Visualizing deformations of complex structures on 4-punctured spheres

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1. Introduction

The space of marked Riemann (or hyperbolic) surfaces modulo natural equivalence relation is called the Teichmüller space. In this article we consider a topological surface S of finite type (i.e. S has finite genus and finite number of punctures). The main focus is more particularly on $S_{0,4}$, 4-punctured sphere. However let us begin with a quick review of the basic facts of Teichmüller space (we denote it by $\mathcal{T}(S)$). Given two marked Riemann surfaces or two points $X, Y \in \mathcal{T}(S)$, Teichmüller showed that there is a unique "best" deformation of complex structures connecting X, Y. The locus of the deformation on $\mathcal{T}(S)$ is called a Teichmüller geodesic because it is actually a geodesic with respect to so-called the Teichmüller distance. Moreover, Teichmüller observed that the "best" deformation can be described in terms of horizontal, and vertical directions, which can be seen as a generalization of Grötzsch's theorem: Given two rectangles with the same area, the best deformation of complex structure is uniquely attained by an affine map (Figure 1). Teichmuller geodesics are also interpreted similarly. That is, for any Teichmuller

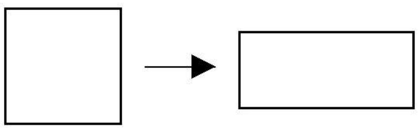

Figure 1. Grötzsch's theorem (depicted by ChatGPT).

geodesic γ, there are horizontal and vertical directions such that the deformation is locally depicted as in Figure 1. Thus we know the existence of such "best" deformations, however how do they actually look like? The goal of this article is to explain the pictures listed Figure 2, Figure 3, and Figure 4.

2. Teichmüller space and Mapping class group

Let S be a surface of finite type. A marked Riemann surface is a Riemann surface X (i.e. S with a complex structure) together with a marking, i.e. a homeomorphism $f : S \to X$. Two marked Riemann surfaces $f : S \to X$ and $g : S \to Y$ are said to be Teichmuller equivalent if there exists a conformal map $\rho : X \to Y$ such that ρ is homotopic to $g \circ f^{-1}$. Then the space of marked Riemann surfaces modulo Teichmuller equivalence is called the Teichmuller space, which is denoted by $\mathcal{T}(S)$. Given $X, Y \in \mathcal{T}(S)$, the Teichmuller distance is defined as

$$d_{\mathcal{T}}(X, Y) = \frac{1}{2} \inf_{h} \log K(h)$$

where h runs over all quasi-conformal mappings compatible with the markings and $K(h)$ is the dilatation of h. Any geodesic with respect to $d_{\mathcal{T}}$ is called a Teichmuller geodesic. To understand Teichmuller geodesics, let us discuss so-called Beltrami differentials and quadratic differentials.

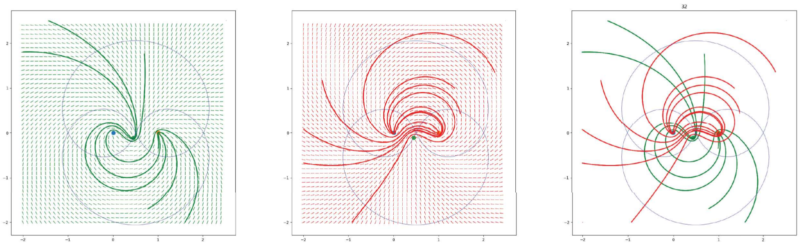

FIGURE 2. $\sigma_1\sigma_2^{-1}$.

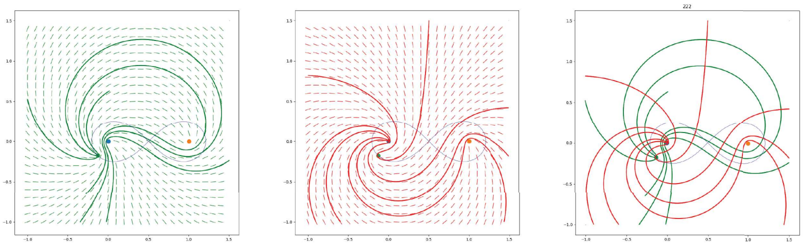

FIGURE 3. $\sigma_1^2\sigma_2^{-2}$.

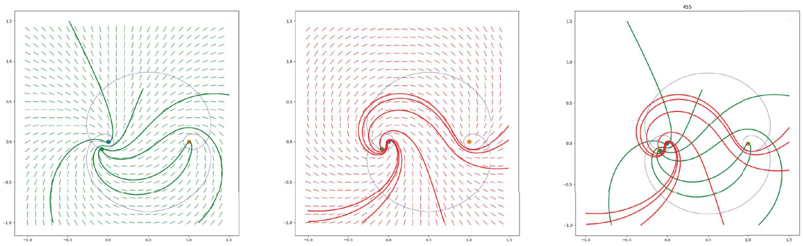

FIGURE 4. $\sigma_1^2\sigma_2^{-1}$.

