcal{P}_U \Vdash \overline{I}$ is* not *S-layered for all stationary $S \subseteq E^{\lambda}_{\geq\mu^+}$.*

The goal of this paper is giving an improvement of Theorems 1.2 and 1.3 by

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Theorem 1.4. *Suppose that μ is a singular cardinal. For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal I over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$),*

1. *If $Z \subseteq \mathcal{P}_\kappa(X)$ then I is* not *S-layered for all stationary $S \subseteq \lambda^+$.*
2. *If $Z \subseteq [X]^\kappa$ and λ is a successor cardinal then I is* not *S-layered for all stationary $S \subseteq \lambda$.*

This theorem ensures

Theorem 1.5. *There is no layered ideal over $\mathcal{P}_\kappa\lambda$ for all regular $\kappa \leq \lambda$ if $\kappa = \mu^+$ for some singular μ.*

The structure of this paper is as follows. In Section 2, we recall the basic facts of saturated ideals and saturation properties. We also introduce Foreman's duality theorem for Section 4. Section 3 is devoted to a proof of Theorem 1.4. The assumption of (2) of Theorem 1.4 for λ being a successor cardinal cannot be removed. In Section 4, we give a model in which there is an inaccessible cardinal λ such that $[\lambda]^{\aleph_{\omega+1}}$ carries an ideal that is S-layered for some stationary $S \subseteq \lambda$.

2. Preliminaries

In this paper, by κ and λ, we mean regular cardinals unless otherwise stated. We use μ to denote infinite cardinals. We use [9] as a reference for set theory in general. First, we introduce the following theorem for Section 4.

Theorem 2.1 (Laver [11]). *If μ is supercompact then there is a poset P such that*

1. *$P \subseteq V_\mu$,*
2. *$P \Vdash \mu$ is supercompact.*
3. *For every P-name $\dot{Q}$ with $P \Vdash \dot{Q}$ is μ-directed closed, $P * \dot{Q} \Vdash \mu$ is supercompact.*

We say that a supercompact cardinal μ is indestructible if, for every μ-directed closed poset Q, $Q \Vdash \mu$ is supercompact. If μ is supercompact and $\kappa > \mu$ is huge then we can force μ to be indestructible without destroying the hugeness of κ.

2.1. Generic ultrapowers.

In this section, we recall basic facts of saturated ideals. For details, we refer to [5]. For an ideal I over $\mathcal{P}_\kappa\lambda$, we always assume that I is normal, fine, and κ-complete. For a normal ideal I over a *non*-empty set Z, by $\mathrm{comp}(I)$, we mean the least κ such that I is not κ^+-complete.

Lemma 2.2. *For a precipitous ideal I over a set Z and $A \in I^+$, the following are equivalent:*

1. *A forces $\mathrm{crit}(\dot{j}) = \kappa$. Here, $\dot{j}$ denotes a $\mathcal{P}(Z)/I$-name for the generic ultrapower mapping.*
2. *$\{B \leq A \mid \mathrm{comp}(I \upharpoonright B) = \kappa\}$ is dense below A.*

Here, $I \upharpoonright A$ is an ideal $I \cap \mathcal{P}(A)$ over A.

We say that I is exactly and uniformly κ-complete if $\mathrm{comp}(I \upharpoonright A) = \kappa$ for all $A \in I^+$. For an exactly and uniformly κ-complete precipitous ideal I, the previous lemma ensures $\mathcal{P}(Z)/I \Vdash \mathrm{crit}(\dot{j}) = \kappa$. Note that every κ-complete ideal over $Z \subseteq \mathcal{P}_\kappa X$(or $[X]^\kappa$) is κ-complete.

