Global well-posedness for a Q-tensor model of nematic liquid crystals

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1 Introduction

This article is a summary of our work [8]. The molecules of nematic liquid crystal flows as in a liquid phase; however, they have the orientation order. In order to analyze the biaxial nematic liquid crystal flows, Beris and Edwards [3] proposed the $N \times N$ symmetric, traceless matrix as the director fields, which is called $\mathbb{Q}$-tensor. In this article, we consider the coupled system by the Navier-Stokes equations with a parabolic-type equation describing the evolution of the director fields $\mathbb{Q}$.

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathfrak{p} = \Delta\mathbf{u} + \mathrm{Div}\,(\tau(\mathbb{Q}) + \sigma(\mathbb{Q})) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \partial_t\mathbb{Q} + (\mathbf{u}\cdot\nabla)\mathbb{Q} - \mathbf{S}(\nabla\mathbf{u},\mathbb{Q}) = \mathbf{H} & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\mathbf{u},\mathbb{Q})|_{t=0} = (\mathbf{u}_0,\mathbb{Q}_0) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

Here, $\partial_t = \partial/\partial t$, t is the time variable, $\mathbf{u}(x,t) = (u_1(x,t),\dots,u_N(x,t))^T$ is the fluid velocity, where $\mathbf{M}^T$ denotes the transposed $\mathbf{M}$, and $\mathfrak{p} = \mathfrak{p}(x,t)$ is the pressure. For a vector of functions $\mathbf{v}$, we set $\mathrm{div}\,\mathbf{v} = \sum_{j=1}^N \partial_j v_j$, and also for $N\times N$ matrix field A with $(j,k)^{\mathrm{th}}$ components A_{jk}, the quantity $\mathrm{Div}\,A$ is an N-vector with j^{th} component $\sum_{k=1}^N \partial_k A_{jk}$, where $\partial_k = \partial/\partial x_k$. The tensors $\mathbf{S}(\nabla\mathbf{u},\mathbb{Q})$, $\tau(\mathbb{Q})$, and $\sigma(\mathbb{Q})$ are

$$\begin{aligned} \mathbf{S}(\nabla\mathbf{u},\mathbb{Q}) &= (\xi\mathbf{D}(\mathbf{u}) + \mathbf{W}(\mathbf{u}))\left(\mathbb{Q} + \frac{1}{N}\mathbf{I}\right) \\ &\quad + \left(\mathbb{Q} + \frac{1}{N}\mathbf{I}\right)(\xi\mathbf{D}(\mathbf{u}) - \mathbf{W}(\mathbf{u})) - 2\xi\left(\mathbb{Q} + \frac{1}{N}\mathbf{I}\right)\mathbb{Q} : \nabla\mathbf{u}, \\ \tau(\mathbb{Q}) &= 2\xi\mathbf{H} : \mathbb{Q}\left(\mathbb{Q} + \frac{1}{N}\mathbf{I}\right) - \xi\left[\mathbf{H}\left(\mathbb{Q} + \frac{1}{N}\mathbf{I}\right) + \left(\mathbb{Q} + \frac{1}{N}\mathbf{I}\right)\mathbf{H}\right] - \nabla\mathbb{Q}\odot\nabla\mathbb{Q}, \\ \sigma(\mathbb{Q}) &= \mathbb{Q}\mathbf{H} - \mathbf{H}\mathbb{Q}, \end{aligned}$$

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where $\mathbf{D}(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ and $\mathbf{W}(\mathbf{u}) = (\nabla\mathbf{u} - (\nabla\mathbf{u})^T)/2$ denote the symmetric and antisymmetric part of $\nabla\mathbf{u}$, respectively. A scalar parameter $\xi \in \mathbb{R}$ denotes the ratio between the tumbling and the aligning effects that a shear flow would exert over the directors. Moreover, $\mathbf{I}$ is the $N \times N$ identity matrix,

$$\mathbf{H} = \Delta\mathbb{Q} - a\mathbb{Q} + b(\mathbb{Q}^2 - \mathrm{tr}(\mathbb{Q}^2)\mathbf{I}/N) - c\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q},$$

and the (i,j) component of $\nabla\mathbb{Q} \odot \nabla\mathbb{Q}$ is $\sum_{\alpha,\beta=1}^N \partial_i Q_{\alpha\beta}\partial_j Q_{\alpha\beta}$.

In this article, we consider the global well-posedness for (1.1) for small initial data in the following solution class:

$$\begin{aligned}
&\mathbf{u} \in \bigcap_{q=q_1,q_2} W_p^1((0,T), L_q(\mathbb{R}^N)^N) \cap L_p((0,T), W_q^2(\mathbb{R}^N)^N),\\
&\mathbb{Q} \in \bigcap_{q=q_1,q_2} W_p^1((0,T), W_q^1(\mathbb{R}^N;\mathbb{S}_0)) \cap L_p((0,T), W_q^3(\mathbb{R}^N;\mathbb{S}_0))
\end{aligned} \tag{1.2}$$

with certain p, q_1, and q_2.

