Examples of Fully Prime Rings

Hisaya Tsutsui

Department of Mathematics, Embry-Riddle Aeronautical University

Definition *A ring R (possibly without identity) in which every ideal is prime (i.e. every proper ideal is a prime ideal and (hence) R is a prime ring) is called a fully prime ring.*

The purpose of this article is to present several examples of fully prime rings with certain conditions. In particular, we give an example of a fully prime right Noetherian domain that has n ideals for any positive integer n, and an example of fully prime non-primitive ring that has countably many ideals.

We first published a series of papers on fully prime rings in 1994 and 1996 (Blair-Tsutsui [2], Tsutsui [10]). More recently, Beachy- Medina-Bárcenas [1] studied module theoretic aspect of fully prime rings. Conditions similar to the fully prime condition have received attention in literature. Hirano [6] studied those rings in which every ideal is completely prime. Courter [4] studied those rings in which every ideal is semiprime, and Koh [7] studied those rings in which every right ideal is prime.

Throughout this article, we assume a ring to be associative but not necessarily commutative. Due to the consideration that an ideal of a ring being a ring of its own, we do not assume the existence of a multiplicative identity on a ring unless otherwise so stated.

Theorem 1 (Blair-Tsutsui [2]). The center of a fully prime ring is either a field or zero, and a ring R is fully prime if and only if every (two sided) ideal of R is idempotent and the set of ideals of R is totally ordered under inclusion.

In particular, as is well known, a commutative fully prime ring with identity is a field. Examples of a non-commutative fully prime ring include the ring of endomorphisms of a vector space V over a division ring D. Every right ideal, and hence every ideal of a regular ring is idempotent and for a regular self-injective rings R, an ideal P is prime if and only if R/P is totally ordered. Since there exists a regular self-injective ring T with a prime ideal P

*The talk was in part supported by the *RIMS*, a Joint Usage/Research Center located in Kyoto University.

*A part of the contents with further explanations may be submitted to elsewhere as a part of our on-going work.

such that the set of ideals of *T/P* is not well-ordered, the set of ideals of a fully prime ring is not necessarily well-ordered.

Blair-Tsutsui [2] gives an example of a non-primitive fully prime ring with exactly one nonzero proper ideal. Mazurek-Roszkowska [9] constructed an example of a domain (with identity) whose lattice of left ideals as well as the lattice of right ideals are linearly ordered. Further, S has the following properties:

(1) S is a F algebra and every ideal of S is a F ideal,

(2) $J(S) = S,$ where $J(S)$ denotes the Jacobson radical of S.

(3) $S/I \approx S$ for each ideal I of S (hence I is a completely prime ideal), and

(4) S has countably many ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_n \cdots \subset I_\omega = S.$

We construct an example of fully prime non-primitive ring with countably many ideals. Let $R = S \oplus F$ where addition is defined component wise and multiplication is given by $(s_1, k_1)(s_2, k_2) = (s_1 s_2 + k_1 s_2 + k_2 s_1, s_1 s_2)$ where $s_1, s_2 \in S,\ k_1, k_2 \in F.$

Let $M = S \oplus 0.$ Then as $R/M \approx F$, M is a maximal right ideal. Hence $J(R) = M \cap J(R).$ But then, since $J(S) = S,\ J(M) = M = M \cap J(R) = J(R).$ As $M \neq 0, R$ is not semiprimitive. As $M = J(R), M$ is a unique maximal right ideal of R, and hence every right ideal of R is contained in M. If $T \oplus 0 \subseteq M$ is an ideal of R, then, T is an ideal of S. On the other hand, for any for any ideal I of S, let $(i, 0) \in I \oplus 0.$ then $(i,0)(s,k) = (is + ki, 0),$ and $(s,k)(i,0) = (si + ki, 0)$ for any $(s,k) \in R.$ As I is an F algebra, this shows that only (two sided) ideals of R are of the form $I \oplus 0$ for ideals I of S.

Theorem 2 (Blair-Tsutsui [2]). Let R be a fully prime ring. Then every ideal of R is fully prime when it is considered as a ring without identity. Every ideal of an ideal of R is an ideal of R. Further, a proper ideal of R cannot be a ring with identity.

If P is a proper ideal of a fully prime ring R with identity and F is a subfield of the center of R, then $S_P = F + P$ is a fully prime ring whose maximal ideal P is also a maximal right and left ideal. Further, proper ideals of S are precisely those ideals of R that are contained in P.