Lemma 2.3. *Suppose that I is a normal, fine, exactly and uniformly κ-complete $|X|^+$-saturated ideal over $Z \subseteq \mathcal{P}(X)$. Let $\dot{j}$ be a $\mathcal{P}(Z)/I$-name for the generic ultrapower mapping $\dot{j} : V \to \dot{M}$. Then the following holds:*

1. *If $Z \subseteq \mathcal{P}_\kappa X$ and $\kappa = \mu^+$ then $\mathcal{P}(Z)/I$ forces that $\dot{j}(\kappa) = |X|^+$.*
2. *If $Z \subseteq [X]^\kappa$ then $\mathcal{P}(Z)/I$ forces that $\dot{j}(\kappa) = |X|$ and $\dot{j}(\kappa^+) = |X|^+$.*

Proof. Let G be a $(V, \mathcal{P}(Z)/I)$-generic and $j : V \to M$ be the generic ultrapower induced by G. Let us show (1). We note ${}^{|X|^V} M \cap V[G] \subseteq M$ by $Z \subseteq \mathcal{P}_\kappa X$. Therefore there are no cardinals between $|X|^V$ and $|X|^+$ in M. By the $|X|^+$-saturation of I, $|X|^+$ is a cardinal in M. By $\{z \in Z \mid |z| < \kappa\} \in I^*$, we have

$$\mu \leq |X|^M = |j``X|^M = |[id]|^M < j(\kappa) \leq |X|^+.$$

By the elementarity of j, $j(\kappa)$ is a cardinal in M. Therefore $j(\kappa) = |X|^+$.

For 2, we consider in the case of $Z \subseteq [X]^\kappa$. By $\{z \in Z \mid |z| = \kappa\} \in I^*$,

$$|X|^{V[G]} = |X|^M = |j``X|^M = |[\mathrm{id}]| = j(\kappa) < (|X|^+)^M = (|X|^+)^{V[G]}.$$

Again, in M, $j(\kappa^+) = (|X|^+)^{V[G]}$. □

2.2. **Saturation properties and duality theorem.** For a poset P and stationary subset $S \subseteq \lambda$,

- P is S-layered if $\{M \in [\mathcal{H}_\theta]^{<\lambda} \mid M \cap \lambda \in S \to M \cap P \prec P\}$ contains a club for all sufficiently large regular θ.
- P is λ-dense if P has a dense subset of size λ

Lemma 2.4. *For a stationary subset $S \subseteq \lambda$ and a λ-dense poset P, the following are equivalent:*

1. *P is S-layered.*
2. *There is an $\subseteq$-increasing sequence $\langle P_\alpha \mid \alpha < \lambda\rangle$ with the following properties:*
   (a) *$\bigcup_{\alpha<\lambda} P_\alpha$ is a dense subset of P.*
   (b) *$P_\alpha \lessdot P$ and $|P_\alpha| < \lambda$ for all $\alpha < \lambda$.*
   (c) *There is a club $C \subseteq \lambda$ such that $P_\alpha = \bigcup_{\beta<\alpha} P_\alpha$ for all $\forall \alpha \in S \cap C$.*
3. *There is an $\subseteq$-increasing continuous sequence $\langle P_\alpha \mid \alpha < \lambda\rangle$ with the following properties:*
   (a) *$\bigcup_{\alpha<\lambda} P_\alpha$ is a dense subset of P.*
   (b) *$P_\alpha \subseteq P$ and $|P_\alpha| < \lambda$ for all $\alpha < \lambda$.*
   (c) *There is a club $C \subseteq \lambda$ such that $P_\alpha \lessdot P$ for all $\alpha \in S \cap C$.*

Our definition of S-layeredness is due to Cox [1] but S-layeredness is induced by Shelah. Shelah defined S-layeredness by (2) of Lemma 2.4.

For an ideal I over Z and a saturation property Ψ, we say that I is Ψ-ideal whenever $\mathcal{P}(Z)/I$ satisfies Ψ.

A Layered ideal is an ideal over $\mathcal{P}_\kappa\lambda$ such that I is S-layered for some stationary subset $S \subseteq E^{\lambda^+}_{\geq\lambda}$. For an ideal I over $\mathcal{P}_\kappa\lambda$, we say that I is strongly layered, dense, and saturated when it is $E^{\lambda^+}_{\geq\lambda}$-layered, λ-dense, and λ^+-saturated, respectively. The following implications are known.

