On numerical semigroups whose quotients by two are generated by two or three elements [1]

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Abstract

Let $\tilde{H}$ be a numerical semigroup generated by five elements whose quotient H by two is generated by three elements. We prove that the numerical semigroup $\tilde{H}$ satisfying a general condition is Weierstrass. Moreover, we give examples of $\tilde{H}$ whose quotient H by two is symmetric such that $\tilde{H}$ is Weierstrass.

1 Terminologies and introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, denoted by $g(H)$. In this paper H always stands for a numerical semigroup. We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\},$$

which is called the *conductor* of H. It is well-known that $c(H) \leqq 2g(H)$. H is said to be *symmetric* if $c(H) = 2g(H)$. H is said to be *quasi-symmetric* if $c(H) = 2g(H) - 1$. We have $(c(H) - 1) + h \in H$ for any $h \in H$ with $h > 0$. The number $c(H) - 1$ is called the *Frobenius number* of H. An element $f \in \mathbb{N}_0 \backslash H$ is called a *pseudo-Frobenius number* of H if $f + h \in H$ for any $h \in H$ with $h > 0$. We denote by $PF(H)$ the set of pseudo-Frobenius numbers.

We explain numerical semigroups in connection with algebraic curves. A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0. For a pointed curve (C, P) we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

where $k(C)$ is the field of rational functions on C. Then $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$. A numerical semigroup H is said to be of *double covering type* , which is abbreviated to *DC*, if there exists a double covering of curves with a ramification point P with $H(P) = H$. Hence, if a numerical

[1] This paper is an extended abstract and the details will be published (see [10])

semigroup is DC, then it is Weierstrass. To find Weierstrass numerical semigroups we use the above property in this paper. To describe DC numerical semigroups we need the following notation: For a numerical semigroup H we set

$$d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\},$$

which is a numerical semigroup. Let $\pi : C \longrightarrow C'$ be a double covering of curves with a ramification point P. Then we have $d_2(H(P)) = H(\pi(P))$.

2 Numerical semigroups generated by four or five elements

There are the following known facts on Weierstrass or non-Weierstrass numerical semigroups:

Known Fact 1 (Classical). Any numerical semigroup generated by two elements is Weierstrass.

Known Fact 2 (Waldi [12]). Any numerical semigroup generated by three elements is Weierstrass.

Known Fact 3 (Buchweitz [2]). There exists a non-Weierstrass numerical semigroup generated by nine elements.

Known Fact 4 ([7]). There exists a non-Weierstrass numerical semigroup generated by six elements.

Combining Known Fact 4 with Torres [11] we get the following:

Known Fact 5. For any $l \geqq 6$ there exists a non-Weierstrass numerical semigroup generated by l elements.

Thus, we pose the following problem:

Problem. Is every numerical semigroup generated by four or five elements Weierstrass?

We have the following results on numerical semigroups generated by four elements:

Known Fact 6 (Buchweitz [2], Waldi [12]). Any symmetric numerical semigroup generated by four elements is Weierstrass.

Known Fact 7 ([5]). Any quasi-symmetric numerical semigroup generated by four elements is Weierstrass.

So, we are interested in the following two cases

(i) Is every neither symmetric nor quasi-symmetric numerical semigroup generated by four elements Weirstrass?

(ii) Is every numerical semigroup generated by five elements Weirstrass?

Here, we introduce the terminologies and the notations as follows: For a numerical semigroup H we denote by $M(H)$ the minimal set of generators for H. For any non-negative integers $a_1, a_2, \cdots, a_n$ we denote by $\langle a_1, a_2, \cdots, a_n \rangle$ the additive monoid generated by $a_1, a_2, \cdots, a_n$. The minimum positive integer in H is denoted by $m(H)$, which is called the *multiplicity* of H. We set

$$s_i = \min\{h \in H \mid h \equiv i \bmod m(H)\}$$

for $i = 1, \ldots, m(H) - 1$. The set $S(H) = \{m(H), s_1, \ldots, s_{m(H)-1}\}$ is called the *standard basis* for H. To explain our aim we give the following remarks:

Remark 1 ([6]). Let H be a Weierstrass numerical semigroup. We set $\tilde{H} = 2H + n\mathbb{N}_0$ with an odd integer $n \geqq c(H) + m(H) - 1$. Then we have the following:

(i) $\sharp M(\tilde{H}) = \sharp M(H) + 1$ and $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2}$.

(ii) If $c(H) = 2g(H) - r$, then $c(\tilde{H}) = 2g(\tilde{H}) - 2r$, because we have $c(\tilde{H}) = 2c(H) + n - 1$.

(iii) If H is Weierstrass and $n \geqq 2g(H) + 1$, then $\tilde{H}$ is DC, hence Weierstrass. Hence, if H is generated by three elements, then $\tilde{H}$ is Weierstrass. In this case, $\tilde{H}$ is generated by four elements. Moreover, if H is not symmetric, then $\tilde{H}$ is neither symmetric nor quasi-symmetric.

