Upsilon and secondary Upsilon invariants of L–space knots

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1 Introduction

This is a survey of two recent papers [25] and [26], where we examine Upsilon and secondary Upsilon invariants for L–space knots.

1.1 Upsilon invariants

In 2017, Ozsváth, Stipsicz and Szabó [23] introduced the Upsilon invariant for any knot in the 3–sphere S^3. For a knot K, the Upsilon invariant $\Upsilon_K(t)$ is a piecewise linear function defined on the interval $[0, 2]$. As the simplest example, let K be the right handed trefoil. Then $\Upsilon_K(t) = -t$ $(0 \le t \le 1), t-2$ $(1 \le t \le 2)$ (see Figure 1).

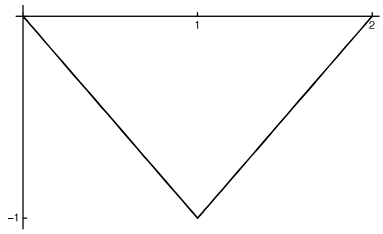

Figure 1: The Upsilon invariant of the right handed trefoil.

The Upsilon invariant has various good properties:

1. $\Upsilon_K(t)$ is a (smooth) concordance invariant.
2. $\Upsilon_K(t) = \Upsilon_K(2-t)$ for $t \in [0, 2]$. That is, it is symmetric about $t = 1$.
3. $\Upsilon_K(0) = 0$.
4. $\Upsilon'(0) = -\tau(K)$, where $\tau(K)$ is the Ozsváth–Szabó τ–invariant.
5. $\Upsilon_{K\#L}(t) = \Upsilon_K(t) + \Upsilon_L(t)$.
6. $\Upsilon_{-K}(t) = -\Upsilon_K(t)$, where $-K$ is the mirror image with reversed orientation.
7. If K is smoothly slice, then $\Upsilon_K(t) = 0$.

8. $\Upsilon_K(t)$ gives lower bounds for the genus, the 4–genus, and the concordance genus.
9. $|\Upsilon_K(1) - \sigma(K)/2| \leq \gamma_4(K)$, where $\gamma_4(K)$ is the smooth 4–dimensional crosscap number.

By using the Upsilon invariants, Ozsváth, Stipsicz and Szabó [23] show that the subgroup of topologically slice knots in the concordance group contains a summand of infinite rank.

For some classes of knots, the Upsilon invariants can be explicitly computed. Let K be an alternating knot (or, more generally, a quasi-alternating knot). Then $\Upsilon_K(t) = (1-|t-1|)\cdot\frac{\sigma(K)}{2}$, where $\sigma(K)$ is the signature of K [23]. For (positive) torus knots, there is an inductive formula [10]. For L–space knots, defined below, there is a description in terms of some integer sequence extracted from the Alexander polynomial [23].

Although the Upsilon invariant was originally defined by using a t–modified knot Floer complex in [23], Livingston [20] later proposed an alternative way on the usual knot Floer complex $\mathrm{CFK}^\infty(K)$. Since then, it seems that most authors adopt Livingston's definition of Upsilon invariant in their research. There is also a grid diagram approach to the Upsilon invariant in [11].

The (full) knot Floer complex $\mathrm{CFK}^\infty(K)$ is a $\mathbb{Z}\oplus\mathbb{Z}$–filtered graded chain complex over the polynomial ring $\mathbb{F}[U, U^{-1}]$, where $\mathbb{F} = \mathbb{Z}_2$ and U is a formal variable, with Maslov (homological) grading and two filtrations called the algebraic (abbreviated as "alg") and Alexander ("Alex") filtrations. The action of U commutes with differential, lowers gradings by 2, and lowers algebraic and Alexander filtrations by 1. It is known that $\mathrm{CFK}^\infty(K)$ is a knot invariant up to graded chain homotopy equivalence. There is a diagrammatic description of $\mathrm{CFK}^\infty(K)$ on a coordinate plane, but we will give it only for L–space knots later (see 1.4). If two knots K_1 and K_2 are concordant, then their knot Floer complexes are stably equivalent. For $-K$, $\mathrm{CFK}^\infty(-K)$ is the dual of $\mathrm{CFK}^\infty(K)$, whose diagrammatic description is simply obtained by rotating that of $\mathrm{CFK}^\infty(K)$ by 180 degree and reversing all maps. There is the global triviality, that is, $H_n(\mathrm{CFK}^\infty(K)) = \mathbb{F}$ if n is even, and 0 otherwise.