Mathematically the Beris-Edwards model (1.1) has been studied by many authors in recent years. Concerning strong solutions, Abels, Dolzmann, and Liu [1] showed the existence of a strong local solution and global weak solutions with higher regularity in time in the case of inhomogeneous mixed Dirichlet/Neumann boundary conditions in a bounded domain without any smallness assumption on the parameter ξ. Liu and Wang [7] improved the spatial regularity of solutions obtained in [1] and generalized their result to the case of anisotropic elastic energy. Abels, Dolzmann, and Liu [2] also proved the local well-posedness with Dirichlet boundary condition for the classical Beris-Edwards model, which means that fluid viscosity depends on the $\mathbb{Q}$-tensor, but for the case $\xi = 0$ only. Cavaterra et al. [4] showed the global well-posedness in the two-dimensional periodic case without any smallness assumption on the parameter ξ. Xiao [13] proved the global well-posedness for the simplified model with $\xi = 0$ in a bounded domain. He constructed a solution in the maximal L_p-L_q regularity class. Recently, Schonbek and the second author in [10] proved the global well-posedness for any ξ and small initial data in the class (1.2). The difference between [10] and the present article is the linear terms of the model. More precisely, the problem (1.1) is separated into the linear part and nonlinear part as follows:

$$\begin{cases}
\partial_t\mathbf{u} - \Delta\mathbf{u} + \nabla\mathfrak{p} + \beta\mathrm{Div}\,(\Delta\mathbb{Q} - a\mathbb{Q}) = \mathbf{f}(\mathbf{u},\mathbb{Q}), & \text{in } \mathbb{R}^N \text{ for } t \in (0,T),\\
\mathrm{div}\,\mathbf{u} = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T),\\
\partial_t\mathbb{Q} - \beta\mathbf{D}(\mathbf{u}) - \Delta\mathbb{Q} + a\mathbb{Q} = \mathbf{G}(\mathbf{u},\mathbb{Q}) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T),\\
(\mathbf{u},\mathbb{Q})|_{t=0} = (\mathbf{u}_0,\mathbb{Q}_0) & \text{in } \mathbb{R}^N,
\end{cases} \tag{1.3}$$

where

$$\begin{aligned}
\beta &= 2\xi/N,\\
\mathbf{f}(\mathbf{u},\mathbb{Q}) &= -(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathrm{Div}\,[2\xi\mathbf{H} : \mathbb{Q}(\mathbb{Q} + \mathbf{I}/N) - (\xi+1)\mathbf{H}\mathbb{Q} + (1-\xi)\mathbb{Q}\mathbf{H} - \nabla\mathbb{Q}\odot\nabla\mathbb{Q}]\\
&\quad - \beta\mathrm{Div}\,\{b(\mathbb{Q}^2 - \mathrm{tr}(\mathbb{Q}^2)\mathbf{I}/N) - c\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q}\},\\
\mathbf{G}(\mathbf{u},\mathbb{Q}) &= -(\mathbf{u}\cdot\nabla)\mathbb{Q} + \xi(\mathbf{D}(\mathbf{u})\mathbb{Q} + \mathbb{Q}\mathbf{D}(\mathbf{u})) + \mathbf{W}(\mathbf{u})\mathbb{Q} - \mathbb{Q}\mathbf{W}(\mathbf{u}) - 2\xi(\mathbb{Q} + \mathbf{I}/N)\mathbb{Q} : \nabla\mathbf{u}\\
&\quad + b(\mathbb{Q}^2 - \mathrm{tr}(\mathbb{Q}^2)\mathbf{I}/N) - c\mathrm{tr}(\mathbb{Q}^2)\mathbb{Q}.
\end{aligned}$$

On the other hand, $\Delta\mathbb{Q} - a\mathbb{Q}$ is removed from the tensor $\tau(\mathbb{Q})$, and so $\mathbf{u}$ part and $\mathbb{Q}$ part of linearized equations are essentially separated in [10]. This is a big difference between [10] and the present article.

Before stating the main result of this article, we summarize several symbols and functional spaces used throughout the article. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural numbers, real numbers and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ = (0, \infty)$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$ with $x = (x_1, \dots, x_N)$. For scalar function f, N-vector of functions $\mathbf{g}$, and $N \times N$ matrix fields $\mathbf{G}$, we set

$$\nabla^k f = (\partial_x^\alpha f \mid |\alpha| = k), \quad \nabla^k \mathbf{g} = (\partial_x^\alpha g_j \mid |\alpha| = k, \ \ j = 1, \dots, N),$$
$$\nabla^k \mathbf{G} = (\partial_x^\alpha G_{j\ell} \mid |\alpha| = k, \ \ j, \ell = 1, \dots, N).$$

For Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y, $\mathcal{L}(X)$ is the abbreviation of $\mathcal{L}(X, X)$, and $\mathrm{Hol}\,(\mathrm{U}, \mathcal{L}(\mathrm{X}, \mathrm{Y}))$ the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions defined on a domain U in $\mathbb{C}$. For any $1 \le p, q \le \infty$, $L_q(\mathbb{R}^N)$, $W_q^m(\mathbb{R}^N)$ and $B_{q,p}^s(\mathbb{R}^N)$ denote the usual Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_q(\mathbb{R}^N)}$, $\|\cdot\|_{W_q^m(\mathbb{R}^N)}$ and $\|\cdot\|_{B_{q,p}^s(\mathbb{R}^N)}$ denote their norms, respectively. We set $W_q^0(\mathbb{R}^N) = L_q(\mathbb{R}^N)$ and $W_q^s(\mathbb{R}^N) = B_{q,q}^s(\mathbb{R}^N)$. $C^\infty(\mathbb{R}^N)$ denotes the set of all C^∞ functions defined on $\mathbb{R}^N$. $L_p((a, b), X)$ and $W_p^m((a, b), X)$ denote the standard Lebesgue space and Sobolev space of X-valued functions defined on an interval (a, b), respectively. The d-product space of X is defined by X^d, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. Let

$$\mathbb{S}_0 = \{\mathbb{Q} \in \mathbb{R}^{N^2} \mid \mathbb{Q} = \mathbb{Q}^T, \mathrm{tr}\mathbb{Q} = 0\}.$$