Theorem 3 (Blair-Tsutsui [2]). $S_P = F + P$ is right primitive if and only if *R* is right primitive. S_P is semiprimitive if and only if *R* is semiprimitive.

Theorem 4 (Tsutsui [11]). A fully prime ring with right Krull dimension is semiprimitive.

Let $A_1(k)$ be the first Weyl algebra over a field of characteristic 0. Then $S = k + xA_1(k)$ is a right and left Noetherian fully prime ring with exactly one nonzero proper ideal $xA_1(k)$. Thus, Noetherian fully prime ring is not necessarily a simple ring. Theorem 2 of Hirano [6] gives a fully (completely) prime ring domain with *n* ideals R_n. For a field *k* of characteristic 0, if we let $D = A_1(k)$, the first Weyl algebra, then R_1 is a fully prime Noetherian domain with exactly one non-zero proper ideal $xA_1(k)$ as shown in Theorem 4.6 of Blair-Tsutsui [2]. Since R_n is the idealizer of $T_k^n(D) \cong A_n(k)$, and $A_n(k)$ is a simple right Noetherian domain, R_n is a fully prime right Noetherian ring with exactly *n* nonzero ideals.

We remark that while the ring R_n is a domain, a fully prime right Noetherian ring is not necessarily a domain as a simple Noetherian ring that is not a domain exists (See Example 14.17 of Chatters-Hajarnavis [3]). Note also that a prime right Noetherian ring is simple Artinian if $\mathrm{Soc}(R) \neq 0$ (See Theorem 1.24 of Chatters-Hajarnavis [3]). Thus Soc(R_n) of the ring R_n is zero.

Since a fully prime right Noetherian ring is in particular a strongly prime ring, it is nonsingular by Proposition II.1 of Handelman-Lawrence [5]. A ring *R* that is not nonsingular is given in (11.21) Example of Lam [8]. *R* has exactly one nonzero proper ideal and the ideal is idempotent. Hence *R* is a fully prime ring that is not nonsingular.

Proposition. A fully prime ring *R* is either a nonsingular primitive ring with Soc(*R*) being the minimum nonzero (two sided) ideal, or a ring with $\mathrm{Soc}(R) = 0$.

Proof: If $\mathrm{Soc}(R) \neq 0$, then *R* being prime implies that *R* is primitive. Further, since Soc(*R*) is the intersection of all essential right ideals; all ideals in a prime ring are essential; and every ideal of a fully prime ring is linearly ordered; we have $0 \neq \mathrm{Soc}(R) \subseteq \bigcap_{0 \neq P_i \lhd R} P_i$. Thus, Soc(*R*) is the minimum nonzero ideal of *R*. As $\mathrm{Sing}(R) \cdot \mathrm{Soc}(R) = 0$, $\mathrm{Sing}(R) = 0$.

References

[1] J.A. Beachy and M. Medina-Bárcenas, *Fully prime modules and fully semiprime modules,* Bulletin of the Korean Mathematical Society vol 57 (2020), issue 5 pp. 1177-1193.

[2] W.D. Blair and H. Tsutsui, *Fully Prime Rings*, Comm. Algebra 22 (1994), no.13, 5389-5400.

[3] A.W. Chatters and C.R. Hajanavis, *Rings with chain conditions*, Research Notes in Mathematics, Pitman Press, Massachusetts, 1980.

[4] R. Courter, *Rings all of whose factor rings are semiprime*. Canadian Math Bull, vol. 12, issue 4 (1969), pp.417-426.

[5] D. Handelman and J. Lawrence, *Strongly Prime Rings,* Trans. AMS, vol.211, 1975.

[6] Y. Hirano, *On rings all of whose factor rings are integral domains*, J. Austral. Math. Soc. (Series A) 55 (1993), 325-333.

[7] K. Koh, *On one-sided ideals of prime type,* Proc. Amer. Math Soc. vol. 28, no. 2 (1971) pp 321-329.

[8] T.Y. Lam*, Lectures on Modules and Rings, Graduate Texts in Mathematics, 189, New York: Springer-Verlag, 1999.*

[9] R. Mazurek and E. Roszkowska, *Example of Chain Domains*, Proc. AMS (1998), vol. 126 no.3, 661-667.

[10] H. Tsutsui, *Fully Prime Rings II*, Comm. Algebra 22 (1996), no.9, 2981-2989.

[11] H. Tsutsui, *On rings in which every ideal is prime*, Contemporary Ring Theory, World Scientific (2011).

Embry-Riddle Aeronautical University
Prescott, AZ 86301-3720
USA
E-mail address: Hisaya.Tsutsui@erau.edu