Proposition 2.5. (1) *Every dense ideal is layered.*
(2) *(Shelah [4]) Every layered ideal is saturated.*
(3) *(Shelah [14]) Every strongly layered ideal is centered.*

Lastly, we introduce Foreman's duality theorem for section 4.

Theorem 2.6 (Foreman [6]). *For a normal, fine, exactly and uniformly μ^+-complete λ^+-saturated ideal over $Z \subseteq \mathcal{P}(X)$ (for some X with $|X| = \lambda > \mu$) and μ^+-c.c. P, there is a dense embedding d such that:*

$$\begin{array}{rccc} d: & P * \dot{\mathcal{P}}(Z)/\bar{I} & \longrightarrow & \mathcal{B}(\mathcal{P}(Z)/I * \dot{j}(P)) \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & \langle p, \dot{A}\rangle & \longmapsto & \tau(p) \cdot ||[\mathrm{id}] \in \dot{j}(\dot{A})|| \end{array}$$

Here, $\tau(p) = \langle 1, \dot{j}(p)\rangle$ is a complete embedding from P to $\mathcal{P}(Z)/I * \dot{j}(P)$ and $\dot{j} : V \to \dot{M}$ denotes the generic ultrapower mapping by $\mathcal{P}(Z)/I$. In particular, $P \Vdash \dot{\mathcal{P}}(Z)/\bar{I} \simeq \mathcal{B}(\mathcal{P}(Z)/I * \dot{j}(P)/\tau``\dot{H}_0)$. Here, $\dot{H}_0$ is the canonical P-name for a generic filter.

3. Proof of Theorems 1.4

Lemma 3.1 (Sakai [13]). *For a poset P and a regular cardinal κ, if P is κ-dense then $P \Vdash \dot{\mathrm{cf}}(\kappa) = |\kappa|$.*

Proof. Let $\{p_\xi \mid \xi < \kappa\}$ be a dense subset. For each P-name $\dot{a}$ for a cofinal subset of κ of size $\dot{\mathrm{cf}}(\kappa)$. Let us define $f : \kappa \to \kappa$ such that $\Vdash f``\dot{a} = \kappa$.

By induction, we can define $\langle \alpha_{\xi\eta}, p_{\xi\eta} \mid \xi, \eta \in \kappa\rangle$ such that

- $p_{\xi\eta} \leq p_\xi$ forces $\alpha_{\xi\eta} \in \dot{a}$.
- $\alpha_{\xi\eta} > \sup\{\alpha_{\xi'\eta'} \mid \xi' < \xi \wedge \eta' < \eta\}$.

Since κ is regular and $\dot{a}$ is forced to be cofinal, this induction is well-done.

Define f by $f(\alpha_{\xi\eta}) = \eta$. For each $\eta < \kappa$ and p, there is a $q \leq p$ that forces $\eta \in f``\dot{a}$. Indeed, we can choose $p_\xi \leq p$ and $p_{\xi\eta} \leq p_\xi$. $p_{\xi\eta}$ forces $f(\alpha_{\xi\eta}) = \eta \in f``\dot{a}$, as desired. □

From this, we have

Lemma 3.2. *Suppose that P is λ-dense, P preserves the cardinality of λ and P forces $|\lambda| = \mu$. Then μ is regular.*

Lemma 3.3. *Suppose that P is S-layered for some stationary $S \subseteq \lambda^+$, P preserves the cardinality of μ and P forces $|\lambda| = \mu$. Then μ is regular.*

Proof. Let $\dot{f}$ be a P-name for a bijection from λ to μ. For each $\xi < \lambda$, let $\mathcal{A}_\xi$ be maximal anti-chain such that every condition in $\mathcal{A}_\xi$ decides the value of $\dot{f}(\xi)$. Since P is S-layered, we can choose an elementary substructure $M \prec \mathcal{H}_\theta$ such that

- $\{P, \mu, \lambda, \dot{f}\} \cup \bigcup_\xi A_\xi \subseteq M$.
- $|M| = \lambda$.
- $M \cap P \lessdot P$.

