Construction and manipulation of Seifert surfaces in knot theory (a note in 2023)

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Abstract. In this note, I present a survey of construction and manipulation of Seifert surfaces in knot theory, including, fiber surfaces with hidden Hopf plumbings (with Murasugi), fiber surfaces with arbitrary enhanced Milnor number (with Rudolph), new fiber-producing twisting operation (with Van Quach), Seifert surfaces preserved by a strong involution (with Sakuma and Hiura) and so on. As an application, given is a demonstration of finding a strongly invertible diagram by manipulating a fiber surface. We start with updating the Seifert algorithm by omitting the step of applying arrows to segments in a knot projection.

1 Introduction

A knot is a closed loop in 3-space, where two knots are identified if one is continuously deformed into another without self-crossings. Knot invariants are useful to conclude that two knots are different. However, when two knots happened to be the same, there are no theoretically complete method to deform one to the other. Knots are too flexible as a topological object when one actually try to deform them. In that case surfaces whose boundary coincide with the knot is useful to fix the knot to some extent while preserving sufficient flexibility, to keep the knot soft, but not too soft. Surfaces are fun to deform providing a lot of topology puzzles, but are also theoretically important.

The author was once asked by Makoto Sakuma and Alexander Mednykh to make a symmetric diagram (to be precise, a strongly invertible diagram) of a knot depicted in Figure 1 (1). The diagram had 17 crossings, and it was said that the knot is related to a hyperbolic 3-manifold obtained by great many copies of icosahedra and is known to be strongly invertible. I used Seifert surfaces to make a symmetric diagram in Figure 1 (2). The knot turned out to be 12n526, and in the knotInfo site [14] it is depicted as Figure 1 (3) in a non symmetric diagram. A detailed process of making a symmetric diagram using Seifert surface is given in Figure 14.

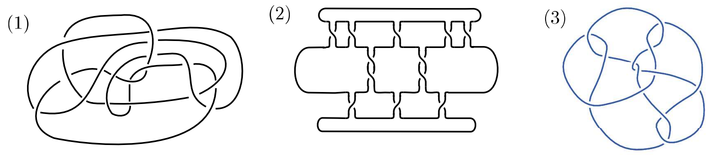

Figure 1: Make a 17 crossing diagram into a symmetric diagram with 12 crossings

2 Updating of the Seifert algorithm

Definition 2.1. *A compact, oriented surface F is a Seifert surface for an oriented link L, if F has no closed component and $\partial F = L$ with coherent orientations.*

The Seifert algorithm is well-known for obtaining a Seifert surface for a link presented by a diagram. It starts with applying arrows to most segments of the diagram to see the orientation of the link everywhere, and then at each crossings, apply smoothing respecting the orientation of the segments. The disjoint union of simple closed curves thus obtained is called the set of *Seifert circles*. Then an oriented surface is built by applying twisted bands (recovering the crossing) to disks spanned by Seifert circles, which is a Seifert surface. However, when there are many crossings, it is awkward to apply arrows to most of the segments in the diagram. So we give an updated algorithm to draw Seifert circles on the knot projection, which we regards as a 4-valent graph whose edges are oriented according to the orientation of the knot.

> Algorithm: Start with any edge and thicken it, and as we travel along the projection, we make every other edges thick. Then connection of thick edges makes a disjoint union of simple closed curves, and so does that of thin edges. The union of those circles compose the set of Seifert circles.

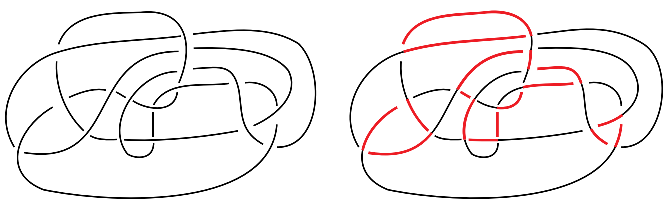

Figure 2: a shortcut to draw Seifert circles

See Figure 2 above. We obtain a system of Seifert circles without applying arrows. There are 17 crossings and 6 circles and hence the canonical Seifert surface has genus $\frac{17-6+1}{2} = 6$.

Proposition 2.2. *The union of simple closed curves thus obtained is the system of Seifert circles.*

Proof. Suppose we apply the Dowker-Thistlethwaite labeling to crossings. We may consider the end points of an edge are labeled with an integer. By the Jordan curve theorem, each crossing is endowed a pair of odd and even integers. The edges whose initial point is odd (resp. even) (and hence the end point is labeled even (resp. odd)) are thickened (resp. remain thin). At each crossings, the parity of the initial points of two outgoing edges are opposite, and hence connecting the thick edges (and connecting the thin edges) has the same effect of smoothing the crossing according to the orientation of edges. □

Thus we need no more application of clumsy arrows to the segments of a knot diagram.

We extended this updated algorithm to link projections, but it is a bit more complicated and we will use it in another occation.

3 Obtaining a minimal genus Seifert surface

A Seifert surface is called a *canonical Seifert surface* if it is obtained by the Seifert algorithm. Not all Seifert surfaces are canonical (for example knotted annuli are not canonical). Murasugi has shown that canonical surfaces obtained from an alternating diagram is of minimal genus. There are some extensions of that theorem, but in general canonical surfaces are not of minimal genus. As an application of the above-mentioned theorem of Murasugi, we can immediately detect the rational number of a 2-bridge knot represented by an alternating diagram which is not apparently of Conway's standard form.

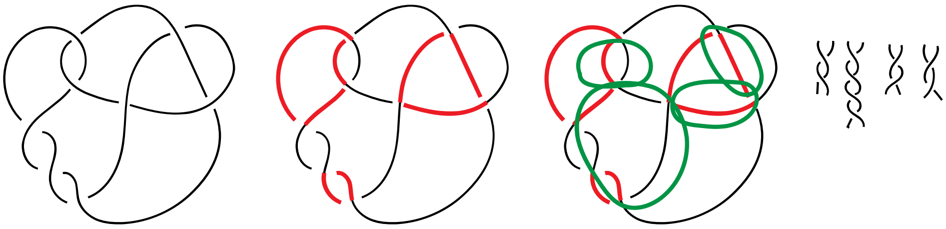

Figure 3: Obtaining the fraction of 2-bridge knot

Let K be the knot in Figure 3. On the minimal genus Seifert surface F obtained above, we see a chain of loops for the basis of the first homology and understand that F is a linear plumbing of unknotted annuli of $2, 4, -2, -2$ half-twists, and hence K is a 2-bridge knot and by putting this sequence into the continued fraction of the "negative format "$r = 1/(a_1 - 1/(a_2 - 1/(a_3 - \cdots - 1/(a_{m-1} - 1/a_m)\ldots)))$, we see that K is represented by the rational number r below:

$$r = \cfrac{1}{2 - \cfrac{1}{4 - \cfrac{1}{-2 - \cfrac{1}{-2}}}} = \frac{14}{25},$$

where the denominator 25 is the knot determinant $|\Delta_K(-1)|$.

