Self-injective inverse semigroups

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In the paper [1]and [2], it was shown that the injective hull of a semilattice is an S-distributive completion $C(S)$ in which S is large. For an inverse semigroup S, B.Schein [3] gave a method of construction of S-distributive completion $C(S)$ of S. By a similar way of the construction of completions, B. Schein [5] described the injective hull of an inverse semigroup as the set $F(S)$ of all filters of S. B.Schein [4] showed that an inverse semigroup S is self-injective if and only if S is a S-distribute completion and E-reflexive. By making use of Schen's comletion, K.Shoji [6] showed that for any E-reflexive inverse semigroup S, there exists an inverse semigroup T such that (1) S is a subsemigroup of T, (2) T is the injective hull of S as a right S-set and (3) T is a self-injective semigroup. In this paper, we shall show that $F(S)$ is not always closed under set products.

1 Completions and injective hulls of inverse semigroups

Definition. Let A be a right S-subset of a right S-set B. Then A is called *large* in B if a homomorphism $\rho : B \to C$, where C is any right S-set, with the restriction $\rho|_A$ being an injection, is itself an injection. A is called *strictly large* in B if for any $b, c \in B$ $(b \neq c)$, $\exists s, t \in S^1$ with $bs, ct \in A$ but $bs \neq ct$.

Definition. A right S-set A is *injective* if for any right S-set B, C, any injection $\alpha : B \to C$ and a homomorphism $\rho : B \to A$, there exists a homomorphism!t $\gamma : C \to A$ with $\gamma\alpha=\rho$.

Definition. Let A be a right S-subset of a right S-set B. A right S-set A is called the *injective envelope* of B if A is large in B and B is an injective right S-set.

(B is the minimal injective right S-set conaining A as a right S-subset.)

Definition Let S be an inverse semigroup and M be a right S-set. Define $a \leq b$ if $a \in bE$ for $a, b \in M$. Then $\leq$ is a partial order relation stable under all operations of A (that is, for all $a, b \in M$ and $s \in S$, $a \leq b \Rightarrow as \leq bs$). The order relation $\leq$ is called *natural order*

Definition A subset A of M is called a *tail* if $a \in A$ and $b \leq a \Rightarrow b \in A$ for all $a, b \in M$. In other words, B is a tail exactly when $BE \subseteq B$ (or, equivalently, $BE^1 = B$).

A subset A of M is called *compatible* if there exists a mapping $B \to E^1$ ($b \to e_b$ for every $b \in B$) such that $be_b = b$ and $b_1 e_{b_2} = b_2 e_{b_1}$.

Definition. If $B \subseteq M$, $a \in M$ and $B \subseteq aE^1$ (that is, $b \le a$ for all $b \in B$), then a is called an *upper bound* of B. A minimal upper bound of B is any minimal (with respect to $\le$) element of the set of all upper bounds of B. A unique minimal upper bound is called the *supremum* of B and is denoted as $\bigvee B$. The supremum $a = \bigvee B \in M$ is called the S-*distributive* supremum if $as = \bigvee(Bs)$ for all $s \in S$.
M is called *complete* if any compatible subset B of M, there exists the supremum of B in M.

Definition. An element a is called a *face* of B, if
(i) a is an upper bound of B and
(ii) for all $s, t \in S^1$, $Bs = Bt$ implies $as = at$.
(a face is a minimal upper bound.)
Result 1([5]). *A right S-set M over an inverse semigroup S is injective if and only if every compatible subset of M has a face.*

Definition. A subset F of a right S-set M is called a *filter* if the followings hold :
(1) F is compatible and a tail,
(2) for each $e \in E$, if Fe has a face a, then $a \in F$.

Let $C(S)$={all the compatible tails of S} and $\mathcal{F}(M)$={all the filters of M}.

Let a mapping $\tau : M \to C(M)$ ($m \to mE$). Then τ is an embedding of M into $\mathcal{F}(M) \subseteq C(M)$. We identify M with the S-subset M of the right S-set $\mathcal{F}(M)$.