Here, θ is sufficiently large regular. By the choice of M, we can regard $\dot{f}$ as a $P \cap M$-name for a bijection from λ to μ. Therefore $M \cap P$ forces $|\lambda| = \mu$. By Lemma 3.2, μ is regular, as desired. □

Proof of Theorem 1.4. (1) Suppose that $Z \subseteq \mathcal{P}_{\mu^+}(X)$ and I is S-layered for some stationary $S \subseteq \lambda^+$. It is enough to prove that μ is regular.

By Lemma 2.3, we can make a list of properties of $\mathcal{P}(Z)/I$ as

- $\mathcal{P}(Z)/I$ preserves all cardinal below μ.
- $\mathcal{P}(Z)/I$ forces $|\lambda| = \mu$.

- $\mathcal{P}(Z)/I$ is S-layered.

Then μ needs to be regular by Lemma 3.3, as desired.

(2) In the case of $Z \subseteq [X]^{\mu^+}$ and λ is a successor cardinal, the same argument works as well. □

4. Layered ideals over $[\lambda]^{\aleph_{\omega+1}}$

By Theorem 1.4, $[\lambda]^{\aleph_{\omega+1}}$ cannot carry an S-layered ideal if λ is a successor cardinal. This assumption is essentially needed. Indeed,

Theorem 4.1. *If there is a supercompact cardinal below a huge cardinal then there is a poset that forces that there is a λ with the following conditions:*

1. *λ is a Mahlo cardinal. In particular, $\lambda \cap \mathrm{Reg}$ is stationary in λ.*
2. *$[\lambda]^{\aleph_{\omega+1}}$ carries a normal, fine, $\aleph_{\omega+1}$-complete $\lambda \cap \mathrm{Reg}$-layered ideal.*

For simplicity, first, we will introduce a proof of Theorem 4.5 and give a proof. After showing this theorem, we give an outline of a proof of Theorem 4.1.

Toward showing Theorem 4.5, we introduce some properties of Prikry forcing, that was introduced by Prikry [12]. For a normal ultrafilter U over μ, Prikry forcing $\mathcal{P}_U$ is $[\mu]^{<\omega} \times U$ ordered by $\langle a, X\rangle \leq \langle b, Y\rangle$ iff $a \supseteq b$, $a \cap (\max b + 1) = b$ and $(a \setminus b) \cup X \subseteq Y$. It is easy to see the following.

- $\mathcal{P}_U$ is μ-centered, and thus, has the μ^+-c.c.
- $\mathcal{P}_U$ forces $\dot{\mathrm{cf}}(\mu) = \omega$.

Since $\mathcal{P}_U$ preserves all regularity (and cardinality) below μ, in the extension by $\mathcal{P}_U$, μ is a singular cardinal with countable cofinality. Preservation of regularity below μ follows by

Lemma 4.2 (Prikry [12])**.** *Suppose that U is a normal ultrafilter over μ. For every $a \in [\mu]^{<\omega}$ and statement σ of the forcing language of $\mathcal{P}_U$, there is a $Z \in U$ such that $\langle a, Z\rangle$ decides σ. That is, $\langle a, Z\rangle \Vdash \sigma$ or $\langle a, Z\rangle \Vdash \neg\sigma$.*

For a detail, we refer to [8]. Here, we use Lemma 4.3 rather than Lemma 4.2.