The knot Floer homology, which is a categorification of the Alexander polynomial, detects the genus of a given knot, but there still exists no practical algorithm to construct a minimal genus Seifert surface for a given knot.

One of the ways to reduce the genus of a given Seifert surface of non-minimal-genus, is the operation called *compression*. A simple closed loop ℓ in a Seifert surface F is the boundary of a *compressing disk* D, if ℓ is essential in F and bounds a disk D in the complement of F in $\mathbb{R}^3$, or equivalently in S^3. Cutting F open along ℓ and capping off by two parallel copies of D is called a *compression* of F, which reduces the genus of the surface by one. Minimal genus Seifert surfaces are incompressible, but the converse is not true.

In the trial to explicitly construct a minimal genus Seifert surface for a given specific knot, compression is sometimes very useful, and again there are some techniques to find a loop which is a boundary of a compressing disk. After finding the boundary of a compression disk, it is another practice to actually apply compression. See Figure 14 for some examples of finding a hidden compressing disk boundary and actually compressing the surface.

4 Fibration of the link complement by a Seifert surface

Let K be a knot in S^3. We say that K is a fibered knot if its complement is a surface bundle over S^1 with fiber a Seifert surface.

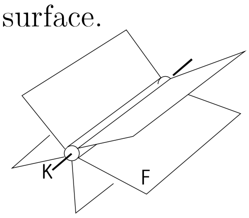

Figure 4: Fibration of the knot complement by Seifert surface

It is known that a fiber is a Seifert surface of minimal genus, and that it is a unique incompressible Seifert surface for K. For a knot K, denote by $G(K)$ the fundamental group of its complement. It was Neuwirth who advocated the study of knots K with $G(K)$ having finitely generated commutator subgroup. Then it was Stallings who showed that condition is equivalent to that K is fibered. Since then here has been two streams in studying fibered knot; one is algebraic and the other geometric. The Alexander polynomial of a fibered knot is monic, but the converse does not hold. Again the knot Floer homology detects fibredness of a knot by its “monicity”. However Floer homology does not give inspiration for the construction of fiber surfaces. At present, there is no general method to construct a fiber surface for a given fibered knot.

Following is the list of classes of knots for which the author and Murasugi determined the genus and fibredness. These are examples of nice collaborations of algebraic and geometric methods. Minimal genus Seifert surfaces and fiber surfaces are actually constructed for these classes. Double torus knot which are band sums of two torus knots [9], Montesinos knots [8], satellite knots of tunnel number one [10].

The author’s favorite class of fibered knots is *divide knots* introduced by N. A’Campo. In [5], we gave an algorithm to convert a divide to the link diagram together with its fiber surface.

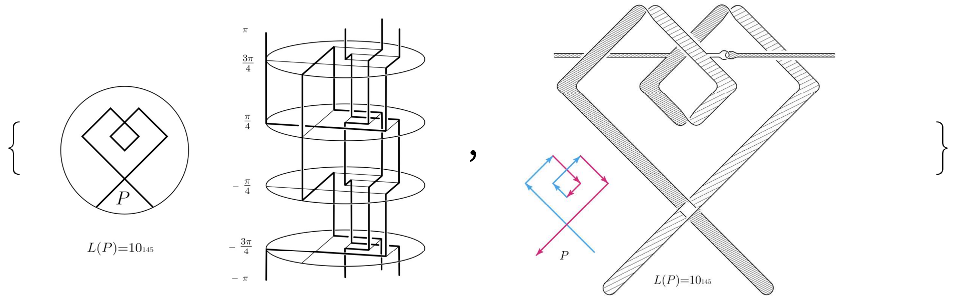

Figure 5: A’Campo’s divide and the fibre surface

As for understanding the fibrations in the complement of links, just constructing a fiber surface is not enough for those eager to see the fibration itself. This task is much more difficult than showing a given surface is a fiber surface, by using, for example, "product decompositions of sutured manifolds"or Hopf plumbings.

Drawing the fibration allowing to deform the link itself is much easier than drawing the fibration fixing the boundary link. Figure 8 is my actual drawing of the fibration of the complement of the Figure-eight knot. Note that around each segment of the knot, a piece of Seifert surfaces sweeps as in Figure 4. Also, remark that in several occasions, Reidemeister moves appear in the contour curves. In particular, a kink is removed through the fifth frame by the cusp-move followed by the R-II move.

In practicing to draw surfaces, the author was most influenced by Tsuyoshi Kobayashi, Makoto Sakuma David Gabai and George Francis.

In his fabulous book, A topological picture book, [2, p.174] Francis mentions Gabai as follows.

> To him, perceiving how a spanning surface squeezes through unlikely places, as in the shear isotopy, is an exercise every low dimensional topologist should do.

In a collaboration with Minoru Yamamoto [12], we made a new method of the sphere eversion, which is turning a sphere in $\mathbb{R}^3$ inside out by regular homotopy, i.e., deformation allowing self-intersection, but not allowing tearing, creasing and pinching. Through that study, the author learned to use the deformation of visual contours of surfaces, which bought a drastic improvement in understanding and drawing continuous deformations of surfaces, like as in Figure 8. I appreciate Yamamoto for introduction of such a method and visiting Francis together in University of Illinois. In drawing, cutting the under path or making the lines behind in dotted lines are particular fashions where what is drawn is not what exactly look like when we really have it, but that is drawn so for those used to understand in some common established manner.