Remark 2. Let H be a numerical semigroup distinct from $\mathbb{N}_0$. We set $\tilde{H} = 2H + \langle n, n + 2\gamma \rangle$ with an odd integer $n \geqq c(H) + m(H) - 1$ and $\gamma \notin H$. Then it is hard to determine the genus $g(\tilde{H})$ of $\tilde{H}$ ([7]). Moreover, there exists a Weierstrass numerical semigroup H such that $\tilde{H}$ is non-Weierstrass.

In this case, we have $M(H) = 4$, hence $M(\tilde{H}) = 6$. So, we are interested in the case $\sharp M(\tilde{H}) = 4$ or 5 with $d_2(\tilde{H}) = H$ as follows:

(1) $\sharp M(\tilde{H}) = 5$ and $\sharp M(H) = 3$.

(2) $\sharp M(\tilde{H}) = 4$ and $\sharp M(H) = 2$.

3 Numerical semigroups whose quotients by two are non-symmetric and generated by three elements

We generalize the concept of the set $PF^*(H)$ of pseudo-Frobenius numbers which are not Frobenius. We set

$$\overline{PF^*(H)} = \{\gamma \in \mathbb{N}_o \backslash H \mid c(H) - 1 - \gamma \in \mathbb{N}_o \backslash H\}.$$

Lemma 1 ([8, Lemma 1.1 ii), iii)]). *We have $PF^*(H) \subseteqq \overline{PF^*(H)}$. Moreover, the cardinality of $\overline{PF^*(H)}$ is equal to $2g(H) - c(H)$.*

Example 1. Let H be a numerical semigroup with $M(H) = \{4, 6, 9, 11\}$. Then $S(H) = \{4, 6, 9, 11\}$. We have $g(H) = 5$ and $c(H) = 11 - 4 + 1 = 8 = 2g(H) - 2$. Moreover, we obtain $PF^*(H) = \{6 - 4, 9 - 4\} = \{2, 5\}$. In this case, we have $PF^*(H) = \overline{PF^*(H)}$

Example 2. Let $M(H) = \{4, 5, 11\}$. Then we have $S(H) = \{4, 5, 10, 11\}$. We obtain $g(H) = 5$ and $c(H) = 11-4+1 = 8 = 2g(H)-2$. Moreover, we get $PF^*(H) = \{10-4\} = \{6\}$ and $\overline{PF^*(H)} = \{1, 6\}$. In this case, we have $PF^*(H) \subset \overline{PF^*(H)}$.

The following is the Key Lemma for investigating a certain numerical semigroup $\tilde{H}$ with $\sharp M(d_2(\tilde{H})) = 3$:

Lemma 2 ([10]). *Let H be a numerical semigroup which is not symmetric. Assume that $PF^*(H)$ consists of only one element t. We set $c(H) = 2g(H) - r$. Let n be an odd integer larger than $c(H) + m(H) - 1$. We set*

$$\tilde{H} = 2H + \langle n, n + 2(c(H) - 1 - t)\rangle$$

Then we have $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} - r$.

We get the following theorem for the above numerical semigroup $\tilde{H}$:

Theorem 1 ([10]). *Let H be a Weierstrass numerical semigroup which is not symmetric. Assume that $PF^*(H)$ consists of only one element t. We set $c(H) = 2g(H) - r$. Let n be an odd integer larger than $\max\{c(H) + m(H) - 1, 2g(H) + 2r\}$. We set*

$$\tilde{H} = 2H + \langle n, n + 2(c(H) - 1 - t)\rangle$$

Then the numerical semigroup $\tilde{H}$ is DC, hence it is Weierstrass.

On the other hand, we have the following:

Proposition 1 ([10]). *Let the notation be as in the above theorem. We note that $\tilde{H}^* = 2H + \langle n, n + 2t\rangle$ is also DC, but $g(\tilde{H}^*) = 2g(H) + \dfrac{n-1}{2} - 1$.*

We have the following known fact:

Remark 3 (Fröberg-Gottlieb-Häggkvist [3, Theorem 11]). Let H be a numerical semigroup which is not symmetric. If $\sharp M(H) = 3$, then the set $PF^*(H)$ consists of only one element.

Hence, combining Theorem 1 with Remark 3 we get the main result in this section.

Corollary 1. Let H be a numerical semigroup generated by three elements which is not symmetric. We set $PF^*(H) = \{t\}$ and $c(H) = 2g(H) - r$. Let n be an odd integer larger than $\max\{c(H) + m(H) - 1, 2g(H) + 2r\}$. We set

$$\tilde{H} = 2H + \langle n, n + 2(c(H) - 1 - t)\rangle.$$

Then $\tilde{H}$ is DC, hence it is Weierstrass. In this case we have $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} - r$.

Example 3 ([7, Theorem 5.2]). Let H be a numerical semigroup with $M(H) = \{4, 6, 9, 11\}$. Then $S(H) = \{4, 6, 9, 11\}$. We have $g(H) = 5$ and $c(H) = 11 - 4 + 1 = 8 = 2g(H) - 2$. Moreover, we have $PF^*(H) = \{6 - 4, 9 - 4\} = \{2, 5\} = \overline{PF^*(H)}$. Let n be an odd integer larger than $10 + 4 = 14$. We set $\tilde{H} = 2H + \langle n, n + 2(7 - 5)\rangle$. Then $\tilde{H}$ is not DC.

