On the vanishing and the non-vanishing of the twisted Alexander polynomial

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1 Introduction

Let K be a knot in S^3. It is well known that the Alexander polynomial $\Delta_K(t)$ is not zero for any knot K. We see it from $\Delta_K(1) = \pm 1$, for example. Then it is natural to ask whether the twisted Alexander polynomial would not be zero.

Friedl and Vidussi showed that for any 3-manifold N and any non-fibered class in $H^1(N;\mathbb{Z})$ there exists a representation such that the corresponding twisted Alexander polynomial is zero. In this paper, we provide several concrete examples with zero twisted Alexander polynomial and some explicit formulas of the twisted Alexander polynomial.

Throughout this paper, we adopt Rolfsen's table [9] to represent a prime knot with 10 or fewer crossings.

2 The twisted Alexander polynomial and the theorem of Friedl-Vidussi

In [11], the twisted Alexander polynomial is defined for finitely presentable group. In this section, we review quickly the definition of the twisted Alexander polynomial of a knot. See [11] for the precise definition and more general settings.

Let $G(K)$ be the knot group of a knot K, namely, the fundamental group of the exterior of K in S^3. We take the Wirtinger presentation

$$G(K) = \langle x_1, x_2, \ldots, x_m \,|\, r_1, r_2, \ldots, r_{m-1} \rangle$$

according to a diagram of K, where x_i represents a meridian of K. The abelianization $\alpha : G(K) \to \mathbb{Z} \simeq \langle t \rangle$ sends x_i to t.

Definition 2.1 *The twisted Alexander polynomial of K associated to a linear representation $\rho : G(K) \to GL(n, \mathbb{Z})$ is defined by*

$$\Delta_K^{\rho}(t) = \frac{\det\left((\rho \otimes \alpha)\frac{\partial r_i}{\partial x_j}\right)_{1 \le i,j \le m-1}}{\det\big((\rho \otimes \alpha)(x_m - 1)\big)}$$

where $\frac{\partial}{\partial x_j}$ is the Fox derivation with respect to x_j.

Remark that the twisted Alexander polynomial is defined up to multiplication of $\pm t^k$ for some $k \in \mathbb{Z}$ and that the twisted Alexander polynomial is not always a genuine polynomial.

For a finite group G and for a homomorphism $f : G(K) \to G$, we can consider the twisted Alexander polynomial $\Delta_K^{\rho\circ f}(t)$, where $\rho : G \to GL(|G|, \mathbb{Z})$ is the regular representation of G. In this situation, Friedl and Vidussi showed the following distinguished theorem.

Theorem 2.2 (Friedl-Vidussi [3]) *A knot K is non-fibered if and only if there exists a finite group G and a surjective homomorphism $f : G(K) \to G$ such that the twisted Alexander polynomial $\Delta_K^{\rho\circ f}(t)$ is zero.*

Then in [6] we define the minimal order $\mathcal{O}(K)$ of a knot K as the smallest order of a finite group G such that there exists a surjective homomorphism $f : G(K) \to G$ with $\Delta_K^{\rho\circ f}(t) = 0$. By Theorem 2.2, $\mathcal{O}(K)$ is finite for any non-fibered knot K. On the other hand, we define $\mathcal{O}(K) = +\infty$ for a fibered knot K.

3 The minimal order $\mathcal{O}(K)$

In general, it is not easy to determine $\mathcal{O}(K)$ for a given non-fibered knot K. However, we obtain the lower bound and the explicit values for some knots.

Theorem 3.1 *For any knot K, we have $\mathcal{O}(K) \geq 24$.*

There are 59 finite groups of order less than 24. We need to show that the twisted Alexander polynomials are not zero for all of them. The following is a sketch of the proof, see [7] in detail. First, if a group G is not normally generated by one element, then there does not exist a surjective homomorphism $f : G(K) \to G$. It implies that we do not need to consider such finite groups to show our statement. Next, it is easy to see that $\Delta_K^{\rho\circ f}(t)$ is not zero for any abelian group. Similarly, we see that $\Delta_K^{\rho\circ f}(t)$ is not zero for any dihedral group D_{p^n} of order $2p^n$ and for any dicyclic group Dic_{p^n} of order $4p^n$, where p is an odd prime number. The remaining groups of order less than 24 are $A_4, D_3 \times C_3, D_3 \ltimes C_3, C_4 \ltimes C_5, C_3 \ltimes C_7$, where C_n is the cyclic group of order n. Finally, we show that $\Delta_K^{\rho\circ f}(t)$ are not zero for these 5 finite groups separately.