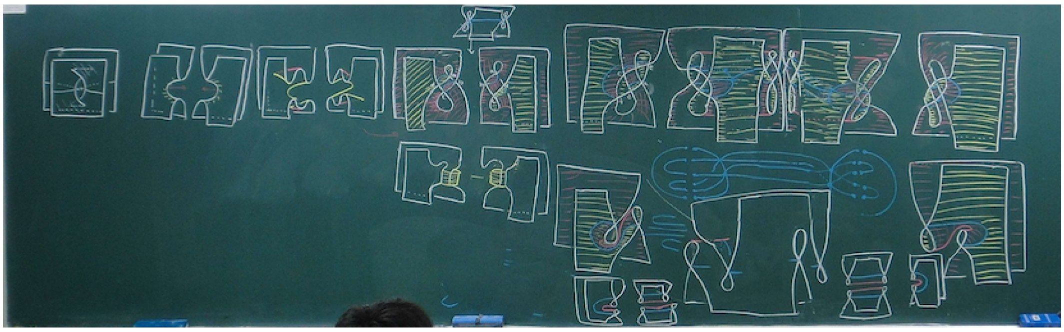

Figure 6: Hirasawa-Yamamoto sphere eversion

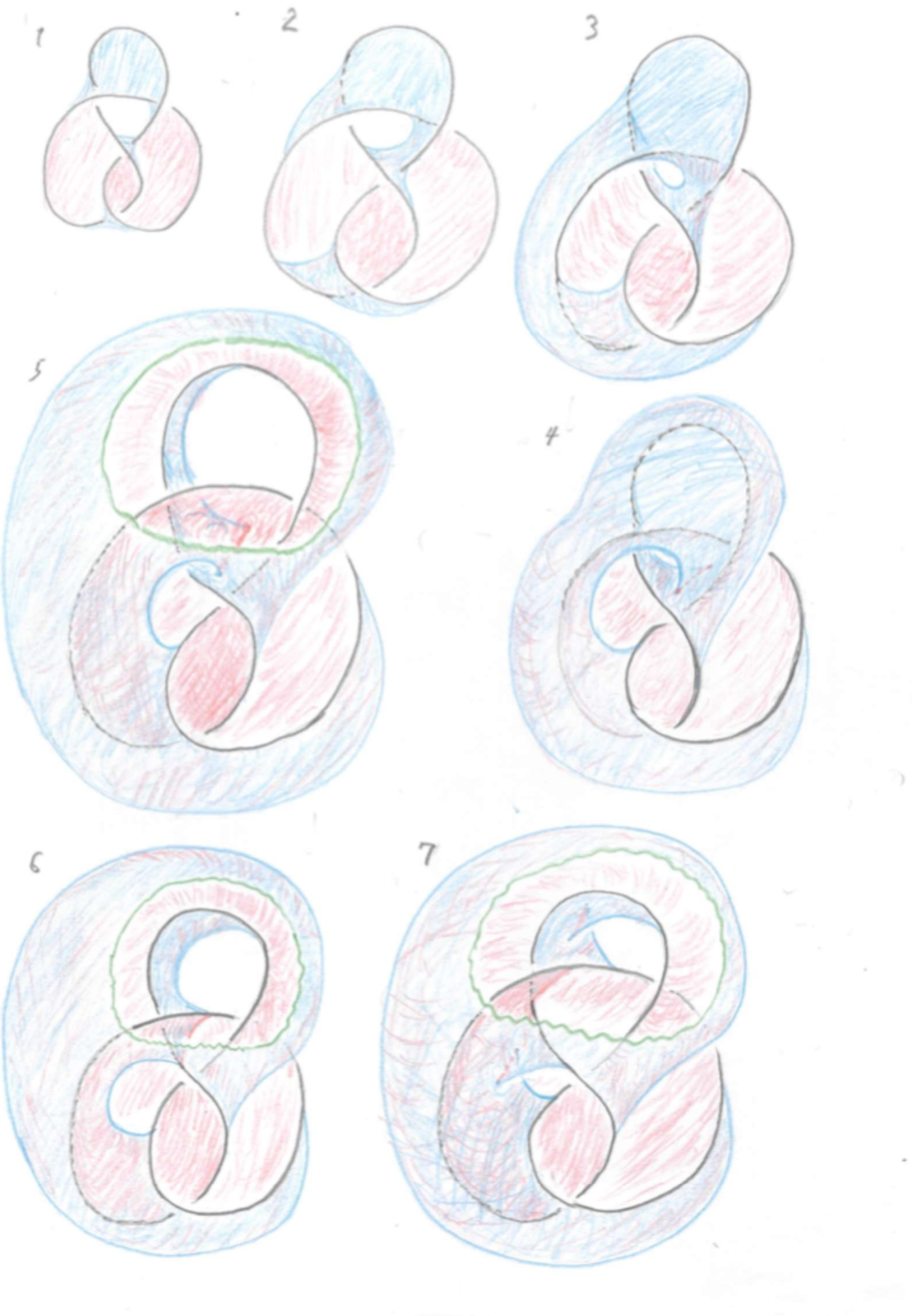

Figure 7: Fibration of the knot 4_1 (1/4. Continues on next pages)