1.2 L–space knots

A knot K is called an *L–space knot* if it admits a positive Dehn surgery yielding an L–space. An L–space Y is a rational homology 3–sphere whose (hat) Heegaard Floer homology has the possible smallest rank, which is equal to $H_1(Y)$. This gives an importance to clarify what an L–space is. The famous L–space conjecture [8] is such an attempt to characterize an L–space in terms of left-ordering or taut foliation. Lens spaces, more generally, Seifert fibered manifolds with finite fundamental groups are L–spaces. We remark that if a knot admits such a negative surgery, then take the mirror image, and that the unknot is the only one that admits both of positive and negative Dehn surgeries yielding L–spaces. Typical examples of L–space knots are torus knots, Berge knots. Among alternating knots, $T(2, 2n+1)$ $(n \geq 1)$ are the only L–space knots [22]. All L–space pretzel knots are determined by Lidman and Moore [19]. It is known that any L–space knot is prime and the Alexander polynomial is a concordance invariant among L–space knots [16]. Notice that any non-trivial L–space knot is not slice.

It is further known that any L–space knot is fibered, and its Alexander polynomial has a form of

$$\Delta(t) = 1 - t^{a_1} + t^{a_2} - \cdots + t^{a_n},$$

where $1 = a_1 < a_2 < \cdots < a_n = 2g(K)$. For example, the right handed trefoil and the $(-2,3,7)$–pretzel knot are L–space knots, and their Alexander polynomials are $1 - t + t^2$ and $1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}$, respectively. The fact that $a_1 = 1$ is highly non-trivial [12]. For L–space knots, the distribution of powers $a_1, a_2, \ldots$ is still mysterious, so its further exploration is expected.

1.3 Raising the problems and results

In this paper, we focus on L–space knots due to the following reasons.

1. For an L–space knot K, the Upsilon invariant $\Upsilon_K(t)$ is the Legendre–Fenchel transform (or, convex conjugate) of a certain function, called the gap function $G(x)$, which has the same information as the Alexander polynomial.
2. The knot Floer complex of an L–space knot has a very special form that is completely determined by the Alexander polynomial.

The first fact was proved by Borodzik and Hedden [7]. As an immediate consequence, we know that the Upsilon invariant of an L–space knot is convex. Since the Legendre–Fenchel transform depends only on the convex hull of the gap function $G(x)$, we come up with the following problems.

1. Find two L–space knots whose gap functions share the same convex full. Then their Upsilon invariants coincide.
2. Find an L–space knot whose Alexander polynomial can be restored from the Upsilon invariant through the Legendre–Fenchel transformation.

Of course, the Upsilon invariant is not strong to distinguish knots, because it is a concordance invariant. All slice knots share the same Upsilon invariant, which is the zero map. On the other hand, we see that there is no duplication of Upsilon invariant among torus knots.

Our first result is an answer to the first problem.

Theorem 1.1 *There are infinitely many pairs of hyperbolic L–space knots K_1 and K_2 such that K_1 and K_2 have distinct Alexander polynomials, but share the same (non-zero) Upsilon invariants.*

For the second problem, we have an answer, which is not satisfactory.

