Spherical functions and local densities for p-adic sesquilinear forms

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§0 Introduction

Let $\mathbb{G}$ be a reductive linear algebraic group defined over k, and $\mathbb{X}$ be an affine algebraic variety defined over k which is $\mathbb{G}$-homogeneous, where and henceforth k stands for a non-archimedian local field of characteristic 0. The Hecke algebra $\mathcal{H}(G,K)$ of G with respect to K acts by convolution product on the space of $\mathcal{C}^\infty(K\backslash X)$ of K-invariant $\mathbb{C}$-valued functions on X, where K is a maximal compact open subgroup of $G = \mathbb{G}(k)$ and $X = \mathbb{X}(k)$. A nonzero function in $\mathcal{C}^\infty(K\backslash X)$ is called *a spherical function on* X if it is a common $\mathcal{H}(G,K)$-eigen function.

Spherical functions on homogeneous spaces are an interesting object to investigate and a basic tool to study harmonic analysis on G-space X. The cases of sesquiliear forms are particular interesting, since spherical functions can be regarded as generating functions of local densities, and the latter have a close connection to global theory of automorphic forms. We will explain this relation in §1. If one has good explicit formulas of spherical functions, one may have a good formulas for local densities, and vica versa.

Although one has most interest in the case of symmetric forms, good explicit formulas of spherical functions nor local densities are not known. On the other hand, one may study some general theory of local densities without their explicit formulas (cf. [BS], [BHS], [H2], [H5], [Ki]). In this note, we summarize such results on local densities on hermitian forms (unramified or ramified hermitian forms over a field and quaternion hermitian forms). In §1, we define spherical functions and local densities and introduce their relations. Then we consider linear independence of local densities in §2, and we give denominators of certain formal power series (Kitaoka series) attached to local densities in §3.

§1 Spherical functions and local densities

Let k a $\mathfrak{p}$-adic field, and set $\mathfrak{o} = \mathcal{O}_k$, $\pi = \pi_k$, $\mathfrak{p} = \pi\mathfrak{o}$, $q = |\mathfrak{o}/\mathfrak{p}|$, as usual. We assume q is odd.

We introduce the notations of unramified hermitian case (U), ramified hermitian case (R), and quaternion hermitian case (D). We set

$$L = \begin{cases} \text{unramified quadratic extension } k' \text{ of } k & \text{for (U)} \\ \text{ramified quadratic extension } k' \text{ of } k & \text{for (R)} \\ \text{division quaternion } D \text{ over } k & \text{for (D),} \end{cases} \tag{1.1}$$

and $\mathcal{O} = \mathcal{O}_L$, and π_L to be the prime element of L such that

$$\pi_L = \begin{cases} \pi & \text{for (U)} \\ \varPi \quad (\varPi^2 = \pi) & \text{for (R) and (D).} \end{cases} \tag{1.2}$$

For matrix $A \in M_{mn}(L)$, we denote its complex conjugate by $A^* \in M_{nm}(L)$, where $(A^*)_{ij} = {A_{ji}}^*$ by nontrivial k-automorphism on k' for (U) and (R), or by the canonical involution on D. One may refer [H7, §1] for basic notions and properties of case (D).

Set $G = G_n = GL_n(L)$ and $K = K_n = GL_n(\mathcal{O})$. Then the spaces of hermitian forms are defined as follows (by the corresponding L)

$$\begin{aligned} V_n &= \{x \in M_n(L) \mid x^* = x\} \supset X_n = \{x \in G_n \mid x^* = x\}, \\ V_n^+ &= V_n \cap M_n(\mathcal{O}) \supset X_n^+ = X_n \cap M_n(\mathcal{O}), \end{aligned} \tag{1.3}$$

where G_n acts on V_n and X_n by $g \cdot x = gxg^*$, and K_n acts on V_n^+ and X_n^+. Set $B = B_n$ the set of lower triangular matrices in G. For a matrix x of size n, we denote by $x^{(i)}$ its upper left $(i \times i)$-block. We define relative B-invariants $d_i(x) (\in k)$, $1 \le i \le n$ on X_n as follows:

$$\begin{aligned} d_i(x) &= \det(x^{(i)}) \quad \text{for (U) and (R),} \\ d_i(x)^2 &= \mathrm{N}_{\mathrm{rd}}(x^{(i)}) \quad \text{for (D),} \end{aligned} \tag{1.4}$$

where $\mathrm{N}_{\mathrm{rd}}(x^{(i)})$ is the reduced norm on $V_i(D)$. The action of $b \in B$ is given by

$$d_i(b \cdot x) = \begin{cases} \mathrm{N}_{k'/k}(\det(b^{(i)}))\, d_i(x) & \text{for (U) and (R)} \\ \mathrm{N}_{\mathrm{rd}}(b^{(i)})\, d_i(x) & \text{for (D)} \end{cases} \tag{1.5}$$