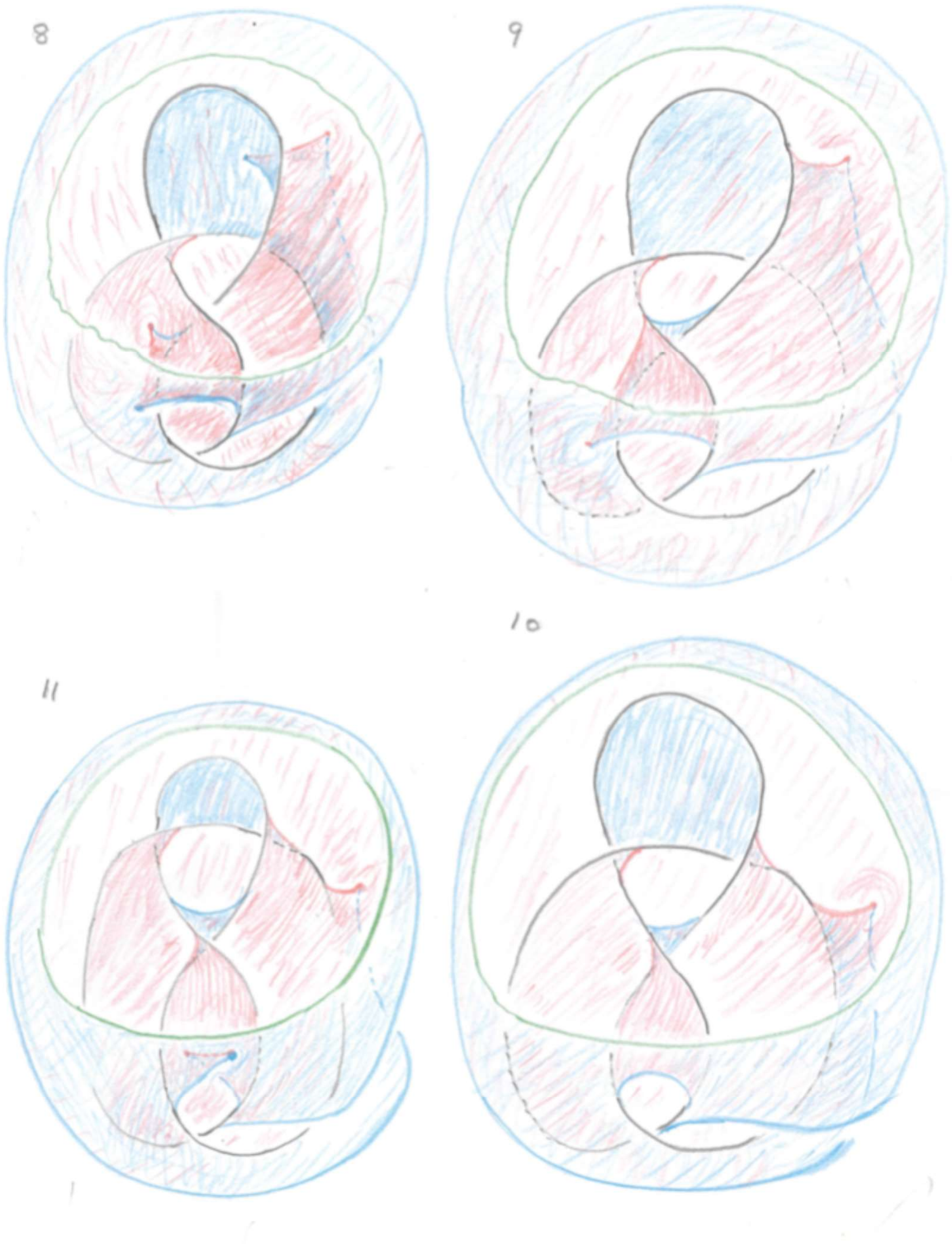

Figure 7: Fibration of the knot 4_1 (2/4, continued)

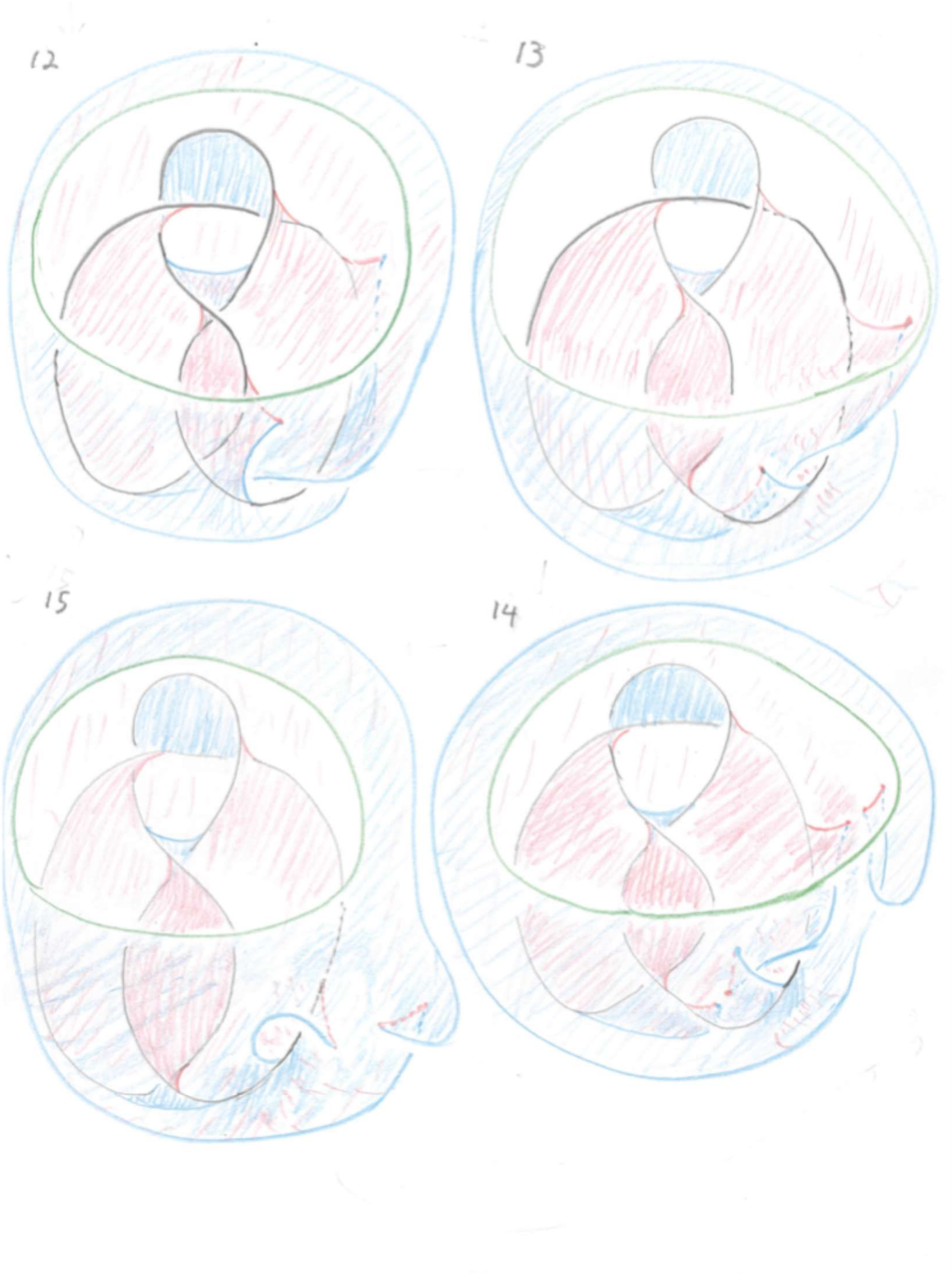

Figure 7: Fibration of the knot 4_1 (3/4, continued)