Theorem 1.2 *Let K be the hyperbolic L–space knot `t09847` or `v2871` in the SnapPy census. Then the Alexander polynomial $\Delta_K(t)$ of K is restorable from the Upsilon invariant $\Upsilon_K(t)$. That is, the equation $\Upsilon_K(t) = \Upsilon_{K'}(t)$ implies $\Delta_K(t) = \Delta_{K'}(t)$ (up to units) for any other L–space knot K'.*

Thus, the remaining problem is to find an infinite family of such hyperbolic L–space knots. It is not hard to give an infinite family of (potentially, Alexander) polynomials, whose gap function is restorable from the convex hull as follows.

Proposition 1.3 *Let $m \geq 3$ be an integer, and let $\Delta(t) = 1 - t + t^m - t^{m+1} + t^{m+2} - t^{2m+1} + t^{2m+2}$. Then its gap function, defined formally, is uniquely determined from the convex hull.*

The polynomial $\Delta(t)$ in Proposition 1.3 satisfies the condition of [17], but it is open whether $\Delta(t)$ is realized as the Alexander polynomial of a hyperbolic L–space knot or not. (When $m = 3$, $\Delta(t)$ is the Alexander polynomial of $T(3,5)$.)

1.4 Secondary Upsilon invariants

To recover a lost information in the Upsilon invariant, Kim and Livingston [15] introduced the secondary Upsilon invariant $\Upsilon^2_{K,t}(s)$ for $t \in (0,2)$ and $s \in [0,2]$. The derivative $\Upsilon'_K(t)$ has finitely many isolated singular points. For example, $\Upsilon'_K(t)$ has singularity at $t = 1$ in Figure 1. The secondary Upsilon invariant is essentially defined at each singularity of $\Upsilon'_K(t)$. It is also a concordance invariant.

Although the secondary Upsilon invariant can be defined for any knot, we restrict ourselves to L–space knots again. To define it, we now need to give the diagrammatic description of the knot Floer complex. The knot Floer complex $\mathrm{CFK}^\infty(K)$ of an L–space knot has a restricted form. We are going to explain the description by using an example.

Let K be the torus knot $T(3,7)$. The Alexander polynomial is $1-t+t^3-t^4+t^6-t^8-t^9+t^{11}-t^{12}$. We record the gaps between the powers as the sequence $[1,2,1,2,2,2,1,2,1]$. Obviously, the symmetry of this sequence is derived from that of the Alexander polynomial. On the (alg, Alex)–plane, put the first black vertex at $(0,g)$, where $g = 6$ is the genus of K. That is, the filtration level of this vertex is $\mathrm{alg} = 0$ and $\mathrm{Alex} = g$. According to the sequence, we go right or down. Hence, go one right step and put a white vertex, and go down two steps and put a black vertex. Repeat this process until we reach $(g,0)$. Finally, draw arrows from each white vertex to adjacent black vertices. This is called the staircase diagram. Figure 2 shows the staircase diagram for $K = T(3,7)$. Each black vertex has Maslov grading 0, but white one has grading 1, and the arrows show boundary maps. Thus each black vertex is a 0–cycle, and represents a generator of $H_0(\mathrm{CFK}^\infty(K))$. All black vertices are homologous. In fact, the full complex is obtained by taking all integer diagonal translates of the staircase diagram. That is, the action of the variable U shifts the vertices a distance of one down and to the left. However, we do not need this structure.

SnapPy [9] can exhibit the knot Floer complex. For example, the input

```
K=Link(braid_closure=[1,2,1,2,1,2,1,2,1,2,1,2,1,2])
K.knot_floer_homology(complex=True}
```

returns 9 generators

$$\begin{array}{lll} x_0 = (-2,-7) & x_1 = (-3,-8) & x_2 = (-5,-11) \\ x_3 = (-6,-12) & x_4 = (0,-4) & x_5 = (5,-1) \\ x_6 = (3,-2) & x_7 = (2,-3) & x_8 = (6,0) \end{array}$$

