Sphericity of the Standard Presentation of Fibonacci Group

Haruki Fukaura and Akihiro Yamamura
Akita University
1-1 Tegata-gakuen, Akita, 010-8502, Japan
m8022310@s.akita-u.ac.jp, yamamura@ie.akita-u.ac.jp

Abstract

We show that the standard presentation of the Fibonnaci group $F(2,n)$ with $n \geq 3$ is spherical, that is, it has a reduced spherical diagram. We discuss reducibility of the diagram.

1 Introduction

The *Fibonacci group* is introduced by Conway [4]. It is presented by

$$F(2,n) = \langle x_1, \ldots, x_n \mid x_1x_2 = x_3,\ x_2x_3 = x_4,\ \ldots,\ x_{n-1}x_n = x_1,\ x_nx_1 = x_2 \rangle. \tag{1.1}$$

It is known that this group has orders $1, 1, 8, 5, 11, 29$ when $n = 1, 2, 3, 4, 5, 7$ and the others have infinity orders. Regarding geometric properties, H. Helling, A.C. Kim and J. L. Mennicke [5] showed that for even $n \geq 8$, $F(2,n)$ are fundamental groups of certain closed hyperbolic 3-manifolds. In addition, they showed that $F(2,n)$ is Noetherian and torsion-free, every abelian subgroup of $F(2,n)$ is cyclic, and $F(2,n)$ have solvable word and conjugacy problems for even $n \geq 8$. C.P. Chalk proved in [3] that $F(2,n)$ is hyperbolic for odd $n \geq 11$. It is shown that $F(2,9)$ is also hyperbolic by using GAP package (for running the Knuth-Bendix completion program) KBMAG developed by D.F. Holt.

D.L. Johnson [6] introduced the *generalized Fibonacci group*

$$F(r,n) = \langle x_1, \ldots, x_n | x_ix_{i+1} \cdots x_{i+r-1} = x_{i+r},\ i \bmod n \rangle.$$

Many authors study the order of $F(r,n)$ for various parameters r and n and the classification has been completed. As a byproduct, the asphericity of $F(r,n)$ is shown for $(r,n) = (4+7k, 7), (6+9k, 9), (9+6k, 6), (l-1+(2l-1)k, 2l-1), (l+(l+2)k, l+2), (l+(2l)k, 2l)$ with $l \geq 4, k \geq 0$ in [2]. We report the irreducibility of the spherical diagrams over $F(2,n)$ in this paper.

2 Spherical diagrams over $F(2,n)$

Spherical diagrams of finitely presented groups are discussed in [8]. We follow terminologies and notations in the textbook. A reader is referred to the textbook for fundamental results on finitely presented groups and diagrams.

2.1 Cells of $F(2,n)$

The Fibonacci group is presented by (1.1). We define $w_i = x_ix_{i+1}x_{i+2}^{-1}$ for $i = 1, \ldots, n$ (where the indexes are and will be either $1, \ldots, n$ in mod n). Then the set of relators of $F(2,n)$ is $\{w_1, \ldots, w_n\}$. The $\mathscr{R}$-cell corresponding to each relator w_i is a triangle, and the three vertices are a_i, b_i, and c_i clockwise from the starting point of the label w_i. See Figure 1.

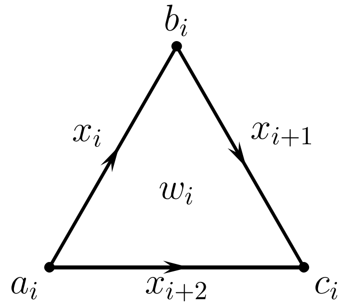

Figure 1: $\mathscr{R}$-cells of $F(2, n)$

2.2 Construction of spherical diagram

The case is divided according to the parity of n. First, we suppose n is odd. Let $n = 2k + 1$ ($k \geq 1$). For each $i = 1, \ldots, n$, all vertices a_i of $\mathscr{R}$-cells Π_i corresponding to w_i are pasted together to form vertex o. However, the order of arrangement is clockwise from Π_1, and for each i, Π_i is followed by Π_{i+2}. Then the adjacent cells Π_i, Π_{i+2} have boundary labels $x_i x_{i+1} x_{i+2}^{-1}, x_{i+2} x_{i+3} x_{i+4}^{-1}$ and the edges oc_i, ob_{i+2} matches the letter (x_{i+2}) and orientation. Thus, by pasting them together for each i, we obtain a circular diagram Δ. ($\partial\Delta = x_2 x_4 \cdots x_{2k} x_1 x_3 \cdots x_{2k+1}$ (cyclic word)).

