An embedding of the Kauffman bracket skein algebra of a surface into a localized quantum torus

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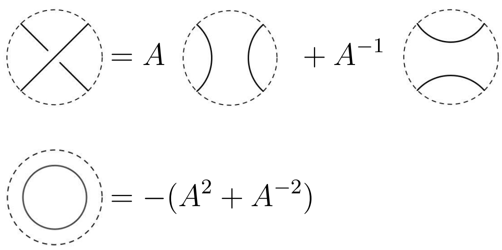
1 Introduction

The goal of these notes is to explain how to construct embeddings of Kauffman bracket skein algebras of surfaces (either closed or with boundary) into localized quantum tori using the action of the skein algebra on the skein module of the handlebody. These embeddings are useful to study representations of Kauffman skein algebras at roots of unity and get a new proof of Bonahon-Wong's unicity conjecture. These methods allow one to explicitly reconstruct the unique representation with fixed classical shadow, as long as the classical shadow is irreducible with image not conjuguate to the quaternion group. The associated articles are [DS22] and [DLS22].

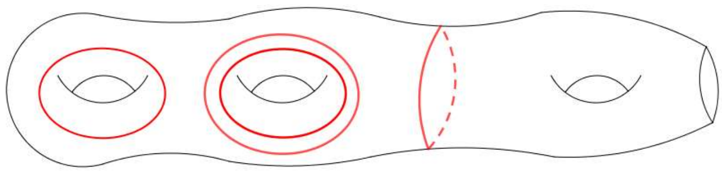
2 Preliminaries

Let M be an oriented compact 3-manifold. The skein module of M denoted by $S(M)$ is the quotient of the free $\mathbb{Z}[A^{\pm 1}]$-module generated by isotopy classes of banded links in the interior of M modulo the following local relations :



In general these modules are very difficult to compute for a given 3-manifold. The easiest example is the 3-sphere. In this case, any banded link can be made into a diagram by a regular projection. By resolving all the crossing and deleting all the trivial components, we can convince our self that $S(S^3) = \mathbb{Z}[A^{\pm 1}]\emptyset$.

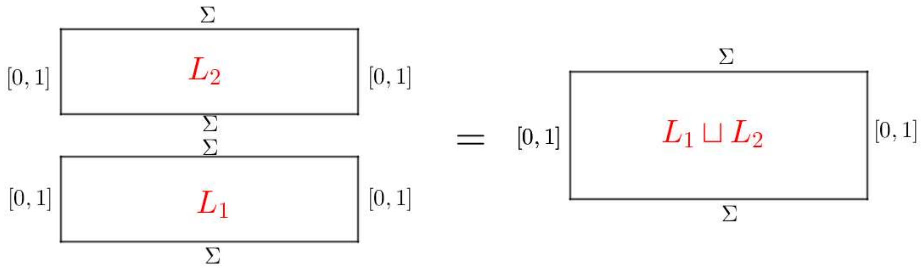
The main interest of these notes is the case of a surface times an interval. Let Σ be an oriented compact connected surface. A multicurve γ on Σ is finite disjoint union of non-null-homotopic simple closed curves in Σ. For instance here is a multicurve on genus 3 surface with one boundary component :



To any multicurve γ on Σ we can define $\gamma \times \left[\frac{1}{3}, \frac{2}{3}\right] \in S(\Sigma \times [0,1])$. This element will still be denoted by γ.

Theorem 2.1 *The isotopy classes of multicurves on Σ give a $\mathbb{Z}[A^{\pm 1}]$-basis of $S(\Sigma \times [0,1])$.*

$S(\Sigma \times [0,1])$ has a $\mathbb{Z}[A^{\pm 1}]$-algebra structure given by the stacking operation.



Equipped with multiplication, $S(\Sigma \times [0,1])$ can be viewed as a non commutative deformation of the algebra of regular functions on $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$. The following theorem is from [PS00].

Theorem 2.2 (Przytycki–Sikora) *$S(\Sigma \times [0,1]) \otimes_{A=-1} \mathbb{C}$ is isomorphic to $\mathbb{C}[X(\Sigma, \mathrm{SL}_2(\mathbb{C}))]$ where*

$$X(\Sigma, \mathrm{SL}_2(\mathbb{C})) = \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C})) /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

3 Bonahon-Wong theory

Bonahon and Wong worked on the following aspects of skein algebras :

1. Embedding $S(\Sigma \times [0,1])$ into a quantum torus
2. Classifying irreducible representations of the skein algebra of $\Sigma \times [0,1]$ at roots of unity

We will resume briefly the work of [BW11],[BW16a], [BW16b] and [BW17].