Example 4. Let H be a numerical semigroup with $M(H) = \{4, 5, 11\}$. Then $S(H) = \{4, 5, 10, 11\}$. We have $g(H) = 5$ and $c(H) = 11 - 4 + 1 = 8 = 2g(H) - 2$. Moreover, $PF^*(H) = \{10 - 4\} = \{6\}$ and $\overline{PF^*(H)} = \{1, 6\}$. Let n be an odd integer larger than $10 + 4 = 14$. We set $\tilde{H} = 2H + \langle n, n+2 \rangle$ and $\tilde{H}^* = 2H + \langle n, n+12 \rangle$. Then $\tilde{H}$ and $\tilde{H}^*$ are DC by Corollary 1 and Prposition 1, respectively.

4 Numerical semigroups whose quotients by two are symmetric

In this section we consider the case $\sharp M(\tilde{H}) = 4$ and 5 with $\sharp M(d_2(\tilde{H})) = 2$ and 3, respectively, and $d_2(\tilde{H})$ is symmetric.

Theorem 2 ([9, Theorem 3.3]). *Let a and b be positive integers with $2 \leqq a < b$ satisfying $(a, b) = 1$. Let n be an odd integer with $n \geqq (a-1)(b-1) + a - 1$. We set*

$$H = 2\langle a, b \rangle + \langle n, n + 2(b - ar) \rangle,$$

where r is a positive integer with $b - ar > 0$. Then we have the following:
(1) $d_2(H) = \langle a, b \rangle$, $g(H) = 2g(\langle a, b \rangle) + \dfrac{n-1}{2} - (a-1)r$ and $c(H) = 2g(H) - 2r$.
(2) If $n \geqq (a-1)(b-1) + 2r(a-1) + 1$, then H is DC, hence it is Weierstrass.

In the above theorem if we replace $H = 2\langle a, b \rangle + \langle n, n+2(b-ar) \rangle$, where r is a positive integer with $b - ar > 0$, by $H = 2\langle a, b \rangle + \langle n, n + 2(lb - ar) \rangle$, where $l \geqq 2$ and r is a positive integer with $lb - ar > 0$, it is hard to show that H is DC. We can show that the following numerical semigroups H are DC.

Theorem 3 ([4, Theorem 2.5]). *Let $d \geqq 4$. Let n be an odd integer with $n \geqq 2(d-1-l)r + (d-1)(d-2) + 1$. We set*

$$H = 2\langle d-1, d \rangle + \langle n, n + 2(ld - (d-1)r) \rangle,$$

where $2 \leqq l \leqq d-2$ and r is a positive integer with $ld - (d-1)r > 0$. Then we have the following:
(1) If $l = d-2$, then H is DC.
(2) If $r = 1$ or 2, then H is DC.

From now on we consider a numerical semigroup H with $\sharp M(H) = 5$ such that $d_2(H)$ is a symmetric numerical semigroup with $\sharp M(d_2(H)) = 3$. Using [3, Corollary after Theorem 14] we can prove the following whose detailed proof will be given in [10]:

Proposition 2. *Let H be a numerical semigroup with $\sharp M(H) = 3$. Then the following are equivalent:*

(i) H is symmetric.

(ii) We have $H = u\langle a, b \rangle + c\mathbb{N}_0$ where $u \geqq 2$, $1 < a < b$ with $(a, b) = 1$ and $c \in \langle a, b \rangle$.

Theorem 4 *([10]). Let $d \geqq 3$ and n be an odd integer with $n \geqq 2(d-1)(d-2)+1$. We set*

$$H = 2\langle d-1, d\rangle + n\mathbb{N}_0,$$

which is a symmetric numerical semigroup by Proposition 2 whose genus $g(H)$ is $(d-1)(d-2) + \frac{n-1}{2}$. We set

$$\tilde{H} = 2H + \langle \tilde{n}, \tilde{n}+4\rangle$$

where $\tilde{n}$ is an odd integer with $\tilde{n} \geqq n + 2d^2 - 2d - 6$. Then $\tilde{H}$ is DC. In this case we have $g(\tilde{H}) = 2g(H) + \frac{\tilde{n}-1}{2} - 2(d-2)$.

Example 5. *Let $d = 3$, $n = 5$ and $\tilde{n} = 23$ in the above theorem. Then $H = \langle 4, 6, 5\rangle$, hence $g(H) = 4$ and $c(H) = 8$. Moreover, we set*

$$\tilde{H} = 2\langle 4, 5, 6\rangle + \langle 23, 23+4\rangle = \langle 8, 10, 12, 23, 27\rangle.$$

Then we have

$$g(\tilde{H}) = 17 = 2 \times 4 + \frac{23-1}{2} - 2(3-2) = 2g(H) + \frac{\tilde{n}-1}{2} - 2(d-2).$$

We obtain that $\tilde{H} = \langle 8, 10, 12, 23, 27\rangle$ is DC.

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