The space for a tensor is defined by

$$X(\mathbb{R}^N; \mathbb{S}_0) = \left\{ \mathbf{G} : \mathbb{R}^N \to \mathbb{S}_0 \mid \|\mathbf{G}\|_X = \sum_{i,j=1}^N \|G_{ij}\|_X < \infty \right\}$$

for the Banach space X. We set

$$W_q^{m,\ell}(\mathbb{R}^N) = \{(\mathbf{f}, \mathbf{G}) \mid \mathbf{f} \in W_q^m(\mathbb{R}^N)^N, \ \ \mathbf{G} \in W_q^\ell(\mathbb{R}^N; \mathbb{S}_0)\},$$
$$\|(\mathbf{f}, \mathbf{G})\|_{W_q^{m,\ell}(\mathbb{R}^N)} = \|\mathbf{f}\|_{W_q^m(\mathbb{R}^N)} + \|\mathbf{G}\|_{W_q^\ell(\mathbb{R}^N)}.$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}_\xi^{-1} = \mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by setting

$$\hat{f}(\xi) = \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

The letter C denotes generic constants and the constant $C_{a,b,\dots}$ depends on $a, b, \dots$. The values of constants C and $C_{a,b,\dots}$ may change from line to line. We use small boldface letters, e.g. $\mathbf{f}$ to denote vector-valued functions and capital boldface letters, e.g. $\mathbf{G}$ to denote matrix-valued functions, respectively. Furthermore, we set spaces and norms:

$$J_q(\mathbb{R}^N) = \{\mathbf{u} \in L_q(\mathbb{R}^N)^N \mid \ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \mathbb{R}^N\},$$
$$D_{q,p}(\mathbb{R}^N) = \{(\mathbf{u}, \mathbb{Q}) \mid \mathbf{u} \in (B_{q,p}^{2(1-1/p)}(\mathbb{R}^N)^N \cap J_q(\mathbb{R}^N)), \ \ \mathbb{Q} \in B_{q,p}^{1+2(1-1/p)}(\mathbb{R}^N; \mathbb{S}_0)\},$$
$$X_{p,q,t} = \{(\mathbf{u}, \mathbb{Q}) \mid \mathbf{u} \in L_p((0,t), W_q^2(\mathbb{R}^N)^N) \cap W_p^1((0,t), L_q(\mathbb{R}^N)^N),$$
$$\mathbb{Q} \in L_p((0,t), W_q^3(\mathbb{R}^N; \mathbb{S}_0)) \cap W_p^1((0,t), W_q^1(\mathbb{R}^N; \mathbb{S}_0))\},$$

$$
\begin{aligned}
&\mathcal{N}(\mathbf{u}, \mathbb{Q})(T) \\
&\quad = \sum_{q=q_1,q_2} \Big(\| < t >^b (\mathbf{u}, \mathbb{Q}) \|_{L_\infty((0,T), W^{0,1}_q(\mathbb{R}^N))} + \| < t >^b \partial_t (\mathbf{u}, \mathbb{Q}) \|_{L_p((0,T), W^{0,1}_q(\mathbb{R}^N))} \Big) \\
&\qquad + \| < t >^b \nabla (\mathbf{u}, \mathbb{Q}) \|_{L_p((0,T), W^{1,2}_{q_1}(\mathbb{R}^N))} + \| < t >^b (\mathbf{u}, \mathbb{Q}) \|_{L_p((0,T), W^{2,3}_{q_2}(\mathbb{R}^N))},
\end{aligned}
$$

where $< t >= (1+t^2)^{1/2}$, b is given in Theorem 1.1 below.

The following theorem is our main result of this article.

Theorem 1.1. *Assume that $N \geq 3$. Let $0 < T < \infty$, $0 < \sigma < 1/2$, and let $p = 2$ or $p = 2 + \sigma$. Let q_1 and q_2 be numbers such that*

$$
q_1 = 2 + \sigma, \quad \begin{cases} q_2 \geq \dfrac{N(2+\sigma)}{N - (2+\sigma)} & \text{if } N = 3, 4, \\ q_2 > N & \text{if } N \geq 5. \end{cases}
$$

Suppose that $b = (N-\sigma)/(2(2+\sigma))$ if $p = 2$ and $b = N/(2(2+\sigma))$ if $p = 2+\sigma$. Then, there exists a small number $\epsilon > 0$ such that for any initial data $(\mathbf{u}_0, \mathbb{Q}_0) \in \bigcap_{i=1}^2 D_{q_i,p}(\mathbb{R}^N) \cap W^{0,1}_{q_1/2}(\mathbb{R}^N)$ with

$$
\mathcal{I} := \sum_{i=1}^2 \|(\mathbf{u}_0, \mathbb{Q}_0)\|_{D_{q_i,p}(\mathbb{R}^N)} + \|(\mathbf{u}_0, \mathbb{Q}_0)\|_{W^{0,1}_{q_1/2}(\mathbb{R}^N)} < \epsilon^2,
$$

problem (1.1) *admits a unique solution* $(\mathbf{u}, \mathbb{Q})$ *with*

$$
(\mathbf{u}, \mathbb{Q}) \in X_{p,q_1,T} \cap X_{p,q_2,T}
$$

satisfying the estimate

$$
\mathcal{N}(\mathbf{u}, \mathbb{Q})(T) \leq \epsilon.
$$

Remark 1.2. $T > 0$ is taken arbitrarily and ϵ is chosen independent of T; therefore, Theorem 1.1 yields the global well-posedness for (1.1).