Lemma 4.3. *Suppose that U is a normal ultrafilter over μ and $\mathcal{A} \subseteq \mathcal{P}_U$ is a maximal anti-chain below $\langle a, X\rangle$. Then there are n and $X \supseteq Z \in U$ such that $\{\langle b, Y\rangle \in \mathcal{A} \mid |b| = n\}$ is a maximal anti-chain below $\langle a, Z\rangle$.*

Proof. Suppose that $\mathcal{A}$ is a maximal anti-chain below $\langle a, X\rangle$. For each $n < \omega$, by Lemma 4.2, there is a $Z_n \in U$ such that $\langle a, Z_n\rangle$ decides $\exists\langle b, Y\rangle \in \dot{G} \cap \mathcal{A}(|b| = n)$. $Z = X \cap \bigcap_n Z_n$ works. □

Lemma 4.4. *For posets $P \lessdot Q$, let $\dot{U}$ and $\dot{W}$ be a P-name and a Q-name for a filter over μ, respectively. Suppose that $Q \Vdash \dot{U} \subseteq \dot{W}$ and $\dot{W}$ is a normal ultrafilter over μ. If $P \Vdash \dot{U}$ is an ultrafilter then $P * \mathcal{P}_{\dot{U}} \lessdot Q * \mathcal{P}_{\dot{W}}$.*

Proof. We may assume that P and Q are Boolean algebras. For a maximal anti-chain $\mathcal{A} \subseteq P * \mathcal{P}_{\dot{U}}$, consider P-name $\dot{\mathcal{B}}$ such that $P \Vdash \dot{\mathcal{B}} = \{\langle a, X\rangle \mid \exists p \in \dot{G}(\langle p, \langle a, X\rangle\rangle \in \mathcal{A})\}$. $\dot{\mathcal{B}}$ is forced to be a maximal anti-chain. It is enough to prove that $Q \Vdash \dot{\mathcal{B}}$ is maximal anti-chain below $\mathcal{P}_{\dot{W}}$. For every $p \Vdash \langle a, \dot{X}\rangle \in \mathcal{P}_{\dot{W}}$, because of $P \Vdash \dot{\mathcal{B}}$ is maximal anti-chain below $\langle a, \emptyset\rangle$, there are $p' \leq p$, n, and, P-name $\dot{Z}$ such that $p' \Vdash \{\langle b, Y\rangle \in \dot{\mathcal{B}} \mid |b| = n\}$ is maximal anti-chain below $\langle a, \dot{Z}\rangle \in \mathcal{P}_{\dot{U}}$. If

$n \leq |a|$, there is a $\dot{Y}$ such that $p' \Vdash \langle b, \dot{Y}\rangle \in \dot{\mathcal{B}} \wedge a \setminus b \subseteq Y)$. Here, b is the first n-th elements in a. Thus, it is forced that $\langle a, \dot{X} \cap \dot{Y}\rangle \leq \langle b, \dot{Y}\rangle, \langle a, \dot{X}\rangle$.

If $n > |a|$, we can choose $p'' \leq p'$ and $\alpha_0, ..., \alpha_{n-|a|-1}$ with $p'' \Vdash \{\alpha_i \mid i < n - |a|\} \in [(\dot{X} \cap \dot{Z}) \setminus (\max a + 1)]^{n-|a|}$. Let $c = a \cup \{\alpha_i \mid i < n - |a|\}$. p'' forces that $\langle c, \dot{Z}\rangle \leq \langle a, \dot{Z}\rangle$ meets with $\dot{\mathcal{B}}$. Because of $|c| = n$, there is a $\dot{Y}$ with $p'' \Vdash \langle c, \dot{Y}\rangle \in \dot{\mathcal{B}}$. In particular, it is forced that $\langle c, \dot{Y} \cap \dot{X}\rangle$ is a common extension of $\langle c, \dot{Y}\rangle$ and $\langle a, \dot{X}\rangle$, as desired. □

Note that the inverse direction of Lemma 4.4 holds. That is, if $P * \mathcal{P}_{\dot{U}} \lessdot Q * \mathcal{P}_{\dot{W}}$ then $\dot{U}$ is forced to be an ultrafilter. Let us show

Theorem 4.5. *Suppose that κ is a huge cardinal and $\mu < \kappa$ is a supercompact cardinal. Then there is a poset that forces that λ is Mahlo, μ is singular, and $Z = [\lambda]^{\mu^+}$ carries a normal, fine, μ^+-complete λ-saturated ideal I such that*