For $x \in X_n$ and $s \in \mathbb{C}^n$, we consider the integral

$$\omega(x; s) = \int_{K_n} |\mathbf{d}(k \cdot x)|^s \, dk, \quad |\mathbf{d}(y)|^s = \begin{cases} \prod_{i=1}^n |d_i(y)|^{s_i} & \text{if } y \in X^{op} \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

where dk is the normalized Haar measure on K_n and $|\ |$ is the absolute value on k and

$$X_n^{op} = \{x \in X_n \mid d_i(x) \neq 0,\ 1 \le i \le n\}.$$

The integral (1.6) is absolutely convergent if $\mathrm{Re}(s_i) \geq 0,\ 1 \leq i \leq n-1$, and continued to a rational function on $q^{s_1}, \ldots, q^{s_n}$. Then it becomes an element of

$$\mathcal{C}^\infty(K\backslash X) = \{\Psi : X \longrightarrow \mathbb{C} \mid \Psi(k \cdot x) = \Psi(x),\ k \in K\}, \tag{1.7}$$

and we use the notation $\omega(x; s)$ in such sense. Let $\mathcal{H}(G, K)$ be the Hecke algebra of G with respect to K and recall the action of $\mathcal{H}(G, K)$ on $\mathcal{C}^\infty(K\backslash X)$

$$f * \Psi(x) = \int_G f(g)\Psi(g^{-1} \cdot x)dg, \quad (f \in \mathcal{H}(G, K),\ \Psi \in \mathcal{C}^\infty(K\backslash X),\ x \in X), \tag{1.8}$$

where dg is the normalized Haar measure on G. We call $\omega(x; s)$ is *a spherical function on* X, since it is a common eigenfunction with respect to the above $\mathcal{H}(G, K)$-action. (This is a general theory for spherical functions on homogeneous spaces, see [H6], for example.)

For $A \in X_m^+$ and $B \in X_n^+$ with $m \geq n$, we define local density of B by A as follows:

$$\begin{aligned} \mu(B, A) &= \lim_{\ell \to \infty} \frac{\sharp \left\{ \overline{v} \in M_{mn}(\mathcal{O}/\pi^\ell \mathcal{O}) \;\middle|\; A[v] - B \in M_n(\pi^\ell \mathcal{O}) \right\}}{q^{\ell n (**)}}, \\ \mu^{pr}(B, A) &= \lim_{\ell \to \infty} \frac{\sharp \left\{ \overline{v} \in M_{mn}^{pr}(\mathcal{O}/\pi^\ell \mathcal{O}) \;\middle|\; A[v] - B \in M_n(\pi^\ell \mathcal{O}) \right\}}{q^{\ell n (**)}}, \end{aligned} \tag{1.9}$$

where we identify $M_{mn}(\mathcal{O}/\pi^\ell\mathcal{O})$ with $M_{mn}(\mathcal{O})/M_{mn}(\pi^\ell\mathcal{O})$, $A[v] = v^*Av \in V_n(\mathcal{O})$,

$$M_{mn}^{pr}(\mathcal{O}) = GL_m(\mathcal{O}) \begin{pmatrix} 1_n \\ 0 \end{pmatrix}, \text{ and } (**) = \begin{cases} 2m - n & \text{for (U) and (R)} \\ 4m - 2n + 1 & \text{for (D).} \end{cases} \tag{1.10}$$

It is known that the above ratios in (1.9) are stable if ℓ is big enough, and for that it is enough to assume that

$$\text{The } K_n\text{-orbit containing } B(\in X_n^+) \text{ decomposes into cosets modulo } V_n(\pi^\ell\mathcal{O}). \tag{1.11}$$

For $A \in X_m^+, B \in X_n^+$ and $e \in \mathbb{N}$, we have the following

$$\mu(\pi^e B, \pi^e A) = q^{(\star)} \cdot \mu(B, A), \quad \mu^{pr}(\pi^e B, \pi^e A) = q^{(\star)} \cdot \mu^{pr}(B, A), \tag{1.12}$$

where

$$(\star) = \begin{cases} n^2 & \text{for (U) and (R)} \\ n(2n-1) & \text{for (D).} \end{cases}$$

Hence we may define local densities for any $A \in X_m$ and $B \in X_n$.

The following relations are known (cf. [H1], [H7]), and by these relations we may regard spherical functions as generating functions of local densities.