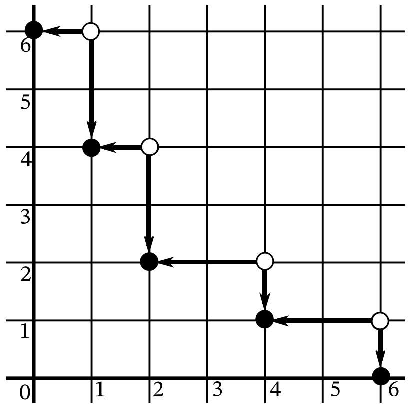

Figure 2: The staircase diagram of the torus knot $T(3,7)$. The corresponding sequence is $[1,2,1,2,2,1,2,1]$.

and differentials

$$\begin{array}{lllll} x_0 \to x_1 & x_0 \to x_4 & x_2 \to x_1 & x_2 \to x_3 & x_5 \to x_6 \\ x_5 \to x_8 & x_7 \to x_4 & x_7 \to x_6 & & \end{array}$$

for $K = T(3,7)$. (SnapPy returns only horizontal and vertical differentials. For non-L–space knots, pay attention to this omission.) For the generators, the first entry is the Alexander grading, but the second is the Maslov grading. All have algebraic grading 0. Thus put these generators on the j–axis. To draw the knot Floer complex, shift $x_0, \dots, x_8$ into the first quadrant by the action of U as follows:

$$\begin{array}{lll} U^{-4}x_0 = (2,1) & U^{-4}x_1 = (1,0) & U^{-6}x_2 = (1,1) \\ U^{-6}x_3 = (0,0) & U^{-2}x_4 = (2,0) & U^{-1}x_5 = (6,1) \\ U^{-1}x_6 = (4,0) & U^{-2}x_7 = (4,1) & x_8 = (6,0) \end{array}$$

Then $U^{-4}x_1$, $U^{-6}x_3$, $U^{-2}x_4$, $U^{-1}x_6$ and x_8 are black vertices.

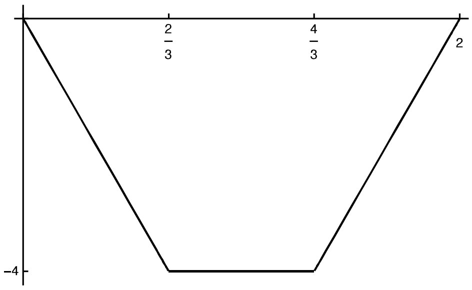

Figure 3: The Upsilon invariant of $T(3,7)$. $\Upsilon'(t)$ is singular at $t = 2/3$ and $4/3$.

Figure 3 illustrates $\Upsilon_K(t)$. Hence $\Upsilon'_K(t)$ is singular at $t = 2/3$ and $4/3$.

Fix $t_0 = 2/3$. On the (alg, Alex)–plane, consider the line L with slope $1 - 2/t_0 = -2$. Move L upwards from south-west, and stop at the first touch with the staircase. Then L

contains three black vertices. Among them, the top most $p^- = (0,6)$ is called the negative pivot point, and the bottom most $p^+ = (2,2)$ is called the positive pivot point. In general, if $\Upsilon'_K(t)$ is singular at t, then the corresponding pivot points are different.

Next, consider the part $\mathbb{S}$ of the staircase diagram between two pivot points. For $s \in [0,2]$, let L_s be the line with slope $1 - 2/s$ touching $\mathbb{S}$ from north-east. (In particular, L_0 is vertical.) Let ξ be the intercept of L_s when $s \neq 0$. Then set

$$\Upsilon^2_{K,t_0}(s) = \begin{cases} -s\,\xi - \Upsilon_K(t_0) & (s \neq 0), \\ -2\,\mathrm{alg}\,(p^+) - \Upsilon_K(t_0) & (s = 0), \end{cases}$$

where $\mathrm{alg}(p^+)$ denotes the algebraic filtration of p^+. This is the secondary Upsilon invariant at the singularity t_0. (The original definition is different, but our definition is equivalent to it for L–space knots.) For any non-singular t, $\Upsilon^2_{K,t}(s)$ is set to be ∞.