2 Idea of the proof of Theorem 1.1

Theorem 1.1 can be proved by the Banach fixed point argument. To explain our idea more precisely, we rewrite symbolically (1.1).

$$
\begin{cases} \partial_t u - \mathcal{A} u = f(u) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N, \end{cases}
$$

where $u = (\mathbf{u}, \mathbb{Q})$ and $f(u) = (\mathbf{f}(u), \mathbf{G}(u))$. Moreover, $\mathcal{A}$ is a linear operator

$$
\mathcal{A} u = (P\Delta \mathbf{u} - \beta P \mathrm{Div}\,(\Delta \mathbb{Q} - a\mathbb{Q}), \beta \mathbf{D}(\mathbf{u}) + \Delta \mathbb{Q} - a\mathbb{Q})
$$

defined for $(\mathbf{u}, \mathbb{Q}) \in D(\mathcal{A})$, where P denotes the solenoidal projection and

$$
D(\mathcal{A}) = (W^2_q(\mathbb{R}^N)^N \cap J_q(\mathbb{R}^N)) \times W^3_q(\mathbb{R}^N; \mathbb{S}_0).
$$

Let p, q_1, and q_2 be exponents given in Theorem 1.1. Let ϵ be a small positive number and let $\mathcal{I}_{T,\epsilon}$ be the underlying space defined as

$$\mathcal{I}_{T,\epsilon} = \{u \in X_{p,q_1,T} \cap X_{p,q_2,T} \ |u|_{t=0} = u_0,\ \mathcal{N}(u)(T) \le \epsilon, \\ \sup_{0<t<T} \|\mathbb{Q}(\cdot, t)\|_{L_\infty(\mathbb{R}^N)} \le 1\}.$$

Given $v \in \mathcal{I}_{T,\epsilon}$, let u be a solution to the equation:

$$\begin{cases} \partial_t u - \mathcal{A}u = f(v) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

The key estimate to apply the Banach fixed point argument is

$$\mathcal{N}(u)(T) \le C\epsilon^2. \tag{2.2}$$

Assuming that we know the linear operator $\mathcal{A}$ is the sectorial operator and the following several theorems concerning the estimates for the solutions of the linearized problem, we explain the outline of the proof of (2.2). For this purpose, we set $u = u_1 + u_2$, where u_1 satisfies the time-shifted equations:

$$\begin{cases} \partial_t u_1 + \lambda_1 u_1 - \mathcal{A}u_1 = f(v), & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ u_1|_{t=0} = 0 & \text{in } \mathbb{R}^N \end{cases} \tag{2.3}$$

with some large constant λ_1 and u_2 satisfies the compensation equations:

$$\begin{cases} \partial_t u_2 - \mathcal{A}u_2 = \lambda_1 u_1, & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ u_2|_{t=0} = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.4}$$

Firstly, we consider (2.3). Replacing $f(v)$ with f, we have the linearized problem:

$$\begin{cases} \partial_t u + \lambda_1 u - \mathcal{A}u = f, & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ u|_{t=0} = 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{2.5}$$

The solution of (2.5) satisfies the maximal L_p-L_q regularity estimates as follows.

Theorem 2.1. *Let $1 < p, q < \infty$. Let $b \ge 0$. Then, there exists a constant $\lambda_1 \ge 1$ such that the following assertion holds: For any $f \in L_p((0,T), W^{0,1}_q(\mathbb{R}^N))$, problem (2.5) admits a unique solution $u \in X_{p,q,T}$ possessing the estimate*

$$\| <t>^b \partial_t u\|_{L_p((0,T),W^{0,1}_q(\mathbb{R}^N))} + \| <t>^b u\|_{L_p((0,T),W^{2,3}_q(\mathbb{R}^N))} \\ \le C\| <t>^b f\|_{L_p((0,T),W^{0,1}_q(\mathbb{R}^N))}.$$

By Theorem 2.1 and the estimates for nonlinear terms, we have

$$\tilde{\mathcal{N}}(u_1)(T) \le C\mathcal{N}(v)(T)^2 \le C\epsilon^2, \tag{2.6}$$

where

$$\tilde{\mathcal{N}}(u_1)(T) = \sum_{q=q_1/2,q_1,q_2} \| <t>^b \partial_t u_1\|_{L_p((0,T),W^{0,1}_q(\mathbb{R}^N))} + \| <t>^b u_1\|_{L_p((0,T),W^{2,3}_q(\mathbb{R}^N))}.$$

Secondly, we consider (2.4). Note that the linear operator $\mathcal{A}$ generates a continuous analytic semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ on

$$X_q(\mathbb{R}^N) = J_q(\mathbb{R}^N) \times W_q^1(\mathbb{R}^N; \mathbb{S}_0)$$

for $1 < q < \infty$ such that $u = e^{\mathcal{A}t}u_0$ is a unique solution of (2.1) with $f(v) = 0$. Moreover, there exist constants γ_1 and C such that

$$\begin{aligned} \|u(t)\|_{X_q(\mathbb{R}^N)} \leq Ce^{\gamma_1 t}\|u_0\|_{X_q(\mathbb{R}^N)}, \quad \|\partial_t u(t)\|_{X_q(\mathbb{R}^N)} \leq Ce^{\gamma_1 t}t^{-1}\|u_0\|_{X_q(\mathbb{R}^N)}, \\ \|\partial_t u(t)\|_{X_q(\mathbb{R}^N)} \leq Ce^{\gamma_1 t}\|u_0\|_{D(\mathcal{A})} \end{aligned} \tag{2.7}$$

for any $t > 0$. (2.7) and a real interpolation method (cf. Shibata and Shimizu [11, Proof of Theorem 3.9]) yield the following theorem.