Theorem 3.2 *For any prime knot K with up to 10 crossings, we have*

- $\mathcal{O}(K) = 24$, *if* $K = 9_{35}, 9_{46}$,
- $\mathcal{O}(K) = 60$, *if* $K = 10_{67}, 10_{120}, 10_{146}$,
- $\mathcal{O}(K) = 96$, *if* $K = 10_{166}$,
- $\mathcal{O}(K) = 120$, *if* $K = 8_{15}, 9_{25}, 9_{39}, 9_{41}, 9_{49}, 10_{58}$,
- $\mathcal{O}(K) \geq 125$, *otherwise* .

The following is a sketch of the proof, see [6] and [7] in detail. We have $\Delta_K^{\rho\circ f}(t)$ of all the non-fibered knots with up to 10 crossings for all the finite groups of order up to 120, with aid of computer. These computations give us that $\Delta_K^{\rho\circ f}(t)$ are zero for

$$\begin{aligned}
&f : G(9_{35}) \to S_4, f : G(9_{46}) \to S_4,\\
&f : G(10_{67}) \to A_5, f : G(10_{120}) \to A_5, f : G(10_{146}) \to A_5,\\
&f : G(10_{166}) \to S_4 \ltimes C_2^2,\\
&f : G(8_{15}) \to S_5, f : G(9_{25}) \to S_5, f : G(9_{39}) \to S_5,\\
&f : G(9_{41}) \to S_5, f : G(9_{49}) \to S_5, f : G(10_{58}) \to S_5
\end{aligned}$$

and that $\Delta_K^{\rho\circ f}(t)$ are not zero for any other finite groups of order less than 24 (respectively 60, 96, 120). Moreover, we see that $\Delta_K^{\rho\circ f}(t)$ of the other knots are not zero for any finite groups of order less than 125.

We can construct infinitely many knots K such that there exist surjective homomorphisms $\varphi : G(K) \to G(9_{35})$ (or $\varphi : G(K) \to G(9_{46})$). These knots K are non-fibered, since Silver-Whitten in [10] showed that the knot group of a fibered knot never surjects to that of a non-fibered knot. Moreover, Kitano-Suzuki-Wada in [5] showed that if there exists a surjective homomorphism $\varphi : G(K) \to G(K')$, then the twisted Alexander polynomial of K contains that of K' as a factor. Therefore we have the following corollary.

Corollary 3.3 *There are infinitely many non-fibered knots K with $\mathcal{O}(K) = 24$.*

4 Some formulas of the twisted Alexander polynomial

In the previous section, we considered only whether the twisted Alexander polynomial is zero or not. In this section, we discuss the explicit formulas of the twisted Alexander polynomial for some classes of finite groups.

First, we consider abelian groups. In this case, the twisted Alexander polynomial $\Delta_K^{\rho\circ f}(t)$ can be described in terms of the (classical) Alexander polynomial $\Delta_K(t)$.

Proposition 4.1 *1. Let A be an abelian group. There exists a surjective homomorphism $f : G(K) \to A$ if and only if A is a cyclic group.*

2. For a surjective homomorphism $f : G(K) \to C_n$ and the regular representation $\rho : C_n \to GL(n, \mathbb{Z})$, the twisted Alexander polynomial is given by

$$\Delta_K^{\rho\circ f}(t) = \prod_{j=1}^{n} \left(\frac{\Delta_K(\alpha^j t)}{\alpha^j t - 1} \right)$$

where $\alpha \in \mathbb{C}$ is a primitive n-th root of unity.

In particular, it follows immediately that the twisted Alexander polynomials are never zero for abelian groups.

Next, we obtain the following formula of the twisted Alexander polynomials for the dihedral groups.