Let $X \in \mathcal{T}(S)$ be a marked Riemann surface, and $T^{1,0}X$ and $T^{0,1}X$ denote the subspaces of the cotangent bundle which corresponds to holomorphic and anti-holomorphic part respectively. A *Beltrami differential* is a section of $T^{0,1}X \otimes (T^{1,0}X)^*$ which is locally expressed by $\beta(z)d\bar{z}/dz$. A *quadratic differential* is a section of $T^{1,0}X \otimes T^{1,0}X$ whose local expression is $q(z)dz^2$. If moreover, $q(z)$ is holomorphic on each local chart, it is called a holomorphic quadratic differential. The space of holomorphic quadratic differentials on X is denoted by $\mathrm{QD}(X)$. By the Riemann-Roch theorem, $\mathrm{QD}(X)$ is isomorphic to the vector space $\mathbb{C}^{3g-3}$. Hence the space of all holomorphic quadratic differentials on S defines a vector bundle over $\mathcal{T}(S)$, that we denote by $\mathrm{QD}(S)$.

A Teichmüller geodesic is determined by a Beltrami differential or a quadratic differential. In this article, we use quadratic differentials. Given a quadratic differential q locally expressed as $q(z)dz^2$ on a Riemann surface X, its horizontal directions is the set of directions $v \in T_zX$ defined by $q(z)v^2 \in \mathbb{R}_{>0}$. The horizontal direction equipped with the transversal measure defined for any transversal arc α by

$$\int_\alpha |\mathrm{Im}\ q(z)^{1/2}dz|$$

determines a measured foliation which is called the *horizontal foliation* of q. Similarly directions $v \in T_zX$ defined by $q(z)v^2 \in \mathbb{R}_{<0}$ equipped with transverse measure

$$\int_\alpha |\mathrm{Re}\ q(z)^{1/2}dz|$$

give the *vertical foliation* of q. Now, the deformation of complex structure on X along a Teichmüller geodesic given by a quadratic differential $q(z)dz^2$ is understood similarly as the Grötzsch's theorem. In Figure 2, Figure 3, and Figure 4, we draw horizontal (left) and vertical directions (middle) of quadratic differentials. In the right most figures, we have drawn horizontal and vertical directions on the same figure. One observes that horizontal and vertical directions form right angles everywhere.

We consider the action of the mapping class group

$$\mathrm{MCG}(S) := \mathrm{Homeo}^+(S)/\mathrm{homotopy}$$

on the Teichmüller space by the change of markings. Let us recall Nielsen-Thurston classification:

Theorem 2.1 ([7]). *Let $S_{g,n}$ be a surface of finite type with $3g-3+n > 0$. Then every mapping class $\varphi \in \mathrm{MCG}(S)$ is homotopic to one of the following:*

- *Periodic: there exists $k \neq 0$ such that φ^k is homotopic to the identidy map.*
- *Reducible: there are simple closed curves $\alpha_1, \ldots, \alpha_m$ and $k \neq 0$ such that $\varphi^k(\alpha_i)$ is isotopic to α_i for each $1 \leq i \leq m$.*
- *Pseudo-Anosov: there are two measured foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ and $\lambda > 1$ such that $\varphi(\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda\mu_u)$ and $\varphi(\mathcal{F}_s, \mu_s) = (\mathcal{F}_s, \mu_s/\lambda)$.*

Now let us consider a pseudo-Anosov mapping class φ. By the work of Bers [2], φ has a unique Teichmüller geodesic axis. By the work of Gardiner-Masur, given measured foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ of φ, there is a quadratic differential whose horizontal and vertical foliations are $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$. Such a quadratic differential determine the unique axis of φ. In Figure 2, Figure 3, and Figure 4, we depicted $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ for certain pseud-Anosov mapping classes.

To draw Figure 2, Figure 3, and Figure 4, we need to compute corresponding quadratic differentials. Fortunately, for four punctured spheres and once punctured tori, Teichmüller space is known to be identified with hyperbolic plane and Teichmüller geodesics are exactly hyperbolic geodesics. Let

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

denote the upper half space model of the hyperbolic plane.

Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, A acts on $\mathbb{H}$ as a Möbius transform:

$$A(z) = \frac{az+b}{cz+d}.$$

If the trace of A is greater than 2, then A has a geodesic axis which is a half circle with center $(a-d)/(2b)$ and radius $\sqrt{(a-d)^2 + 4bc}/(2b)$.