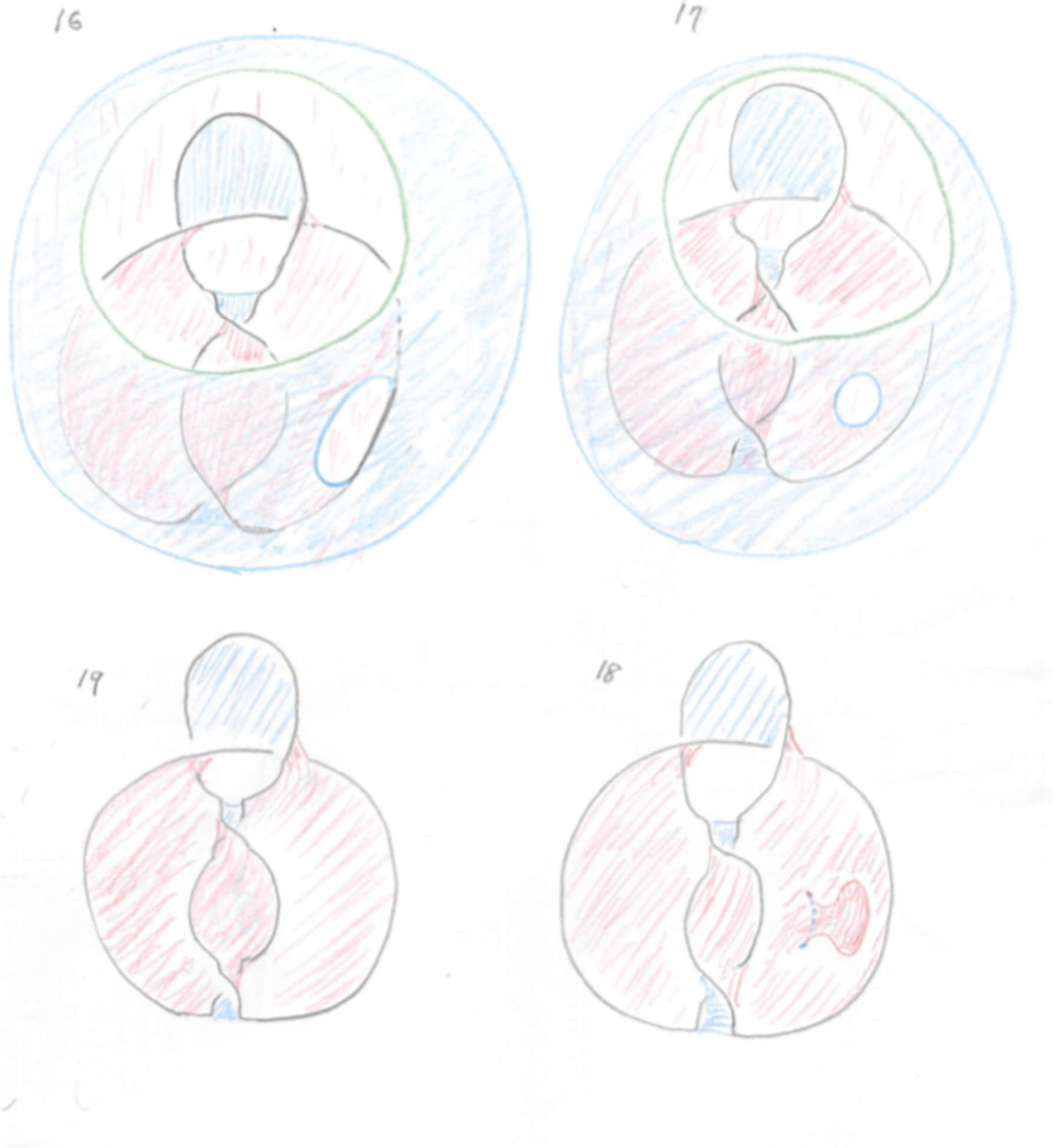

Figure 7: Fibration of the knot 4_1 (4/4, continued)

5 Deformation of fiber surfaces and their invariants

There are several operations to construct a fiber surfaces from fiber surfaces. The first of such operations is the Murasugi sum, which is a generalization of plumbing of two fiber surfaces. The simplest example is plumbing a Hopf band to a fiber surface. Another operation is twisting, introduced by Stallings, and then by Harer. Here a twisting operation is to take a loop ℓ in a fiber surface such that ℓ is unknotted in S^3 and the linking number of ℓ and ℓ^+ is k. Then apply an $1/n$-surgery along ℓ. The total space remains S^3 and the effect of the surgery on F is as follows: push ℓ off F to the normal direction and cut F along the disk spanned by the pushed loop and glue it back after twisting n times.

5.1 Plumbing and deplumbing Hopf bands

The following question was posed by Harer [5]:

Question 5.1. *Is any fiber surface in S^3 obtained from a disk by plumbing and deplumbing Hopf bands ?*

This question was affirmatively answered by Giroux and Goodman [3], using the theory of contact structure of 3-manifolds. But again how to do plumbing and deplumbing was never to be explained.

Sometimes, plumbed Hopf band is deeply hidden in a surface F, and we need to detect it to show that F is a fiber surface. In these cases, the sutured manifold decomposition is sometimes clumsy since the result of a product decomposition yields a manifold which has no more structure of Seifert surface times an interval (though the complement stays so).

The following is an example from [10] of deplumbing a hidden Hopf band in a Seifert surface which was a candidate of a fiber surface. Cutting the band labeled B or depicted as a small square are results of deplumbing hidden Hopf bands.