Definition 1 *Let $N \geq 1$ an integer and $\delta \in \mathcal{M}_N(\mathbb{Z})$ a skew-symmetric matrix. The quantum torus associated to δ is the $\mathbb{Z}[A^{\pm 1}]$-algebra generated by $X_1^{\pm 1}, \ldots X_N^{\pm 1}$ with relations*

$$X_i X_j = A^{\delta_{i,j}} X_j X_i$$

Theorem 3.1 (Bonahon-Wong) *When $\partial\Sigma \neq \emptyset$, $S(\Sigma\times[0,1])$ embeds in the Chekov–Fock quantization of the Teichmuller space (which is a quantum torus).*

Let us now briefly discuss about the representation side. Let $p \geq 3$ be a odd integer and ξ be a $2p$-th primitve root of unity.

$$S_\xi(\Sigma \times [0,1]) := S(\Sigma \times [0,1]) \otimes_{A=\xi} \mathbb{C}$$

Bonahon and Wong associated to an irreducible complex finite dimensional representation of $S_\xi(\Sigma \times [0,1])$ a point in $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ called the classical shadow (associated to the representation).

Remark 3.1 *Two remarks :*

- *If ξ is not a root of unity then $S_\xi(\Sigma \times [0,1])$ does not have finite dimensional representations*
- *The theory for $2p$-th primitive roots of unity with p odd gives the theory for any roots of unity*

The classical shadow is defined via the following theorem.

Theorem 3.2 (Bonahon-Wong) *If $\rho : S_\xi(\Sigma \times [0,1]) \to \mathrm{End}(V)$ is an irreducible finite dimensional representation, then there exists $r_\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ such that for any simple closed curve $T_p(\rho(\gamma)) = -\mathrm{Tr}(r_\rho(\gamma))\,\mathrm{Id}_V$.*

Here T_p is the first kind p-th Chebyshev polynomial. The class $[r_\rho] \in X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ is called the classical shadow of ρ. Of course two isomorphic representations will have the same classical shadow. We can naturally ask the following question : Is the map $\rho \mapsto [r_\rho]$ a bijection?

Theorem 3.3 (Bonahon-Wong) *The above map is sujective.*

Remark 3.2 *When Σ has boundary, $\rho \mapsto [r_\rho]$ is not injective and extra data is needed. In these notes we will forget about this detail.*

The natural object to introduce to study injectivity is the Azumaya locus. Let $\mathfrak{A}$ be the set of points $[r] \in X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ such that there exists a unique irreducible representation $\rho : S_\xi(\Sigma) \to \mathrm{End}(V)$ with $[r] = [r_\rho]$. $\mathfrak{A}$ is called the Azumaya locus of $S_\xi(\Sigma)$.

The first major theorem about the Azumaya locus is in [FKL19] :

Theorem 3.4 (Frohman, Kania-Bartoszynska, Lê 2016) *$\mathfrak{A}$ contains a Zariski open dense set in $X(\Sigma, \mathrm{SL}_2(\mathbb{C}))$.*

Another crucial result is in [GJS19] :

Theorem 3.5 (Ganev, Jordan, Safronov 2019) *$\mathfrak{A}$ contains $X^{\mathrm{irr}}(\Sigma, \mathrm{SL}_2(\mathbb{C}))$.*

4 Localized quantum torus

Now let us discuss how to embed $S(\Sigma)$ in a localized quantum torus. For simplicity $\partial\Sigma = \emptyset$. Suppose that Σ has genus $g \geq 2$.

Let $\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a pants decomposition of Σ and H be the canonical handlebody associated to $\mathcal{P}$. The embedding is obtained by understanding the canonical action of $S(\Sigma)$ on $S(H) \otimes \mathbb{Q}(A)$ given by gluying.

To understand this action, we need to introduce a canonical basis of H. An element $(u_1, \dots, u_{3g-3}) \in \mathbb{N}^{3g-3}$ is admissible if whenever $\alpha_i, \alpha_j, \alpha_k \in \mathcal{P}$ bound a pair of pants :

$$u_i + u_j \leq u_k,$$

$$u_k + u_i \leq u_j,$$

$$u_j + u_k \leq u_i,$$

$$u_i + u_j + u_k \text{ even}$$

Let Δ be the set of all admissible elements of $\mathbb{N}^{3g-3}$.