Theorem 2.2. *Let $1 < p, q < \infty$. Then, for any $u_0 \in D_{q,p}(\mathbb{R}^N)$, (2.1) with $f(v) = 0$ admits a unique solution $u = e^{\mathcal{A}t}u_0$ possessing the estimate:*

$$\|e^{-\gamma t}\partial_t u\|_{L_p(\mathbb{R}_+, W_q^{0,1}(\mathbb{R}^N))} + \|e^{-\gamma t}u\|_{L_p(\mathbb{R}_+, W_q^{2,3}(\mathbb{R}^N))} \leq C\|u_0\|_{D_{q,p}(\mathbb{R}^N)} \tag{2.8}$$

for any $\gamma \geq \gamma_1$.

Moreover, $\{e^{\mathcal{A}t}\}_{t\geq 0}$ satisfies the decay properties and a standard estimate.

Theorem 2.3. *Let u be the solution of (2.1) with $f(v) = 0$ for $u_0 \in X_q(\mathbb{R}^N)$. Then, u satisfies the following estimate:*

$$\begin{aligned} \|\nabla^j u\|_{W_p^{0,1}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{j}{2}}(\|u_0\|_{W_q^{0,1}(\mathbb{R}^N)} + \|u_0\|_{W_p^{0,1}(\mathbb{R}^N)}), \\ \|\partial_t u\|_{W_p^{0,1}(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-1}(\|u_0\|_{W_q^{0,1}(\mathbb{R}^N)} + \|u_0\|_{W_p^{0,1}(\mathbb{R}^N)}) \end{aligned} \tag{2.9}$$

for $t \geq 1$, $1 < q < 2 \leq p < \infty$, $j = 0, 1, 2$. Moreover,

$$\|u\|_{W_p^{2,3}(\mathbb{R}^N)} + \|\partial_t u\|_{W_p^{0,1}(\mathbb{R}^N)} \leq C\|u_0\|_{W_p^{2,3}(\mathbb{R}^N)} \tag{2.10}$$

for $0 < t < 1$.

Remark 2.4. (2.10) follows from the fact that $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is the continuous analytic semigroup.

The Duhamel's principle implies that

$$u_2 = e^{\mathcal{A}t}u_0 + \lambda_1 \int_0^t e^{\mathcal{A}(t-s)}u_1(\cdot, s)\, ds. \tag{2.11}$$

For the first term of (2.11), we use (2.9) if $t > 1$ and (2.8) if $0 < t < 1$. On the other hand, for the second term of (2.11), we use (2.9) and (2.10) in order to estimate $\int_0^{t-1} e^{\mathcal{A}(t-s)}u_1(\cdot, s)\, ds$ and $\int_{t-1}^t e^{\mathcal{A}(t-s)}u_1(\cdot, s)\, ds$, respectively. For the later part, what $u_1(t) \in D(\mathcal{A})$ for $t > 0$ is a key observation. Therefore, by (2.6), we have

$$\mathcal{N}(u_2)(T) \leq C(\mathcal{I} + \tilde{\mathcal{N}}(u_1)(T)) \leq C\epsilon^2. \tag{2.12}$$

Combining (2.6) and (2.12), we have (2.2).

3 $\mathcal{R}$-boundedness of solution operators

According to section 2, the key statements to prove Theorem 1.1 are the maximal L_p-L_q regularity for the time-shifted equations, the generation of the continuous analytic semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$, and the estimates for $\{e^{\mathcal{A}t}\}_{t\geq 0}$. In order to show these statements, we analyze the following resolvent problem:

$$\begin{cases} \lambda\mathbf{u} - \Delta\mathbf{u} + \nabla\mathfrak{p} + \beta\mathrm{Div}\,(\Delta\mathbb{Q} - a\mathbb{Q}) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = 0 & \text{in } \mathbb{R}^N, \\ \lambda\mathbb{Q} - \beta\mathbf{D}(\mathbf{u}) - \Delta\mathbb{Q} + a\mathbb{Q} = \mathbf{G} & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

where λ is the resolvent parameter varying in a sector

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| < \pi - \epsilon, |\lambda| \geq \lambda_0\}$$

for $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. Moreover, we set an angle $\sigma_0 \in (0, \pi/2)$ by

$$\sigma_0 = \begin{cases} 0 & \text{if } \beta = 0, \\ \arg(1 + i|\beta|) & \text{if } \beta \neq 0. \end{cases} \tag{3.2}$$

In this section, we especially consider $\mathcal{R}$-boundedness of solution operator families for (3.1). Here, we introduce the definition of $\mathcal{R}$-boundedness of operator families.

Definition 3.1. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, we have the inequality:

$$\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u)T_jf_j\|_Y^p\,du\Big\}^{1/p} \leq C\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u)f_j\|_X^p\,du\Big\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

The following theorem is the main result of this section.

Theorem 3.2. *Let $1 < q < \infty$ and $\lambda_0 > 0$. Then, for any $\sigma \in (\sigma_0, \pi/2)$, there exist operator families*

$$\mathcal{A}(\lambda) \in \mathrm{Hol}(\Sigma_{\sigma,\lambda_0}, \mathcal{L}(W_q^{0,1}(\mathbb{R}^N), W_q^2(\mathbb{R}^N)^N))$$
$$\mathcal{B}(\lambda) \in \mathrm{Hol}(\Sigma_{\sigma,\lambda_0}, \mathcal{L}(W_q^{0,1}(\mathbb{R}^N), W_q^3(\mathbb{R}^N; \mathbb{S}_0)))$$

such that for any $\lambda = \gamma + i\tau \in \Sigma_{\sigma,\lambda_0}$, $\mathbf{f} \in L_q(\mathbb{R}^N)^N$, and $\mathbf{G} \in W_q^1(\mathbb{R}^N; \mathbb{S}_0)$,

$$\mathbf{u} = \mathcal{A}(\lambda)(\mathbf{f}, \mathbf{G}), \quad \mathbb{Q} = \mathcal{B}(\lambda)(\mathbf{f}, \mathbf{G})$$