In our case, $\Upsilon_K(t_0) = \Upsilon(2/3) = -4$. Hence it is easy to see that

$$\Upsilon^2_{K,t_0}(s) = \begin{cases} -2s & (0 \leq s \leq 2/3), \\ -5s + 2 & (2/3 \leq s \leq 2). \end{cases}$$

We have a simple observation that the secondary Upsilon invariant is a concave conjugate of a certain restriction of the gap function.

Define $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(x,y) = (x - y, 2x)$. For $p \in \mathbb{R}^2$, let $\Phi_1(p)$ denote the first coordinate of $\Phi(p)$.

Theorem 1.4 *Let K be an L–space knot and $\Upsilon_K(t)$ its Upsilon invariant. Let $t_0 \in (0,2)$ be a singularity of $\Upsilon'_K(t)$, and let p^- and p^+ be the corresponding negative and positive pivot points on the staircase diagram. Then the secondary Upsilon invariant $\Upsilon^2_{K,t_0}(s)$ at t_0 is given by*

$$\Upsilon^2_{K,t_0}(s) = G^*(s) - \Upsilon_K(t_0),$$

where $G^*(s)$ is the concave conjugate of the restriction of the gap function $G(x)$ on the interval $I = [\Phi_1(p^-), \Phi_1(p^+)]$.

Here, $G^*(s) = \min_{x \in I}\{sx - G(x)\}$. Since $s \in [0,2]$ and $G(x)$ is bounded on I, the minimum value exists.

2 Our construction

In this section, we sketch the construction of pairs of hyperbolic L–space knots for Theorem 1.1.

For each $n \geq 1$, consider the closures of 4–braids:

$$K_1 : (\sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_1\sigma_2\sigma_3)^{4n}\sigma_2^{-1}(\sigma_2\sigma_3)^6,$$

$$K_2 : (\sigma_2\sigma_1\sigma_3\sigma_2)(\sigma_1\sigma_2\sigma_3)^{4n}\sigma_3^{-1}(\sigma_2\sigma_3)^6.$$

Since both are transformed into the closures of positive braids, they are fibered, and have genus $6n+6$. We remark that when $n = 1$, K_1 and K_2 are `m240` and `t10496`, respectively, in the SnapPy census.

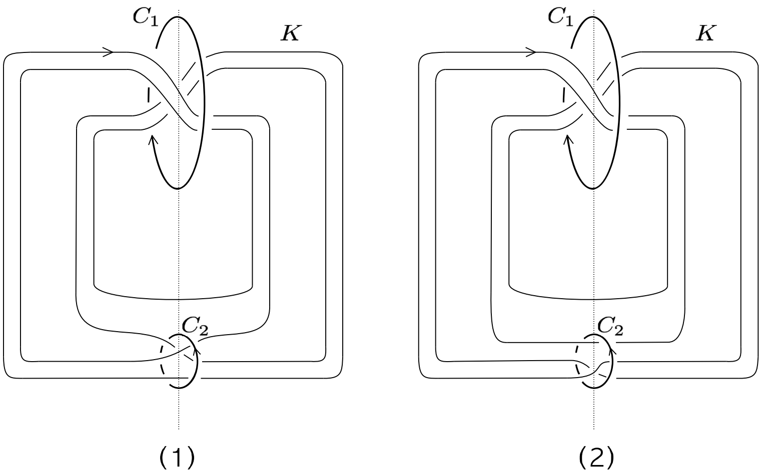

Figure 4: The knots K_1 (left) and K_2 (right). After performing $-1/n$–surgery on C_1 and $-1/2$–surgery on C_2, K will be our knots.

Proposition 2.1 *For $n \geq 1$, K_1 and K_2 satisfy the following*:

(1) *They are hyperbolic.*

(2) $(16n+21)$*–surgery on* K_1, $(16n+20)$*–surgery on* K_2 *yield* L*–spaces.*

(3) *Their Alexander polynomials are distinct.*

(4) *Their Upsilon invariants coincide.*

We will sketch the outline of the proof.