1. *I is $\lambda \cap \mathrm{Reg}$-layered.*
2. *I is λ-dense.*

Proof. We may assume that μ is indestructible supercompact by Theorem 2.1. Let $j : V \to M$ be a huge embedding induced by normal ultrafilter over $[\lambda]^\kappa$. Then $\mathrm{Coll}(\mu, < \kappa) \Vdash [\lambda]^\kappa$ carries a normal, fine, and κ-complete ideal I such that $\mathcal{P}([j(\kappa)]^\kappa)/I \simeq \mathrm{Coll}(\mu, < j(\kappa))$ (See [5, Example 7.25]). Note that such λ is Mahlo (Moreover, it is weakly compact). Since $\mathrm{Coll}(\mu, < \kappa)$ is μ-directed closed, there is a $\mathrm{Coll}(\mu, < \kappa)$-name $\dot{U}$ for a normal ultrafilter over μ. By Theorem 2.6, $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}} \Vdash \mathcal{P}([j(\kappa)]^\kappa)/\overline{I} \simeq \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}/\dot{G} * \dot{H}$. It is easy to see that $\overline{I}$ is forced to be $j(\kappa)$-dense and $j(\kappa)$-saturated. It remains to show that $\overline{I}$ is forced to be $\lambda \cap \mathrm{Reg}$-layered. We claim that it is forced that $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}} \Vdash \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}/\dot{G} * \dot{H}$ is $(\mathrm{Reg} \cap j(\kappa))^V$-layered.

For $\mathrm{Coll}(\mu, < j(\kappa))$-name $\dot{X}$ for a subset of μ, there is a maximal anti-chain $\mathcal{A}_{\dot{X}}$ such that every $q \in \mathcal{A}_{\dot{X}}$ decides $\dot{X} \in j(\dot{U})$. Let $\rho(\dot{X})$ be the least $\alpha < j(\kappa)$ such that $\mathcal{A}_{\dot{X}} \subseteq \mathrm{Coll}(\mu, < \alpha)$. For $\beta < j(\kappa)$, define $\rho(\beta) < j(\kappa)$ by $\sup\{\rho(\dot{X}) \mid \dot{X}$ is $\mathrm{Coll}(\mu, < \beta)$-name for a subset of $\mu\} \cup \{2^\beta\}$. Let C be a club generated by ρ. For every $\alpha \in C \cap \mathrm{Reg}$, $\mathrm{Coll}(\mu, < \alpha)$ has the α-c.c. since α is inaccessible. In particular, $\mathrm{Coll}(\mu, < \alpha) \Vdash \dot{U}_\alpha := \dot{j}(U) \cap V[\dot{G}_\alpha]$ is an ultrafilter. Here, $\dot{G}_\alpha$ is the canonical name for a generic filter of $\mathrm{Coll}(\mu, < \alpha)$. By Lemma 4.4, we have

$$\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}} \lessdot \mathrm{Coll}(\mu, < \alpha) * \mathcal{P}_{\dot{U}_\alpha} \lessdot \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}.$$

It is easy to see that $\mathrm{Coll}(\mu, < \alpha) * \mathcal{P}_{\dot{U}_\alpha}/\dot{G} * \dot{H} \lessdot \mathrm{Coll}(\mu, < j(\kappa)) * \mathcal{P}_{j(\dot{U})}/\dot{G} * \dot{H}$ is forced by $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}}$ for each $\alpha \in C \cap \mathrm{Reg}$. Let $\dot{P}_\alpha$ be a $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U}}$-name for $\mathrm{Coll}(\mu, < \alpha) * \mathcal{P}_{\dot{U}_{f(\alpha)}}$. Then $\langle \dot{P}_\alpha \mid \alpha < j(\kappa)\rangle$ is forced to satisfy the condition of (3) of Lemma 2.4. □

Lastly, we give an outline of a proof of Theorem 4.1.