Result 2([5]). (1) $C(S)$ *is an injective right S-setcotaining S as a right S-subset.*
(2) $\mathcal{F}(M)$ *is the injective hull of S, for every right S-set S.*

Let $C'(S)$={all the compatible tails H of S satisying $HH^{-1}, H^{-1}H \subseteq E$}.

Result 3([3]). *Let S be an inverse semigroup. Then $C'(S)$ is an inverse semigroup and is an S-ditributive completion of S.*

Definition. An inverse semigroup S with the semilattice E of idempotents is called E-*reflexive* if for any $a, b \in S$, $ab \in E$ implies $ba \in E$.

Result 4([4]). *Let S be an inverse semigroup. Then S is self-injective semigroup if and only if S is complete, S-distibutive and E-reflexive.*

If S is E-reflexive, then $C'(S)$=$C(S)$. By Result 2 and Result 3, $C(S)$ is a self-invective semigroup.

Result 5$([6])$.*Let* S *be an* E*-reflexive inverse semigroup with semilattice* E *of idempotents. Let* $\tau : S \to C(S)$ $(s \to sE)$. *Define a relation* $\equiv$ *on* $C(S)$ *as follows :*

$H \equiv F(H, F \in C(S))$ *if and only if the followings hold :*

(1) $HF^{-1} \subseteq E$ *(equivalently,* $FH^{-1} \subseteq E$*),*

(2) $HS \cap FS$ *is strictly large in* HS *and* FS.

Then (i) *the relation* $\equiv$ *a congruence on* $C(S)$*,*

(ii) $C^0(S) = C(S)/\equiv$ *is a complete,* S*-distributive,* E*-reflexive inverse semigroup, (equivalently,* $C^0(S)$ *is a self-injective inverse semigroup) and*

(iii) *there exists the embedding* $\tau_0 : S \to C^0(S)$ *and* $C^0(S)$ *is the injective hull of* S*, where* τ_0 *denotes the composite mapping of* τ *and the natural mapping induced by the congruence* $\equiv$.

By Result 2 and Result 5, there exists an S*-isomohism* $\xi : \mathcal{F}(S) \to C^0(S)$ *with the restricion of* ξ *to* S *is the identity mapping of* S*. Consquenly, the operation of* S *on the right* S*-set* $\mathcal{F}(S)$ *extends to a multiplication of* $\mathcal{F}(S)$ *so that* $\mathcal{F}(S)$ *becomes a self-injective inverse semigroup.*

So we have

Question *Whether or not* $\mathcal{F}(S)$ *is closed under set product?*

Example *Let* $\Gamma = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *be a semilattice depicted in Figure 1. Let* S *be a semilattice* Γ *of groups* G_i *(*i*=1,2,3,4,5,6,7,8), where* G_i *(*i*=1, 2,4,5,6,7,8) is a copy of the free grpup generated by* a *and* b *and* G_3 *is a copy of the cyclic group generated by* ab*. The multiplication of* S *is defined by* $x_i x_j = x_{ij}$*, where for any* $x \in G$*,* x_i *is the image of* x *in* G_i*. Let* H*=*$\{a_1, a_4, a_6, a_7, a_8\}$ *and* F*=*$\{b_2, b_5, b_6, b_7, b_8\}$*. Then* H *and* F *are filters of* S*, but* HF *is not a filter. Actually,* $HF \cup \{ab_3\}$ *is a filter of* S*, Hence* $\mathcal{F}(M)$ *is not closed under set product.*

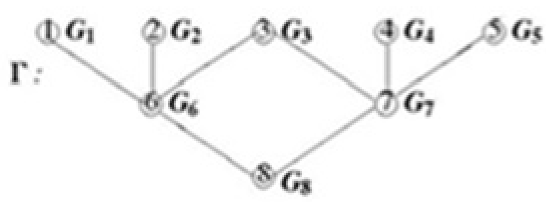

Figure 1

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