As seen from Figure 4, both knots are strongly invertible. The Montesinos trick [21] reveals that the above surgeries yield L–spaces. In fact, $(16n+21)$–surgery on K_1 yields the Seifert fibered manifold $M(0;-3/7,-1/3,-1/n)$, which is shown to be an L–space by a well known criterion. On the other hand, $(16n+20)$–surgery on K_2 seems to be not a Seifert fibered manifold. By using resolutions, it will be shown to be an L–space.

Next, we calculate the multivariable Alexander polynomials of the 3–component links $K \cup C_1 \cup C_2$ as shown in Figure 4. Perform the surgeries on C_1 and C_2. Then the Torres condition [27] gives the desired Alexander polynomials of K_1 and K_2, which are seen to be distinct.

From the Alexander polynomials, we can determine the formal semigroups of K_1 and K_2. In general, the formal semigroup is defined for any L–space knot K [28]. Let $\Delta_K(t)$ be the Alexander polynomial of the form $\Delta(t) = 1 - t^{a_1} + t^{a_2} - \cdots + t^{a_n}$. Then expand the Alexander function into a formal power series as

$$\frac{\Delta_K(t)}{1-t} = \sum_{s \in \mathcal{S}} t^s.$$

The set $\mathcal{S}$ is a subset of non-negative integers, which is called the formal semigroup of K. For example, the formal semigroup of a positive torus knot $T(p,q)$ is an actual semigroup $\langle p,q \rangle = \{ap+bq \mid a,b \geq 0\}$. For hyperbolic L–space knots, it is hardly a semigroup.

For our knots K_1 and K_2, their formal semigroups are not semigroups. This immediately implies that neither is a torus knot. For example, the formal semigroup of K_1 with $n = 1$ is

$$\mathcal{S} = \{0, 4, 7, 10, 11, 14, 15, 17, 18, 20, 21, 22\} \cup \{24, 25, 26, \dots\}.$$

Hence $4 \in \mathcal{S}$, but $4 + 4 = 8 \notin \mathcal{S}$. Similarly, the formal semigroup of K_2 with $n = 1$ is

$$\mathcal{S} = \{0, 4, 7, 10, 12, 14, 15, 17, 18, 20, 21, 22\} \cup \{24, 25, 26, \dots\}.$$

This is not a semigroup either.

To confirm their hyperbolicity, suppose that K_i is a satellite knot. Since K_i has bridge number 4, the companion is a 2–bridge knot and the pattern has wrapping number 2. Also, the companion knot and the pattern knot are L–space knots [14, 3]. In addition, the pattern knot is braided [3]. Thus, we can conclude that the companion is a 2–bridge torus knot and K_i is its 2–cable. By [28], the formal semigroup of an iterated torus L–space knot is a semigroup, which is a contradiction.

Finally, to confirm that K_1 and K_2 share the Upsilon invariants, we need to determine their gap functions, and verify that their convex hull coincide.

Here is the definition of gap function. Let K be an L–space knot with formal semigroup $\mathcal{S}$. Put $\mathcal{G} = \mathbb{Z} - \mathcal{S}$, which is called the gap set. In facf, $\mathcal{G} = \mathbb{Z}_{<0} \cup \{b_1, b_2, \dots, b_g\}$, where $g = g(K)$ and $0 < b_1 < b_2 < \cdots < b_g$. The sequence $b_1, b_2, \dots, b_g$ is called a gap sequence.

Then the Alexander polynomial is restored as

$$\Delta_K(t) = 1 + (t-1)(t^{b_1} + t^{b_2} + \cdots + t^{b_g}).$$

By using the gap set $\mathcal{G}$, we define a function $I\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ by

$$I(m) = \#\{i \in \mathcal{G} \mid i \geq m\},$$

and let $J(m) = I(m + g)$. Next, we extend $J(m)$ linearly to a piecewise linear function on $\mathbb{R}$, and set $G(x) = 2J(-x)$. This is the gap function of K by [7]. (See Figure 5 for an example.) Borodzik and Hedden show that $\Upsilon_K(t)$ is the Legendre–Fenchel transform of $G(x)$, that is

$$\Upsilon_K(t) = \max_{x \in \mathbb{R}} \{tx - G(x)\}.$$

From the gap sets of K_1 and K_2, we can confirm that their gap functions share the same convex hull.