It is known (see e.g. [3]) that the mapping class group of $S_{0,4}$ contains $\mathrm{SL}(2, \mathbb{Z})$. Recall that the braid group B_3 with three strands has a representation

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

Fixing one puncture of $S_{0,4}$, we may regard each 3-braid as a mapping class of $S_{0,4}$. The group B_3 has a representation in $\mathrm{SL}(2,\mathbb{Z})$ given by

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and hence for a given braid, we may consider corresponding matrix in $\mathrm{SL}(2,\mathbb{Z})$. The geodesic axis is exactly the Teichmüller geodesic axis regarding a braid as a mapping class acting on the Teichmüller space.

Using this data, one obtains corresponding Beltrami differentials (see e.g. [6, Chapter V, Section 6]) which relates points on the axis of A. Given a Beltrami differential μ, on the once punctured torus corresponding quadratic differential is given as $q(z) = Cdz^2$ where $\mu(z) = \|q\|\frac{\bar{C}}{|C|}$. Hence we may determine $C \in \mathbb{C}$ for a given μ.

Now one uses so called Weierstrass' pe-function to relate Cdz^2 and a quadratic differential on $S_{0,4}$ (c.f. [5]).

3. Definition and properties of $\wp$

We recall some properties of Weierstrass' pe-function. See [1, Section 7] for more discussion. We fix $\tau \in \mathbb{H}$. Let

$$L_\tau := \{n + m\tau \mid n, m \in \mathbb{Z}\}$$

denote the lattice group generated by 1 and τ.

Definition 3.1. *The Weierstrass pe-function $\wp_\tau : \mathbb{C} \to \hat{\mathbb{C}}$ with period lattice L_τ is defined as*

$$\wp_\tau(z) := \frac{1}{z^2} + \sum_{0 \neq \omega \in L_\tau} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Proposition 3.2. *Let $\wp := \wp_\tau$. We define*

- $e_1 = e_1(\tau) := \wp(1/2)$
- $e_2 = e_2(\tau) := \wp(\tau/2)$
- $e_3 = e_3(\tau) := \wp\left(\frac{1+\tau}{2}\right)$.

Then

(1) *$\wp$ is an even function, i.e. $\wp(z) = \wp(-z)$.*
(2) *$\wp$ is doubly periodic, i.e. $\wp(z+1) = \wp(z) = \wp(z+\tau)$.*
(3) *The deferential $\wp'(z)$ has the following expression:*

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

(equation (20) in [1, Section 7]*)*

(4) *Let $Z := (\mathbb{C} \setminus \frac{1}{2}L_\tau)/L_\tau$ be a 4-punctured torus. Then $\wp : Z \to \hat{\mathbb{C}}$ defines a double covering*

$$\wp : Z \to \hat{\mathbb{C}} \setminus \{\infty, e_1, e_2, e_3\} = \mathbb{C} \setminus \{e_1, e_2, e_3\}$$

which corresponds to the quotient with respect to $z \mapsto -z$ on Z (c.f. [5, Section 3]*).*

As is well known, every holomorphic quadratic differential on a once-punctured torus

$$X := (\mathbb{C} \setminus L_\tau)/L_\tau$$

is represented by Cdz^2 for some $C \in \mathbb{C}$. We consider the lift of Cdz^2 to Z, which is again represented as Cdz^2. By Proposition 3.2 (3), the push-forward of Cdz^2 to $\mathbb{C} \setminus \{e_1, e_2, e_3\}$ is

$$\frac{Cdz^2}{4(z-e_1)(z-e_2)(z-e_3)},$$

where by abuse of notations, we again use z as a parameter on $\mathbb{C} \setminus \{e_1, e_2, e_3\}$. Then by the Möbius transformation

$$z \mapsto \frac{z - e_2}{e_1 - e_2},$$

$\mathbb{C} \setminus \{e_1, e_2, e_3\}$ is mapped to $\mathbb{C} \setminus \{0, 1, \lambda(\tau)\}$, where

$$\lambda(\tau) := \frac{e_3 - e_2}{e_1 - e_2}$$

is called *the modular function.* The push-forward of the quadratic differential above is then

$$\frac{1}{4(e_1 - e_2)} \frac{C \cdot dz^2}{z(z-1)(z - \lambda(\tau))}. \tag{1}$$

Remark 3.3. By definition, each e_i $(i = 1, 2, 3)$ depends on τ.

4. Summary

Once the quadratic differential (1) is obtained one may compute horizontal and vertical direction using the definition $q(z)v^2 \in \mathbb{R}_{>0}$ and $q(z)v^2 \in \mathbb{R}_{<0}$. One need to be careful when one draws flow lines because quadratic differentials.

Also one may generate a movie of Teichmüller geodesic as we indeed have computed geodesic axises of pseudo-Anosov mapping classes. Some movies by the author are available at `https://www.youtube.com/playlist?list=PLQLzkZ9xZDXiVz-fS7EqU6RYh6L2h1wVB`. Those movies take the Teichmüller distance $d_{\mathcal{T}}$ into account, and hence movies are "unit-speed" with respect to $d_{\mathcal{T}}$.

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