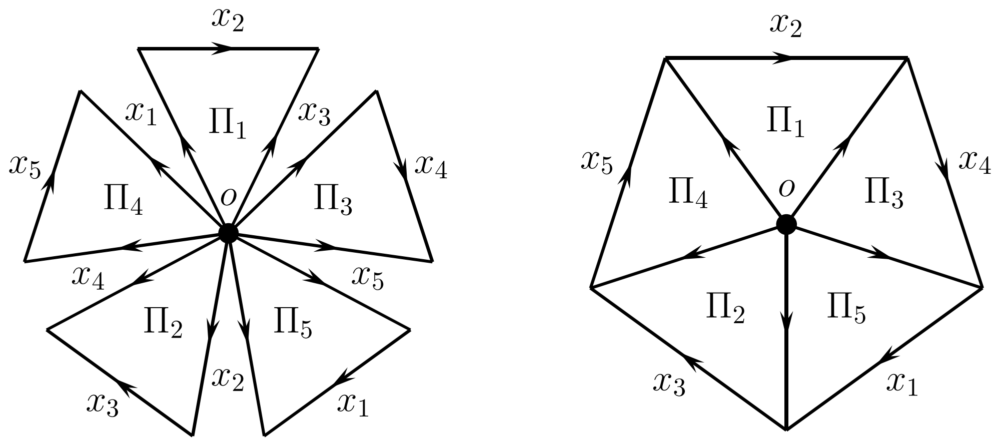

Figure 2: Δ of odd ($n = 5$)

Moreover, we can paste the edge $a_{i-1}b_{i-1}$ of the new $\mathscr{R}$-cell Π'_{i-1} corresponding to w_{i-1} onto the edge $b_i c_i$ on $\partial\Delta$ of each Π_i in Δ (since both letters are x_{i+1}). Therefore, by pasting $\Pi'_1, \ldots, \Pi'_n$ to each edge of $\partial\Delta$ in such a way, we obtain a circular diagram Δ'. Then the labels that the adjacent cells Π'_i, Π'_{i+2} of Δ' form on $\partial\Delta'$ are $x_i x_{i+1}, x_{i+2} x_{i+3}$, which means that $\partial\Delta' = x_1 \cdots x_n x_1 \cdots x_n$ (cyclic word). Finally, Δ' and its copy rotated by 1/2 are pasted together at their respective boundaries to obtain the spherical diagram $\Delta(n)$.

Next we suppose n is even. Let $n = 2k$ ($k \geq 2$). First, as in the odd case, the edges of the $\mathscr{R}$-cell Π_i corresponding to each w_i are pasted together so that clockwise Π_i, Π_{i+2} are adjacent to obtain a circular diagram. However, we get two diagrams Δ_1, which started to be arranged from Π_1, and Δ_2, which started to be arranged from Π_2. (They consist of $\Pi_1, \Pi_3, \ldots, \Pi_{2k-1}$ and $\Pi_2, \Pi_4, \ldots, \Pi_{2k}$ respectively and the boundary labels are $\partial\Delta_1 = x_2 x_4 \cdots x_{2k}, \partial\Delta_2 = x_1 x_3 \cdots x_{2k-1}$ (cyclic word).)

Then, for Δ_1 and Δ_2, by pasting $\Pi_4, \ldots, \Pi_{2k}, \Pi_2$ on each edge of $\Pi_1, \Pi_3, \ldots, \Pi_{2k-1}$ on $\partial\Delta_1$ and Π_1, Π_3, $\ldots$, Π_{2k} on each edge of $\Pi_2, \Pi_4, \ldots, \Pi_{2k}$ on $\partial\Delta_2$, we obtain circular diagrams Δ'_1, Δ'_2.