3 Restorable Alexander polynomials

Here are the braid words of our knots mentioned in Theorem 1.2. Both knots are the closures of positive 4–braids.

$$\texttt{t09847} : (\sigma_2\sigma_1\sigma_3\sigma_2)^3(\sigma_2\sigma_1^2\sigma_2)\sigma_1,$$

$$\texttt{v2871} : (\sigma_2\sigma_1\sigma_3\sigma_2)^3(\sigma_2\sigma_1^2\sigma_2)\sigma_1^3.$$

Their Alexander polynomials are $1 - t + t^4 - t^5 + t^7 - t^9 + t^{10} - t^{13} + t^{14}$ and $1 - t + t^4 - t^5 + t^7 - t^8 + t^9 - t^{11} + t^{12} - t^{15} + t^{16}$, respectively, Hence the formal semigroups are $\{0, 4, 7, 8, 10, 11, 12\} \cup \mathbb{Z}_{\geq 14}$ and $\{0, 4, 7, 9, 10, 12, 13, 14\} \cup \mathbb{Z}_{\geq 16}$. By using them, we can determine the gap functions and verify that they are uniquely determined from the convex hulls.

4 Secondary Upsilon invariants

Figure 5 shows the gap function $G(x)$ of $T(3, 7)$.

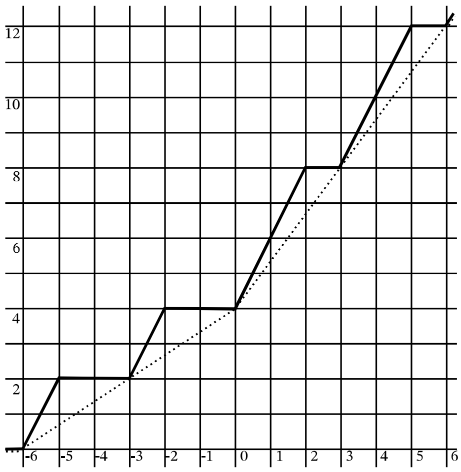

Figure 5: The gap function $G(x)$ of $T(3, 7)$, where $g = 6$.

The key observation to prove Theorem 1.4 is the fact that the graph of $G(x)$ restricted on $[-g, g]$, where $g = g(K)$, is the image of the staircase diagram under the map Φ defined in Subsection 1.4. (Compare Figures 2 and 5.)

In Figure 5, the graph of the gap function $G(x)$ consists of segments with slope 0 or 2. Let $\boldsymbol{a} = (1, 2)$ and $\boldsymbol{b} = (1, 0)$ be the vectors. Starts at the point $(-6, 0)$, where $g(T(3, 7)) = 6$. According to the sequence $[1, 2, 1, 2, 2, 1, 2, 1]$, which records the gaps of powers of the Alexander polynomial, we use $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus the structure of the graph of $G(x)$ coincides with that of the staircase diagram.

5 Comments

Recently, some generalizations of Upsilon invariant emerged.

- Use a south–west region on the plane [1].
- Sato's invariant $\mathcal{G}_0$ [24].
- Extend Upsilon-type invariants to null-homologous knots in rational homology 3–spheres, in particular, cyclic branched covers of knots [2].
- Use involutive Heegaard Floer homology [13].

- Extend to balanced spatial graphs by using grid homology [18].

In this note, we restricted ourselves to L–space knots, but Baldwin and Sivek [4] introduce almost L–space knots which can yield almost L–spaces by Dehn surgery. Also, Binns [5, 6] examines their knot Floer complexes. It might be interesting to study their Upsilon invariants.

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