are unique solutions of problem (3.1), and

$$\mathcal{R}_{\mathcal{L}(W_q^{0,1}(\mathbb{R}^N), A_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^n\mathcal{S}_\lambda\mathcal{A}(\lambda) \mid \lambda \in \Sigma_{\sigma,\lambda_0}\}) \leq r, \tag{3.3}$$
$$\mathcal{R}_{\mathcal{L}(W_q^{0,1}(\mathbb{R}^N), B_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^n\mathcal{T}_\lambda\mathcal{B}(\lambda) \mid \lambda \in \Sigma_{\sigma,\lambda_0}\}) \leq r \tag{3.4}$$

for $n = 0, 1$, where $\mathcal{S}_\lambda\mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u})$, $\mathcal{T}_\lambda\mathbb{Q} = (\nabla^3\mathbb{Q}, \lambda^{1/2}\nabla^2\mathbb{Q}, \lambda\mathbb{Q})$, $A_q(\mathbb{R}^N) = L_q(\mathbb{R}^N)^{N^3+N^2+N}$, $B_q(\mathbb{R}^N) = L_q(\mathbb{R}^N; \mathbb{R}^{N^5}) \times L_q(\mathbb{R}^N; \mathbb{R}^{N^4}) \times W_q^1(\mathbb{R}^N; \mathbb{S}_0)$, and r is a constant independent of λ.

Remark 3.3. (i) Since Definition 3.1 with $n = 1$ implies the uniform boundedness of the operator family $\mathcal{T}$, solutions $\mathbf{u}$ and $\mathbb{Q}$ of equations (3.1) satisfy the resolvent estimate for any $\lambda \in \Sigma_{\sigma,\lambda_0}$; therefore, the linear operator $\mathcal{A}$ generates a continuous analytic semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ satisfying (2.7).

(ii) By Theorem 3.2 and the Weis operator valued Fourier multiplier theorem [12], we have Theorem 2.1.

From now, we explain the outline of the proof of Theorem 3.2. Firstly, we calculate a solution formula of (3.1). Taking divergence of the first equation of (3.1), we have

$$\mathfrak{p} = -\beta(\operatorname{div}\operatorname{Div}\mathbb{Q} - a\Delta^{-1}\operatorname{div}\operatorname{Div}\mathbb{Q}) + \Delta^{-1}\operatorname{div}\mathbf{f}. \tag{3.5}$$

Inserting (3.5) into the first equation of (3.1), we have

$$(\lambda - \Delta)\mathbf{u} - \beta\nabla\operatorname{div}\operatorname{Div}(\mathbb{Q} - a\Delta^{-1}\mathbb{Q}) + \beta\operatorname{Div}(\Delta\mathbb{Q} - a\mathbb{Q}) = \mathbf{f} - \nabla\Delta^{-1}\operatorname{div}\mathbf{f}. \tag{3.6}$$

Since the second equation of (3.1) yields

$$(\lambda - \Delta + a)\mathbb{Q} = \beta\mathbf{D}(\mathbf{u}) + \mathbf{G},$$

$\mathbb{Q}$ can be represented by $\mathbf{u}$; therefore, we only consider a solution formula for $\mathbf{u}$ below. Applying $\lambda - \Delta + a$ to (3.6) we have

$$\begin{aligned}\mathbf{u} = P(\lambda)^{-1}\{&(\lambda - \Delta + a)(\mathbf{f} - \nabla\Delta^{-1}\operatorname{div}\mathbf{f})\\ &+ \beta\nabla(\operatorname{div}\operatorname{Div}\mathbf{G} - a\Delta^{-1}\operatorname{div}\operatorname{Div}\mathbf{G}) - \beta\operatorname{Div}(\Delta\mathbf{G} - a\mathbf{G})\},\end{aligned}$$

where $P(\lambda) = (\lambda - \Delta)(\lambda - (\Delta - a)) + \beta^2(\Delta^2 - a\Delta)$.Thus $\mathbf{u} = (u_1, \ldots, u_N)$ has form:

$$u_j = \mathcal{A}_j(\lambda)(\mathbf{f}, \mathbf{G})$$

with

$$\begin{aligned}\mathcal{A}_j(\lambda)(\mathbf{f}, \mathbf{G}) = &\mathcal{F}^{-1}\left[\frac{\lambda + |\xi|^2 + a}{P(\xi, \lambda)}\left(\hat{f}_j - \frac{\xi_j}{|\xi|^2}\xi \cdot \hat{\mathbf{f}}\right)\right]\\ &- \mathcal{F}^{-1}\left[\frac{\beta}{P(\xi, \lambda)}\left(i\xi_j\sum_{\ell,m=1}^{N}\xi_\ell\xi_m\hat{G}_{\ell m} - \sum_{\ell=1}^{N}i\xi_\ell|\xi|^2\hat{G}_{j\ell}\right)\right]\\ &- a\mathcal{F}^{-1}\left[\frac{\beta}{P(\xi, \lambda)}\left(i\xi_j\sum_{\ell,m=1}^{N}\frac{\xi_\ell\xi_m}{|\xi|^2}\hat{G}_{\ell m} - \sum_{\ell,m=1}^{N}i\xi_\ell\hat{G}_{j\ell}\right)\right],\end{aligned}$$

where

$$\begin{aligned}P(\xi, \lambda) &= (\lambda + |\xi|^2)(\lambda + |\xi|^2 + a) + \beta^2(|\xi|^4 + a|\xi|^2)\\ &= (\lambda - \lambda_+(|\xi|))(\lambda - \lambda_-(|\xi|))\end{aligned}$$

with

$$\lambda_\pm(|\xi|) = -\left(|\xi|^2 + \frac{a}{2}\right) \pm \sqrt{\frac{a^2}{4} - a\beta^2|\xi|^2 - \beta^2|\xi|^4}$$

which has the following expansions :

$$\begin{cases}\lambda_+(|\xi|) = -(1 + \beta^2)|\xi|^2 + O(|\xi|^4),\\ \lambda_-(|\xi|) = -(1 - \beta^2)|\xi|^2 - a + O(|\xi|^4) \quad \text{as } |\xi| \to 0,\end{cases} \tag{3.7}$$

$$\lambda_\pm(|\xi|) = (-1 \pm i|\beta|)|\xi|^2 + O(1) \quad \text{as } |\xi| \to \infty. \tag{3.8}$$

By (3.7) and (3.8), $P(\xi, \lambda)$ has the following estimate.