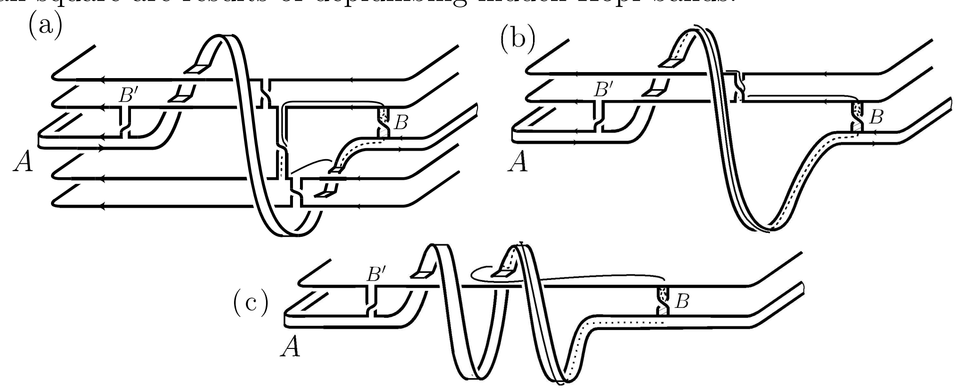

Figure 8: Deplumbing hidden Hopf bands, continues

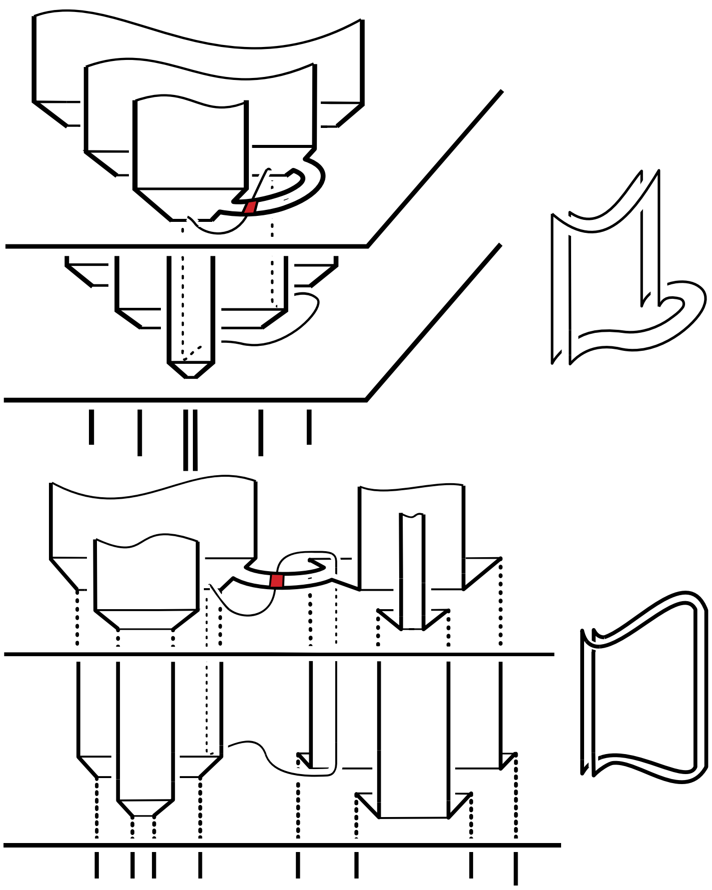

Figure 8: Deplumbing hidden Hopf bands,continued

Concerning Question 5.1, Rudolph introduced an integer invariant $\lambda(F)$ of a fiber surface F. It has been studied in the context of the "enhanced Milnor number"(μ, λ) of F, where μ is the 1st Betti number of F, and λ is called the enhancement. Now let us call λ the Rudolph number of a fiber surface.

Neumann and Rudolph showed that λ is additive under the Murasugi sum. According to their new interpretation, we can formulate the invariant λ by the following, where D, H^+, H^- denote a disk, a positive and a negative Hopf band.

$$\lambda(D) = \lambda(H^+) = 0, \lambda(H^-) = 1, \text{ and } \lambda \text{ is addtive under plumbing.}$$

Neumann and Rudolph asked the following question.

Question 5.2. *Can there be a fiber surface F such that $\lambda(F)$ in negative ?*

They solved this question affirmatively, and again a simple example had been expected. Since there is no deplumbing of a negative Hopf band from a disk, to obtain such an example, we need to plumb on a positive Hopf band, and then find a deplumbable negative Hopf band in the resulting surface.

In the September of 2001, Rudolph kindly invited the author to his home and we spend a long time to find an actual plumbing-deplumbing procedure. While sparing some time in the airport of Toronto after many cancellations of flights, an example was finally constructed. The figure below is the first found procedure where we deplumb a negative Hopf band after plumbing positive Hopf bands so that λ goes down. In the first step, we plumb on two positive Hopf bands to a surface which locally look like the first figure, for example, a connected sum of two positive and two negative Hopf bands. Then, we can deplumb negative Hopf bands twice. Here the result of deplumbing a Hopf band is cutting a band as specified in the figure.

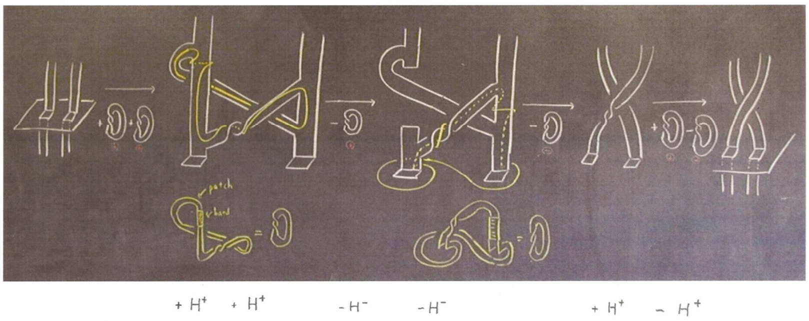

Figure 9: Making Rudolph number negative

The last part of untwisting a band by plumbing a Hopf band then deplumbing a Hopf band was introduced by Melvin-Morton [15, Fig.8]. I would call this whole sequence of moves a *Stallings' half twist*. This operation is repeatable and hence we immediately have fiber surfaces with negative λ.

5.2 One more, last twisting operation

Let ℓ be a simple loop in a fiber surface F. Suppose ℓ in unknotted in S^3 and the self-linking number of ℓ in F is k. Consider applying a t-twist on F. Then the known twistings which produce a new fiber surface is as follows:

Stallings twist [17], where $k = 0$ and $t = \pm 1$. Harer twist [5], where $k = \pm 2$ and $t = \mp 1$.

See below for an example of Harer twist.

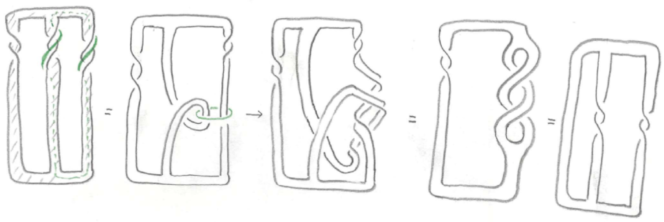

Figure 10: Harer twist

We showed [11] that there is one more fiber-producing twist, and there is no more twistings after that.

Theorem 5.3. *The twist with $k = \pm 1$ and $t = \mp 2$ produces a new fiber surface, and this twist and the above-mentioned two kinds of twisting are the only fiber-producing twists.*

See below for examples of our new twists. Note that our twist changes a positive Hopf band to a negative one, and vice versa.