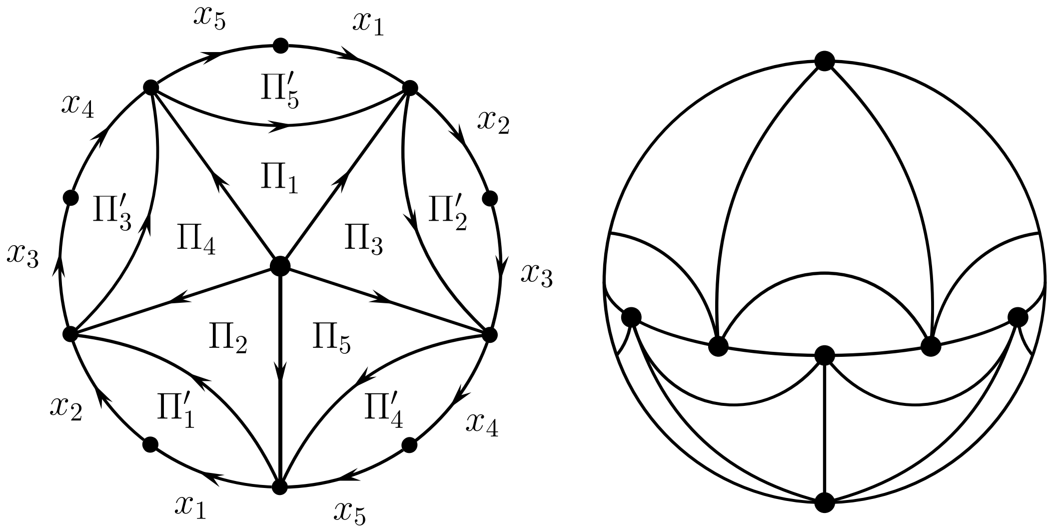

Figure 3: Δ' and $\Delta(n)$ of odd $(n = 5)$

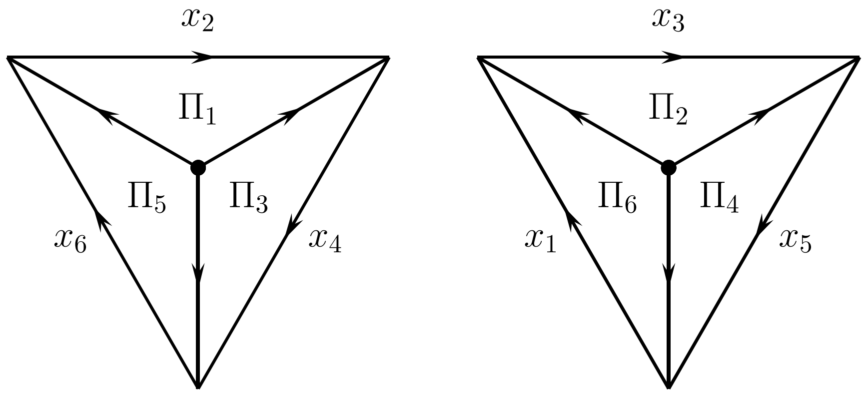

Figure 4: Δ_1 and Δ_2 of even $(n = 6)$

Finally, we know that the boundary labels of Δ'_1, Δ'_2 are $\partial\Delta'_1 = \partial\Delta'_2 = x_1x_2\cdots x_n$, so we get the spherical diagram $\Delta(n)$ by gluing the boundaries with the same labels.

3 Irreducibility of $\Delta(n)$

3.1 A condition for irreducibility

To prove that $\Delta(n)$ is reduced, we show the following lemma.

Lemma 3.1. *If the diagram (that is not a trivial spherical diagram) is not reduced, there is some vertex where two edges with the same letter and orientation join.*

Proof. Assume that there is a cancellable pair of cells in the diagram, that is, there are two $\mathscr{R}$-cells π_1, π_2 whose labels starting from vertices u, v are mutually inverse, and u, v are connected by a path t without self-intersection such that the label equal to 1 in the free group.

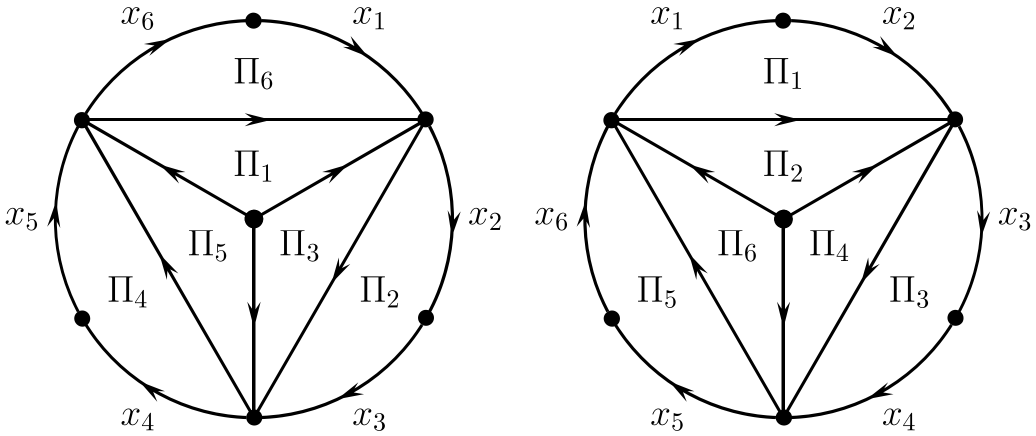

Figure 5: Δ_1' and Δ_2' of even ($n = 6$)

($|t| = 0$) If all edges of $\partial\pi_1, \partial\pi_2$ are glued together, then the diagram contains a trivial spherical subdiagram consisting only of π_1, π_2, which is a contradiction. So path $\partial\pi_1 \cap \partial\pi_2$ has endpoints and they satisfy the condition.