Lemma 3.4. *Let σ_0 be the angle defined in* (3.2). *Then, for any $\sigma \in (\sigma_0, \pi/2)$ and $(\xi, \lambda) \in \mathbb{R}^N \times \Sigma_{\sigma,0}$, we have*

$$|P(\xi,\lambda)| \geq C_{\sigma,\beta}(|\lambda|^{1/2} + |\xi|)^4$$

with some constant $C_{\sigma,\beta}$ independent of ξ and λ.

Secondly, we prove $\mathcal{R}$-boundedness of $\mathcal{A}(\lambda)$, where we have set $\mathcal{A}(\lambda)(\mathbf{f}, \mathbf{G})$ is a vector whose j^{th} component is $\mathcal{A}_j(\lambda)(\mathbf{f}, \mathbf{G})$. We introduce the following lemma proved by [5, Lemma 2.1], [6, Theorem 3.3], and [9, Lemma 2.5].

Lemma 3.5. *Let $1 < q < \infty$, $\delta > 0$. Assume that $k(\xi, \lambda)$, $\ell(\xi, \lambda)$, and $m(\xi, \lambda)$ are functions on $(\mathbb{R}^N \setminus \{0\}) \times \Sigma_{\sigma,0}$ such that for any $\sigma \in (\sigma_0, \pi/2)$ and any multi-index $\alpha \in \mathbb{N}_0^N$ there exists a positive constant $M_{\alpha,\sigma}$ such that*

$$\begin{aligned}
&|\partial_\xi^\alpha k(\xi,\lambda)| \leq M_{\alpha,\sigma}|\xi|^{1-|\alpha|}, \quad |\partial_\xi^\alpha \ell(\xi,\lambda)| \leq M_{\alpha,\sigma}|\xi|^{-|\alpha|},\\
&|\partial_\xi^\alpha m(\xi,\lambda)| \leq M_{\alpha,\sigma}(|\lambda|^{1/2} + |\xi|)^{-1}|\xi|^{-|\alpha|}
\end{aligned}$$

for any $(\xi, \lambda) \in (\mathbb{R}^N \setminus \{0\}) \times \Sigma_{\sigma,0}$. Let $K(\lambda)$, $L(\lambda)$, and $M(\lambda)$ be operators defined by

$$\begin{aligned}
&[K(\lambda)f](x) = \mathcal{F}^{-1}[k(\xi,\lambda)\hat{f}(\xi)](x) \qquad (\lambda \in \Sigma_{\sigma,0}),\\
&[L(\lambda)f](x) = \mathcal{F}^{-1}[\ell(\xi,\lambda)\hat{f}(\xi)](x) \qquad (\lambda \in \Sigma_{\sigma,0}),\\
&[M(\lambda)f](x) = \mathcal{F}^{-1}[m(\xi,\lambda)\hat{f}(\xi)](x) \quad\; (\lambda \in \Sigma_{\sigma,\delta}).
\end{aligned}$$

Then, the following assertions hold true:

(1) *The set $\{K(\lambda) \mid \lambda \in \Sigma_{\sigma,0}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(W_q^1(\mathbb{R}^N), L_q(\mathbb{R}^N))$ and there exists a positive constant $C_{N,q}$ such that*

$$\mathcal{R}_{\mathcal{L}(W_q^1(\mathbb{R}^N),L_q(\mathbb{R}^N))}(\{K(\lambda) \mid \lambda \in \Sigma_{\sigma,0}\}) \leq C_{N,q} \max_{|\alpha|\leq N+1} M_{\alpha,\sigma}.$$

(2) *Let $n = 0, 1$. Then, the set $\{L(\lambda) \mid \lambda \in \Sigma_{\sigma,0}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(W_q^n(\mathbb{R}^N))$ and there exists a positive constant $C_{N,q}$ such that*

$$\mathcal{R}_{\mathcal{L}(W_q^n(\mathbb{R}^N))}(\{L(\lambda) \mid \lambda \in \Sigma_{\sigma,0}\}) \leq C_{N,q} \max_{|\alpha|\leq N+1} M_{\alpha,\sigma}.$$

(3) *The set $\{M(\lambda) \mid \lambda \in \Sigma_{\sigma,\delta}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N), W_q^1(\mathbb{R}^N))$ and there exists a positive constant $C_{N,q}$ such that*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N),W_q^1(\mathbb{R}^N))}(\{M_\lambda \mid \lambda \in \Sigma_{\sigma,\delta}\}) \leq C_{N,q,\delta} \max_{|\alpha|\leq N+1} M_{\alpha,\sigma}.$$

By Lemma 3.4 and Bell's formula, $P(\xi, \lambda)$ satisfies the following estimates.