There is a canonical basis of $S(H) \otimes \mathbb{Q}(A)$ indexed by elements in Δ, we denote it by $(\Gamma_u)_{u\in\Delta}$. This basis is obtained by inserting appropriate Jones-Wenzl idempotents to a trivalent graph dual to $\mathcal{P}$ inside H.

Theorem 4.1 (Detcherry-S.) *Let γ be a multicurve on Σ. There exists a family of rational functions $F_k^\gamma(A, X_1, \dots, X_{3g-3}) \in \mathbb{Q}(A, X_1, \dots, X_{3g-3})$ indexed by elements $k \in \mathbb{Z}^{3g-3}$ such that for almost all $u \in \Delta$:*

$$\gamma.\Gamma_u = \sum_k F_k^\gamma(A, A^{u_1}, \dots, A^{u_{3g-3}})\Gamma_{u+k}$$

Moreover F_k^γ is zero for all but finitely many $k \in \mathbb{Z}^{3g-3}$.

Moreover each F_k^γ is the quotient of an element in $\mathbb{Z}[A^{\pm 1}][X_1^{\pm 1}, \dots X_{3g-3}^{\pm 1}]$ by a finite product of elements of the form $A^m X_j^2 - A^{-m} X_j^{-2}$ for $m \in \mathbb{Z}$ and $1 \leq j \leq 3g-3$.

This Theorem is obtained using fusion rules (see [MV94]).

Let $\mathcal{T}(\mathcal{P})$ the quantum torus over $\mathbb{Z}[A^{\pm 1}]$ with variables $Q_1^{\pm 1}, \dots Q_{3g-3}^{\pm 1}, E_1^{\pm 1}, \dots, E_{3g-3}^{\pm 1}$ where all variables commute except Q_j and E_j that satisfy $Q_j E_j = A E_j Q_j$. Viewed as a ring, the quantum torus $\mathcal{T}(\mathcal{P})$ is an integral domain so we can define $\mathcal{A}(\mathcal{P})$ to be a $\mathbb{Z}[A^{\pm 1}]$-algebra containing $\mathcal{T}(\mathcal{P})$ where all the $A^l Q_j^2 - A^{-l} Q_j^{-2}$ are invertible for $l \in \mathbb{Z}$. $\mathcal{A}(\mathcal{P})$ is what we call a localized quantum torus.

Theorem 4.2 (Detcherry-S.) *The map $\sigma : S(\Sigma) \to \mathcal{A}(\mathcal{P})$ defined for a multicuve γ by*

$$\sigma(\gamma) = \sum_{k\in\mathbb{Z}^{3g-3}} E^k F_k^\gamma(A, Q_1, \dots, Q_{3g-3}) \in \mathcal{A}(\mathcal{P})$$

is a $\mathbb{Z}[A^{\pm 1}]$-algebra embedding.

5 Application to representations theory

Definition 2 *Let $\mathcal{P}$ be a pants decomposition of Σ and $r : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$. We say that r is compatible with $\mathcal{P}$ if the restriction of r to each pair of pants of $\mathcal{P}$ is irreducible.*

Definition 3 *A pants decomposition of Σ is sausage type if it has $g-1$ separating curves.*

Theorem 5.1 (Detcherry-S.) *Let $\rho : S_\xi(\Sigma) \to \mathrm{End}(V)$ be a finite dimensional irreducible representation with classical shadow r_ρ. Suppose that r_ρ is compatible with a sausage type pants decomposition $\mathcal{P}$. Then there exists $\tilde{\rho} : \mathcal{A}_\xi(\mathcal{P}) \to \mathrm{End}(V)$ such that*

$$\tilde{\rho} \circ \sigma_\xi = \rho$$

Theorem 5.2 (Detcherry-Le Fils-S.) *There is an explicit finite set of $[r] \in X^{\mathrm{irr}}(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ such that r is compatible with* ***no*** *sausage type pants decomposition.*

The representation theory of a localized quantum torus is very easy to understand, we get the following corollary.

Corollary 5.3 *The Azumaya locus of $S_\xi(\Sigma)$ contains the complement of a finite set in $X^{\mathrm{irr}}(\Sigma, \mathrm{SL}_2(\mathbb{C}))$.*

This result is stronger than the result from Frohman, Kania-Bartoszynska, Lê but weaker than the result from Ganev, Jordan, Safronov. Note that with our method we have explicit formulas for the irreducible representations.

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