Theorem 4.2 *Let p be an odd prime number and $q = p^n$. We denote by $\rho : D_q \to GL(2q, \mathbb{Z})$ the regular representation of D_q. If there exists a surjective homomorphism $f : G(K) \to D_q$, then*

$$\Delta_K^{\rho\circ f}(t) \equiv \left(\frac{\Delta_K(t)}{t-1} \cdot \frac{\Delta_K(-t)}{t+1}\right)^q \pmod{p}.$$

Let $G(m, p|k)$ be a finite group of order mp defined by the following presentation:

$$G(m, p|k) = \langle x, y \,|\, x^m = y^p = 1,\, xyx^{-1} = y^k \rangle$$

where p is an odd prime number, $m \in \mathbb{N}$ such that $p \equiv 1 \pmod{m}$, and k is a primitive m-th root of unity $\pmod{p}$. In [2], Fox called this group K-metacyclic group when $m = p-1$. Hirasawa-Murasugi and Boden-Friedl discussed the twisted Alexander polynomial for $G(m, p|k)$ in [4], [1] respectively. Note that $G(m, p|k)$ is a semi-direct product $C_m \ltimes C_p$ and that $G(2, p|p-1)$ is the dihedral group D_p.

Theorem 4.3 *Let $k_j \in \{1, 2, \ldots, p-1\}$ be m-th roots of unity $\pmod{p}$, namely, ${k_j}^m \equiv 1 \pmod{p}$ $(j = 1, 2, \ldots, m)$ and $\alpha \in \mathbb{C}$ a primitive m-th root of unity. We denote by $\rho : G(m, p|k) \to GL(mp, \mathbb{Z})$ the regular representation of $G(m, p|k)$. If there exists a surjective homomorphism $f : G(K) \to G(m, p|k)$, then*

$$\Delta_K^{\rho\circ f}(t) \equiv \prod_{j=1}^{m}\left(\frac{\Delta_K(k_j t)}{k_j t - 1}\right)^{p-1} \cdot \prod_{j=1}^{m}\left(\frac{\Delta_K(\alpha^j t)}{\alpha^j t - 1}\right) \pmod{p}.$$

Finally, we get the twisted Alexander polynomial for the dicyclic group Dic_n of order $4n$ which is defined by the presentation:

$$\mathrm{Dic}_n = \langle a, x \,|\, a^{2n} = 1, x^2 = a^n, xax^{-1} = a^{-1} \rangle.$$

The dicyclic group Dic_n can be considered as an extension of C_2 by C_{2n}, namely, we have a short exact sequence:

$$1 \to C_{2n} \to \mathrm{Dic}_n \to C_2 \to 1.$$

Furthermore, $\mathrm{Dic}_n/\langle x^2 \rangle$ is isomorphic to D_n. Then Dic_n is also called the binary dihedral group.

Theorem 4.4 *Let p be an odd prime number and $q = p^n$. We denote by $\rho : \mathrm{Dic}_q \to GL(4q, \mathbb{Z})$ the regular representation of Dic_q. If there exists a surjective homomorphism $f : G(K) \to \mathrm{Dic}_q$, then*

$$\Delta_K^{\rho\circ f}(t) \equiv \left(\frac{\Delta_K(t)}{t-1}\right)^q \cdot \left(\frac{\Delta_K(-t)}{t+1}\right)^q \cdot \left(\frac{\Delta_K(it)}{it-1}\frac{\Delta_K(-it)}{it+1}\right)^q \pmod{p}.$$

See [8] for the proof of Theorem 4.2, 4.3, and 4.4.

Remark 4.5 *We obtain Theorem 3.1 by showing the non-vanishing of the twisted Alexander polynomial. Besides, we can also use Theorem 4.2, 4.3, and 4.4. Among 59 finite groups of order less than 24, there are 12 finite groups which are non-abelian and normally generated by one element. We can apply Theorem 4.2 for $D_3, D_5, D_7, D_9, D_{11}$ and Theorem 4.3 for $C_4 \ltimes C_5 = G(4, 5|2), C_3 \ltimes C_7 = G(3, 7|2)$ and Theorem 4.4 for $\mathrm{Dic}_3, \mathrm{Dic}_5$. They follow that the twisted Alexander polynomials are not zero. Then the remaining groups are only $A_4, D_3 \times C_3, D_3 \ltimes C_3$.*

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