Lemma 3.6. *Let $a > 0$ and $n = 0, 1$. Then, for any $\sigma \in (\sigma_0, \pi/2)$ and any multi-index $\alpha \in \mathbb{N}_0^N$, there exists a positive constant C depending on at most α, ϵ and b such that for any $(\xi, \lambda) \in \mathbb{R}^N \times \Sigma_{\sigma,0}$ with $\lambda = \gamma + i\tau$,*

$$|\partial_\xi^\alpha\{(\tau\partial_\tau)^n P(\xi,\lambda)^{-1}\}| \leq C(|\lambda|^{1/2} + |\xi|)^{-4-|\alpha|}.$$

Therefore, we can verify $(\lambda, \lambda^{1/2}\xi_j, \xi_j\xi_k)\mathcal{A}(\lambda)\mathbf{f}$ with $j, k = 1, \ldots, N$ satisfies the assumption of Lemma 3.5, which yields (3.3) in Theorem 3.2. Similarly, we have (3.4) in Theorem 3.2.

Finally, we mention (2.9) in Theorem 2.3. Theorem 3.2 implies that there exist operators

$$S(t) \in \mathcal{L}(X_q(\mathbb{R}^N), W_q^2(\mathbb{R}^N)^N), \ \ T(t) \in \mathcal{L}(X_q(\mathbb{R}^N), W_q^3(\mathbb{R}^N; \mathbb{S}_0))$$

such that for any $(\mathbf{u}_0, \mathbb{Q}_0) \in X_q(\mathbb{R}^N)$, $\mathbf{u} = S(t)(\mathbf{u}_0, \mathbb{Q}_0)$ and $\mathbb{Q} = T(t)(\mathbf{u}_0, \mathbb{Q}_0)$ satisfy (1.3) with $(\mathbf{f}(\mathbf{u}, \mathbb{Q}), \mathbf{G}(\mathbf{u}, \mathbb{Q}))$ replaced by $(0, O)$. In this article, we only explain the L_p-L_q decay estimates of the operator $S(t)$. For this purpose, we decompose the solution into low and high-frequency parts. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a function such that $0 \le \varphi(\xi) \le 1$, $\varphi(\xi) = 1$ if $|\xi| \le 1/3$ and $\varphi(\xi) = 0$ if $|\xi| \ge 2/3$. Let φ_0 and φ_∞ be functions such that

$$\varphi_0(\xi) = \varphi(\xi/A_0), \ \ \varphi_\infty(\xi) = 1 - \varphi(\xi/A_0),$$

where $A_0 \in (0, 1)$ is a sufficiently small number. Moreover, we set

$$\begin{aligned} S_n(t)(\mathbf{u}_0, \mathbb{Q}_0) = & \frac{1}{2\pi i} \sum_{j=1}^{2} \mathcal{F}^{-1} \left[\int_\Gamma e^{\lambda t} \ell_j(\xi, \lambda) \varphi_n \hat{\mathbf{u}}_0 \, d\lambda \right](x) \\ & + \frac{1}{2\pi i} \sum_{j=1}^{4} \mathcal{F}^{-1} \left[\int_\Gamma e^{\lambda t} m_j(\xi, \lambda) \varphi_n \widehat{\mathrm{Div}\, \mathbb{Q}_0} \, d\lambda \right](x), \end{aligned}$$

where $n = 0, \infty$,

$$\ell_1(\xi, \lambda)\hat{\mathbf{u}}_0 = \frac{\lambda + |\xi|^2 + a}{P(\xi, \lambda)} \hat{\mathbf{u}}_0, \qquad \ell_2(\xi, \lambda)\hat{\mathbf{u}}_0 = -\frac{\lambda + |\xi|^2 + a}{P(\xi, \lambda)} \frac{\xi}{|\xi|^2} \xi \cdot \hat{\mathbf{u}}_0,$$

$$m_1(\xi, \lambda)\widehat{\mathrm{Div}\, \mathbb{Q}_0} = -\frac{\beta\xi}{P(\xi, \lambda)} \xi \cdot \widehat{\mathrm{Div}\, \mathbb{Q}_0}, \qquad m_2(\xi, \lambda)\widehat{\mathrm{Div}\, \mathbb{Q}_0} = \frac{\beta|\xi|^2}{P(\xi, \lambda)} \widehat{\mathrm{Div}\, \mathbb{Q}_0},$$

$$m_3(\xi, \lambda)\widehat{\mathrm{Div}\, \mathbb{Q}_0} = -\frac{a\beta}{P(\xi, \lambda)} \frac{\xi}{|\xi|^2} \xi \cdot \widehat{\mathrm{Div}\, \mathbb{Q}_0}, \qquad m_4(\xi, \lambda)\widehat{\mathrm{Div}\, \mathbb{Q}_0} = \frac{a\beta}{P(\xi, \lambda)} \widehat{\mathrm{Div}\, \mathbb{Q}_0}.$$

Here, we set the integral path $\Gamma = \Gamma^+ \cup \Gamma^-$ as follows:

$$\Gamma^\pm = \{\lambda \in \mathbb{C} \mid \lambda = \tilde{\lambda}_0(\sigma) + s e^{\pm i(\pi - \sigma)}, s : 0 \to \infty\}$$

for $\sigma_0 < \sigma < \pi/2$ with $\tilde{\lambda}_0(\sigma) = 2\lambda_0 / \sin \sigma$, where λ_0 is the same as in Theorem 3.2. In view of (3.7), the L_p-L_q estimates of the heat kernel are helpful for the low-frequency part. On the other hand, since we have the resolvent estimates by using Theorem 3.2 if the integral path belongs to $\Sigma_{\sigma_0, \lambda_0}$ and otherwise Fourier multiplier theorem, we have exponential decay estimates for the high-frequency part if $t \ge 1$. Therefore, we have (2.9) in Theorem 2.3.

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