Sketch of Proof of Theorem 4.1. We may assume that μ is indestructible supercompact and $2^\mu = \mu^+$. If μ is measurable and $2^\mu = \mu^+$ then, for every normal ultrafilter U over μ, there is a $(M_U, \mathrm{Coll}(\mu^+, < j_U(\mu))^{M_U})$-generic filter $\mathcal{G}$. Here, $j_U : V \to M_U$ is an ultrapower mapping induced by U. Indeed, since $\mathrm{Coll}(\mu^+, < j_U(\mu))^{M_U}$ has the $j_U(\mu)$-c.c. in M_U and $|j_U(\mu)^{<j_U(\mu)}| = |j_U(\mu)| = \mu^+$, we can enumerate every anti-chain of $\mathrm{Coll}(\mu^+, < j_U(\mu))^{M_U}$ belongs to M_U as $\langle \mathcal{A}_\alpha \mid \alpha < \mu^+\rangle$. Since

$\mathrm{Coll}(\mu^+, < j_U(\mu))^{M_U}$ is μ^+-closed, the standard argument takes a filter $\mathcal{G}$ that meets with any $\mathcal{A}_\alpha$.

By using $\mathcal{G}$, let us introduce a modification $\mathcal{P}_{U,\mathcal{G}}$ of $\mathcal{P}_U$ that was used in [7]. We call this poset "Woodin's modification". $\mathcal{P}_{U,\mathcal{G}}$ is the set of all quadruplet $\langle a, f, X, F\rangle$ such that

- $a = \{\alpha_1, ..., \alpha_{n-1}\} \in [\Psi]^{<\omega}$.
- $f = \langle f_0, ..., f_{n-1}\rangle \in \prod_{i<n} \mathrm{Coll}(\alpha_i^+, < \alpha_{i+1})$. But α_0 and α_n denote ω and μ, respectively.
- $X \in U$ and $X \subseteq \Psi$.
- $F \in \prod_{\alpha \in X} \mathrm{Coll}(\alpha^+, < \mu)$ and $[F] \in \mathcal{G}$.

Here, $\Psi = \{\alpha < \mu \mid \alpha \text{ is an inaccessible and } 2^\alpha = \alpha^+\}$. $\mathcal{P}_{U,\mathcal{G}}$ is ordered by $\langle a, f, X, F\rangle \leq \langle b, g, Y, H\rangle$ if and only if $\langle a, X\rangle \leq \langle b, Y\rangle$ in $\mathcal{P}_U$, $\forall i \in [|b|, |a|)(h(i) \supseteq F(\beta_i))$, and $\forall \alpha \in X(F(\alpha) \supseteq H(\alpha))$.

$\mathcal{P}_{U,\mathcal{G}}$ has properties that are similar with Lemmas 4.3 and 4.4. Importance is μ being $\aleph_\omega$ in the extension by $\mathcal{P}_{U,\mathcal{G}}$. Let $\dot{U}$ and $\dot{\mathcal{G}}$ be $\mathrm{Coll}(\mu, < \kappa)$-names for a normal ultrafilter over μ and a $(M_{\dot{U}}, \mathrm{Coll}(\mu^+, < j_U(\mu))^{M_{\dot{U}}})$-generic filter, respectively. Then we can show that $\mathrm{Coll}(\mu, < \kappa) * \mathcal{P}_{\dot{U},\dot{\mathcal{G}}}$ is a required poset. □

Note that λ is weakly compact in the models of Theorems 4.5 and 4.1. We note

Theorem 4.6 (Cox–Lücke [2]). *For an uncountable cardinal λ, the following are equivalent:*

(1) *λ is weakly compact.*
(2) *If P has the λ-c.c. then P is S-layered for some stationary $S \subseteq \lambda$.*

It was known that having the λ-c.c. is equivalent to λ-Knasterness if λ is weakly compact. In [1], Cox also showed that if P is S-layered for some stationary $S \subseteq \lambda$ then P is λ-Knaster.

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