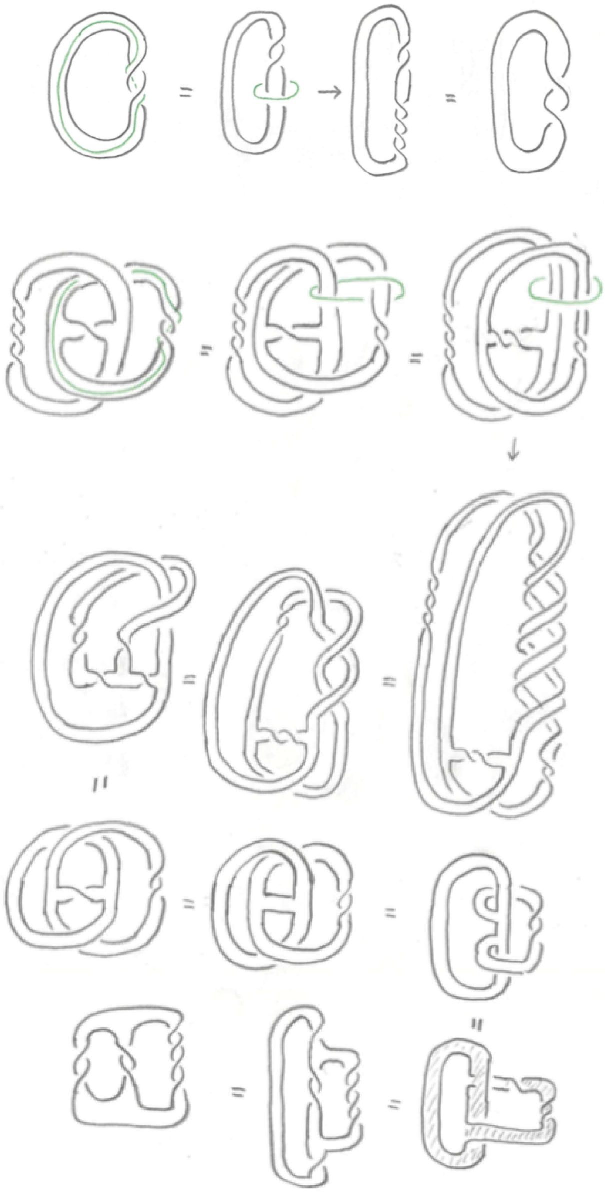

Figure 11: new twist

5.3 Generalized Hopf banding by Shimokawa et al.

In [1], Dorothy Buck, Kai Ishihara, Matt Rathbun and Koya Shimokawa introduced a genuine generalization of a Hopf plumbing.

Definition 5.4. *Let ℓ be an arc in a Seifert surface F such that ℓ has a single self-intersection point and $\ell \cap F = \partial \ell$. Then generalized Hopf banding is an operation to* *attach an untwisted band B to F which is parallel to F with one overpass at the site of the self-intersection point of ℓ.*

Theorem 5.5. *[1] Suppose that F' is obtained from F by a generalized Hopf banding. Then F is a fiber surface if and only if so is F'.*

Their proof is by product sutured manifold decomposition, but I have found a surface manipulating proof that the banding does not affect the fibredness of F.

If their banding is done in a trivial way, then it is the same as Hopf plumbing, but in general, the result is not obtained by Hopf plumbing.

The figure below (which is re-rendering of [1, Figure 5]) depicts generalized Hopf banding applied a Hopf band. Notice the similarity to Figure 11 above.

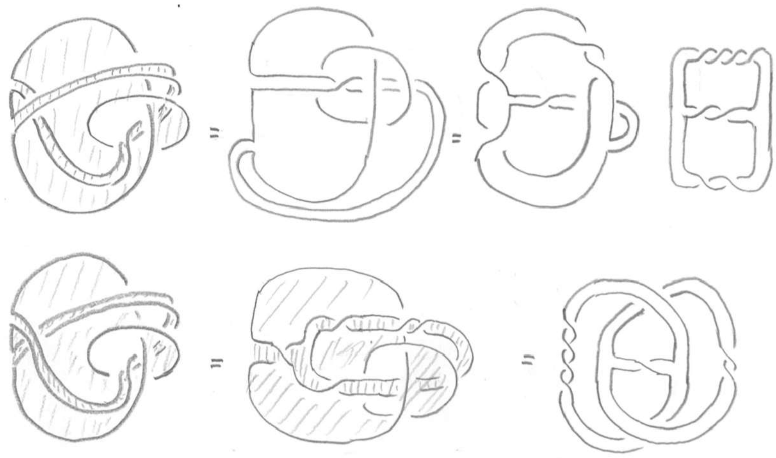

Figure 12: Generalised Hopf banding

We noticed that a special case of their generalized Hopf banding is related to our new twist, when the self-linking number of the loop obtained from their self-intersecting loop is equal to 0. In that case, their banding is realized by the composition of two consecutive Stallings twist, one Hopf plumbing and one application of our twist, as in the figure below:

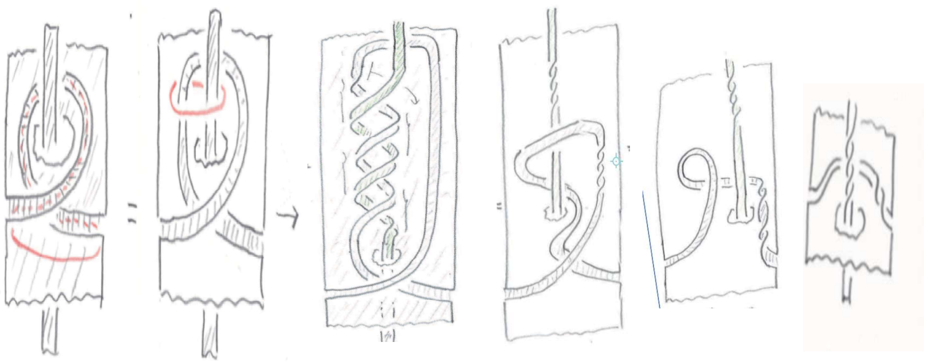

Figure 13: Twisting and banding

6 Seifert surfaces preserved by a strong involution

By a *marked strongly invertible knot*, we mean a triple (K, h, δ) where (K, h) is a strongly invertible knot with involution h and δ is a subarc of $\mathrm{Fix}(h)$ bounded by $\mathrm{Fix}(h) \cap K$. Two marked strongly invertible knots (K, h, δ) and (K', h', δ') are regarded to be *equivalent* if there is an orientation-preserving diffeomorphism φ of S^3 mapping K to K' such that $h' = \varphi h \varphi^{-1}$ and $\delta' = \varphi(\delta)$.

Definition 6.1. *By an invariant Seifert surface for a marked strongly invertible knot (K, h, δ), we mean a Seifert surface S for K such that $h(S) = S$ and $\mathrm{Fix}(h) \cap S = \delta$. The equivariant genus $g(K, h, \delta)$ of (K, h, δ) is defined to be the minimum of the genera of invariant Seifert surfaces for (K, h, δ).*

There are some strongly involution of a knot which preserves no Seifert surface of minimal genus. This makes a clear contrast to Edmonds' result that every periodic knot admits an invariant minimal genus Seifert surface.

It has been known by Hiura [13] that can be arbitrarily large gap between $g(K)$ and $g(K, h, \delta)$ for some 2-bridge knot K with specific δ.

In [6], we first showed the following:

Theorem 6.2. *A fiber surface for a strongly invertible knot is preserved by the strong involution.*

We then showed that even if a minimal genus Seifert surface is not preserved by a strong involution, there is a disjoint pair of minimal genus Seifert surfaces interchanged by the involution, which helps in the determination of $g(K, h, \delta)$.

Namely, we showed the following:

Theorem 6.3. *Let (K, h) be a strongly invertible knot. Then there is a minimal genus Seifert surface F for K such that F and $h(F)$ have disjoint interiors.*

Theorem 6.4. *Let (K, h, δ) be a marked strongly invertible knot, and let F be a minimal genus Seifert surfaces for K such that F and $h(F)$ have disjoint interiors. Then there is a Seifert surface S of quivariant genus for (K, h, δ) whose interior is disjoint from the interiors of F and $h(F)$.*

This result made it possible for us to determine the minimal genus of Seifert surfaces for 2-bridge knot preserved by a strong involution [7] with the conclusion that the gap between the ordinary genus and the *equivariant genera* can be arbitrarily large.