($|t| > 0$) Since the label of path t is equal to 1 in the free group and has at least one letter, it contains a subpath with label $x^{-1}x$, whose middle vertex satisfies the condition.

□

3.2 Proof of irreducibility

Using the contraposition of the above lemma, that is, by checking that for every vertex of $\Delta(n)$ there are no vertices where two edges with the same letter and orientation join, we show that $\Delta(n)$ is reduced.

Suppose n is odd. $\Delta(n)$ has 2 vertices of degree n (o and its copy) and $2n$ vertices of degree 5 (on $\partial\Delta'$).

- For the vertices of degree n, by the construction of $\Delta(n)$, o and its copies have n edges with letters $x_1, \ldots, x_n$ and outgoing orientation, and the letters are all different.
- For the vertices of degree 5, the $2n$ vertices consist of n vertices connected to o by edges and n unconnected vertices in Δ'.
  - For connected vertices v, suppose edge ov is the common edge of cells Π_i and Π_{i+2}. For v, the edge ov is "x_{i+2} & In", the other two edges of Π_i, Π_{i+2} joining v are "x_{i+1} & In" and "x_{i+3} & Out", the two edges on $\partial\Delta'$ are "x_i & In" and "x_{i+1} & Out". If $n \geq 3$ then for each $i = 1, \ldots, n$ there is no edge with the same letter and orientation.
  - For unconnected vertices, by construction of $\Delta(n)$, the vertices is connected to a copy of o in a copy of Δ'. Therefore, it comes down to the above case.

Suppose n is even. $\Delta(n)$ has 2 vertices of degree $n/2$ (central vertices of Δ_1', Δ_2') and n vertices of degree 5.

- For the vertices of degree $n/2$, by the construction of $\Delta(n)$, $n/2$ edges with letters $x_1, x_3, \ldots, x_n - 1$ join to the central vertex of Δ_1' and $n/2$ edges with letters $x_2, x_4, \ldots, x_n$ join to the central vertex of Δ_2', and the letters are all different.
- For the vertices of degree 5, if $n \geq 4$ then exactly the same as the odd case.

From the above, all vertices of $\Delta(n)$ do not have two edges with the same letter and orientation. Therefore $\Delta(n)$ is reduced for any $n \geq 3$ by Lemma.

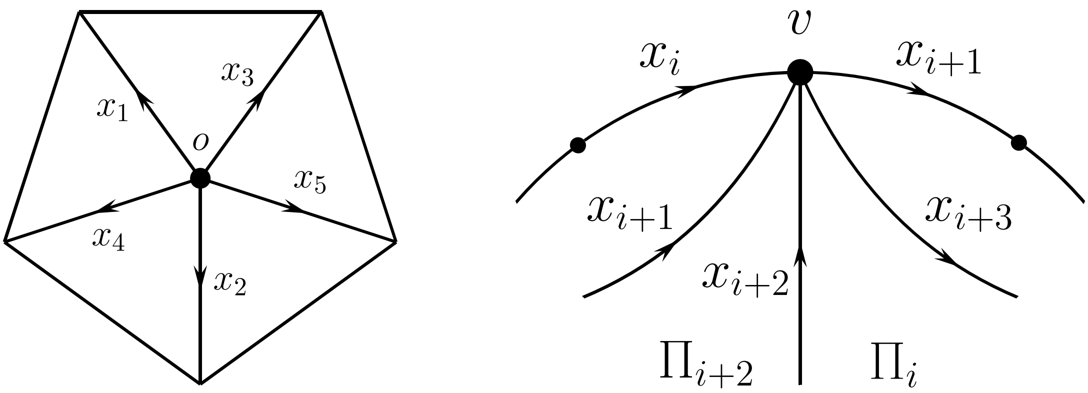

Figure 6: The vertices o ($n = 5$) and v

3.3 Conclusion

Theorem 3.2. *For any $n \geq 3$, the presentation*

$$F(2, n) = \langle x_1, \ldots, x_n | x_1x_2 = x_3, \ldots, x_{n-1}x_n = x_1, x_nx_1 = x_2\rangle$$

is spherical, that is, it has a reduced spherical diagram containing $\mathscr{R}$-cells.

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