SOME QUESTIONS ON PRIMITIVITY AND THOMPSON'S GROUP

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The FC Center of Thompson's Group F

R. Thompson introduced the group now known as Thompson's group F in 1965 as a potential counterexample to the von Neumann conjecture, which asserts that a group is amenable if and only if it has no free subgroup of rank 2. Thompson's group F was quickly found to have no such free subgroup, but the question of its amenability remains open, long after the von Neumann conjecture was settled in the negative by Ol'Shaskii [10]. The group F continues to be of broad interest, and a number of questions concerning its structure remain open.

For instance, it is not known for which domains R, if any, the group ring RF is right-primitive. Recall that a ring R is *right-primitive* if there exists a faithful irreducible right R-module.

Group ring primitivity saw its first majors results in the 1970's by Domanov [7], Farkas-Passman [8], and Roseblade [11], leading to a classification of primitivity for polycyclic-by-finite group rings over fields. More recently, the techniques of Formanek [9] have allowed Alexander-Nishinaka [2], the author [12], and Abbott-Dahmani [1] to expand the characterization of primitivity to an extremely broad class of group rings over countable domains.

A common theme of the above results is that the nontriviality of a particular characteristic subgroup is an obstruction to primitivity. The *FC center* of a group G, denoted $\Delta(G)$, is the set of elements of G with finite conjugacy class. It is thus natural to ask whether the FC center of Thompson's group F is trivial.

The following results on Thompson's group are well established in the literature. We follow Burillo's book [4], which is an excellent introduction to F and contains all of the results we will need here.

For our purposes, Thompson's group F is defined as the group of piecewise-linear homeomorphisms of the unit interval such that:

- the derivative is a power of two whenever it is defined, and
- the derivative is undefined only at finitely many dyadic rationals: numbers of the form $\frac{k}{2^n}$ for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

So defined, the group's binary operation is function composition.

The points at which $f \in F$ fails to be differentiable are called the *breakpoints* of f. As usual, the *support* of f, denoted $\operatorname{supp}(f)$, is the set of points that are not fixed by f.

It is well-known that the commutator subgroup $F' = [F, F]$ is simple [4, Theorem 3.3.1]. Consequently, every normal subgroup of F contains F' and every proper quotient of F is Abelian. The Abelianization map of F is particularly simple, given by

$$\begin{aligned} \phi &: F \to \mathbb{Z}^2 \\ f &\mapsto (\log_2(f'(0^+)), \log_2(f'(1^-)) \end{aligned}$$

.

Here, $f'(0^+)$ means the right-hand limit of f' at 0, and likewise for $f'(1^-)$ [4, Theorem 3.2.1]. The Abelianization map immediately yields a simple geometric characterization of F':

Proposition ([4, Theorem 3.2.2])**.** *The commutator subgroup F' is precisely the set of homomorphisms in F which are the identity when restricted to the neighborhoods of both 0 and 1.*

We can now prove the following:

Theorem. *The FC center of F is trivial.*

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Proof. Let $[a,b] \subseteq I$ be any closed interval with dyadic rational endpoints. The subgroup of elements of F with support contained in $[a,b]$ is denoted $F_{[a,b]}$. The scaling homeomorphism between $[0,1]$ and $[a,b]$ induces an isomorphism $F \to F_{[a,b]}$ [4, Theorem 3.1.3].

Let $\iota : F \to F_{[1/8,3/8]}$ be such an isomorphism. Since elements of $F_{[1/8,3/8]}$ restrict to the identity outside $[1/8,3/8]$, it is clear that $F_{[1/8,3/8]} \leq F'$.

Let $x_0 \in F$ be the piecewise homomorphism

$$x_0(t) = \begin{cases} 2t & 0 \leq t < 1/4 \\ t + 1/4 & 1/4 \leq t < 1/2 \\ t/2 + 1/2 & 1/2 \leq t \leq 1 \end{cases} .$$

Consider $\iota(x_0) \in F_{[1/8,3/8]}$. The leftmost breakpoint of $\iota(x_0)$ is at $t = \frac{1}{8}$. Direct calculation shows that $x_0^{-n}\iota(x_0)x_0^n$ has its leftmost breakpoint at $t = \frac{1}{8\cdot 2^n}$ for $n \in \mathbb{N}$. Therefore, the nontrivial element $\iota(x_0) \in F_{[1/8,3/8]} \subseteq F'$ has infinitely many conjugates.

If $\Delta(F)$ were a nontrivial normal subgroup, it would contain F' and hence $\iota(x_0)$, which contradicts that $\iota(x_0)$ has infinitely many conjugates. We conclude that $\Delta(F)$ must be the trivial subgroup. □

Primitivity and Cellular Automata

As the question of the primitivity of RF for a domain R remains open, we wish to introduce an alternative approach which may be of interest.

Let G be a group with finite generating set X. A *cellular automaton* on G consists of a finite set of states S, the Cayley graph $\Gamma = \Gamma_X(G)$ of G with respect to X, and a local map $\mu : S^{B_1(1)} \to S$, where $B_1(1)$ is the unit ball in Γ. Note that G is the vertex set for Γ, so we define a *configuration* to be a map $c : G \to S$. By applying the local map at each vertex of Γ, we construct a *transition map* $\tau : G^S \to G^S$.

Recent work by Ceccherini-Silberstein, Machi, and Scarabotti gives a cellular-automaton-theoretic characterization of the amenability of a discrete group G: G is amenable if and only if a Garden of Eden theorem holds for every cellular automaton on the group [6]. (A Garden of Eden theorem holds for a cellular automaton if the existence of Garden of Eden patterns is equivalent to the existence of mutually erasable patterns; see [6] for details.) Furthermore, results of Kielak and Bartholdi demonstrate that the amenability of G is directly tied to the existence of solutions to linear equations in its group ring over a field [3]. Recent preliminary work by Nishinaka suggests that the primitivity is also related to solutions to certain equations in RG, so it may be possible to use cellular-automaton-theoretic tools to explore the primitivity of certain group rings.

By allowing a cellular automaton on G to take on a domain R as its (infinite) state set and imposing R-linearity conditions on the space of configurations, we obtain a representation of RG in terms of R-linear cellular automata. This representation is already known to detect certain algebraic characteristics of R, for instance the presence or absence of zero divisors and an equivalences between amenability of G and Garden of Eden theorems for R-linear cellular automata [5]. Given this, we ask the following:

Question. *Does the primitivity of RG have a characterization in terms of R-linear cellular automata on G?*

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