Since then we are wondering how strong or effective our theory is to actually finding an invariant surface.

The symmetry group of knots up to 12 crossings are listed in KnotInfo database site [14]. When one can determine (by theory) that a given knot is strongly invertible, it is in general quite difficult to find a strongly invertible diagram for the knot. The main problem is that a knot is to flexible and one can not image which part corresponds to which part when the knot is isotoped into a strongly invertible form. To challenge this problem, Seifert surfaces sometimes can very helpful, as in Figure 14.

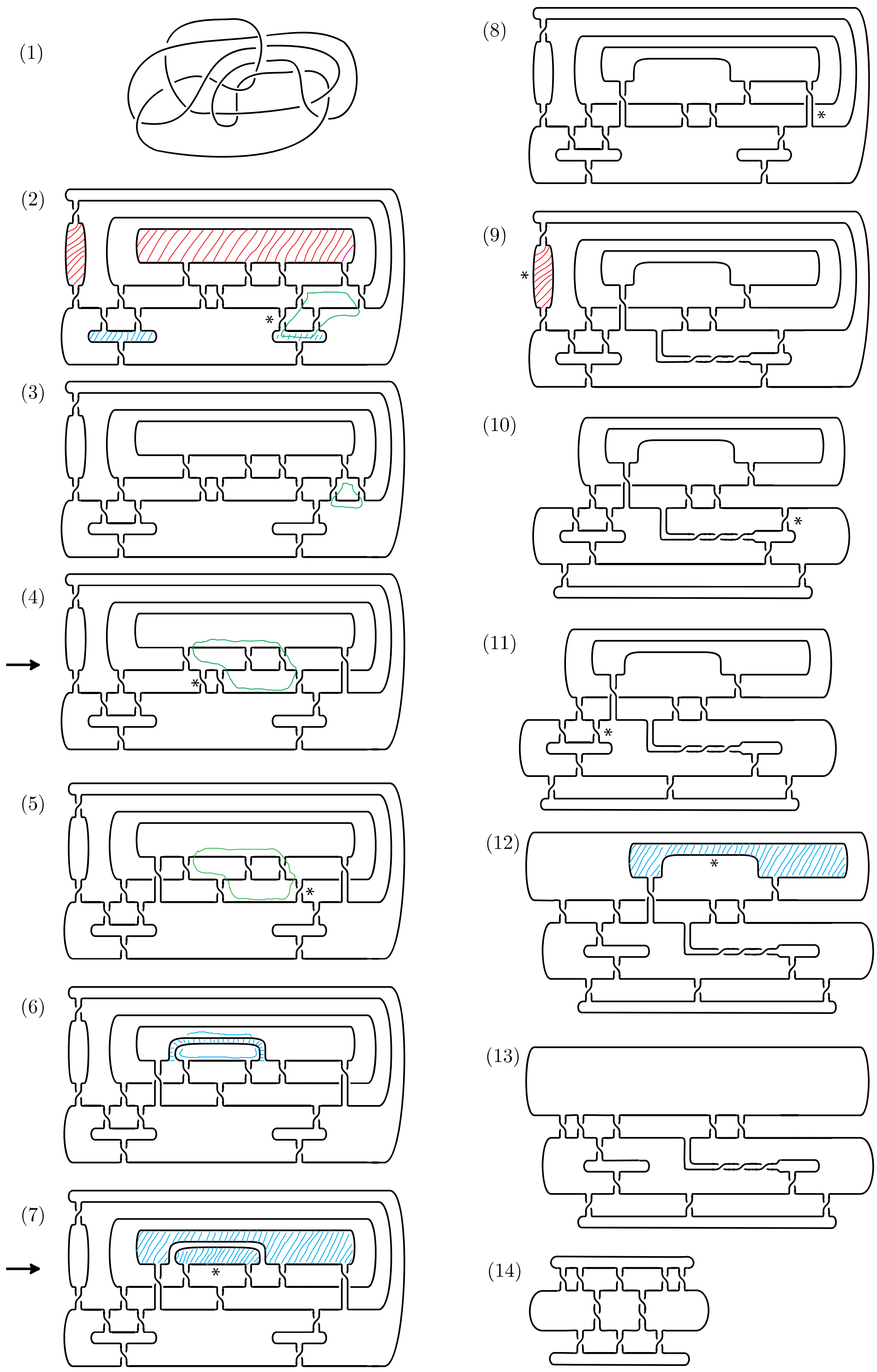

Figure 14: Making a strongly invertible diagram using Seifert surfaces

In Figure 14, we start with a diagram of the knot mentioned in Introduction. Figure 14 (1) has 17 crossings, and as seen in Figure 2, it has 6 Seifert surfaces. So the canonical Seifert surface has genus $(17 - 6 + 1)/2 = 6$. First we try to make compressions so that we have a surface of lower genus. The loop in Figure 14 (2) is the boundary of a compressing disk. It is depicted as a slight push off from the surface and shrinks to a point in the complement of the surface. We move the band marked with $*$ so that the loop gets smaller, and we can apply compression from (3) though (4). In (4), we find another compressing disk boundary, with more complicated compressing disk before. We slide the bands marked with $*$ though (5) and (6) so that the surface has an untwisted band attached, where the compressing disk is obvious and the result (7) of a compression is cutting the surface along the core of the untwisted band. At this moment, we can conclude this surface of genus 4 is a fiber surface, since it is a plumbing of Hopf bands. We keep moving the bands marked with $*$ to make the diagram simpler. Sometimes we slide the feet of the bands in the opposite way along the boundary of the base surface, as in (12) to (13). By Theorem 6.2, a fiber surface is preserved by a strong involution. Note that fiber surfaces of a fibered knot are unique. So this surface (7) can be isotoped so that it is preserved by a strong involution. But the drawback a fiber surface is that a Hopf band can be hidden in the surface in many ways, as in Figures 8 and 9. Anyway, we obtained (14) which is symmetric. As a conclusion, this knot has a symmetric diagram with the minimal crossing number 12. We leave it as an exercise for the reader to show that the diagram from the knotInfo (Figure 1 (3)) is deformed into the symmetric diagram of Figure 14 (14) by applying Reidemeister III move twice.

There are many techniques in finding a compressing disk and deforming the band. Only some is shown in the pictures above, and there are other techniques to reduce the genus of a surface other than compressions. I hope someday, these techniques are composed into a technology and are implemented in a computer program. Until then or even after that, the author hopes people keep practicing construction and deformation of surfaces.

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