Theorem 1.1 (Induction Theorem) *Assume that* $m \geq n$ *and* $\mathrm{Re}(s_i) \geq 0,\ 1 \leq i \leq n$ *and take any*$A \in X_m^+$.

$$\begin{aligned} &\omega(A; s_1, \ldots, s_n, 0, \ldots, 0) \\ &= c_{n,m}(\star) \cdot \sum_{B \in K_n \backslash X_n^+} \frac{\mu^{pr}(B, A)}{\mu(B, B)} \cdot \omega(B; x_1, \ldots, x_n) \\ &= c_{n,m}(\star) \cdot \prod_{i=1}^n (1 - q^{-h_i(s)}) \cdot \sum_{B \in K_n \backslash X_n^+} \frac{\mu(B, A)}{\mu(B, B)} \cdot \omega(B; x_1, \ldots, x_n), \end{aligned}$$

where

$$c_{n,m}(t)=\frac{\prod_{i=1}^{n}(1-t^i)\prod_{i=1}^{m-n}(1-t^i)}{\prod_{i=1}^{m}(1-t^i)},\quad (\star)=\sharp(\mathcal{O}/(\Pi)=\begin{cases} q^{-1} & \textit{for (R)}\\ q^{-2} & \textit{for (U) and (D)},\end{cases}$$

$$h_i(s)=\begin{cases} 2s_i+\cdots+2s_n+m-i+1 & \textit{for (R)}\\ 2s_i+\cdots+2s_n+2m-2i+2 & \textit{for (U)}\\ s_i+\cdots+s_n+2m-2i+2 & \textit{for (D)}.\end{cases}$$

The values of $\omega(x;s)$ and $\mu(x,y)$ are determined by the K_n orbit of $x\in X_n$ and K_m-orbit of $y\in X_m$. Set

$$\Gamma_k=\left\{\gamma\in\mathbb{Z}^k \mid \gamma_1\geq\cdots\geq\gamma_k\right\}\supset\Gamma_k^+=\left\{\gamma\in\Gamma_k \mid \gamma_k\geq 0\right\},\quad \mathcal{E}=\{1,\delta\} \tag{1.13}$$

where $\delta\in\mathfrak{o}^\times\backslash\mathfrak{o}$. Then the orbit space $K_n\backslash X_n$(resp. $K_n\backslash X_n^+$) corresponds bijectively to Λ_n (resp. Λ_n^+), where, for each case of (U), (R) or (D), it is given as

$$\begin{aligned}\Lambda_n(U) &=\Gamma_n,\\ \Lambda_n(R) &=\left\{(\lambda,\epsilon)=(\lambda_1^{n_1}\ldots\lambda_t^{n_t},(\epsilon_1,\ldots,\epsilon_t)\;\middle|\;\begin{array}{l}\lambda\in\Gamma_n,\ \lambda_1>\cdots>\lambda_t,\ n_i>0\,(1\leq i\leq t)\\ \epsilon\in\mathcal{E}^t,\ n_i \text{ is even if } \lambda_i \text{ is odd}\end{array}\right\}\\ \Lambda_n(D) &=\left\{\lambda\in\Gamma_n \;\middle|\; \sharp\{j\mid\lambda_j=\lambda_i, 1\leq j\leq n\} \text{ is even if } \lambda_i \text{ is odd}\right\}\end{aligned} \tag{1.14}$$

The set $\Lambda_n^+(\ )$ is defined by using Γ_n^+ instead of Γ_n for each case. Here, for (U), the representatives are diagonal consisting of shape $\langle\pi^r\rangle$ or $\langle\pi^r\epsilon\rangle$, while for (R) and (Q), they are orthogonal sums of shape $\langle\pi^r\rangle$ or $\langle\pi^r\epsilon\rangle$ if $\lambda_i=2r$ and shape $\begin{pmatrix}0 & \Pi\pi^r\\ -\Pi\pi^r & 0\end{pmatrix}$ if $\lambda_i=2r+1$. Because of the shape $\Lambda_n(R)$, we should consider spherical functions with characters for (R) (cf. [H1]).

We define the pairing $\langle\ ,\ \rangle : V_n\times V_n\longrightarrow k$ by

$$\langle A,\, B\rangle=\begin{cases}\mathrm{trace}(AB) & \text{for (U) and (R)}\\ \sum_i A_{ii}B_{ii}+\sum_{i<j}\mathrm{T_{rd}}(A_{ij}B_{ji}) & \text{for (Q)},\end{cases} \tag{1.15}$$

where $\mathrm{T_{rd}}$ is the reduced trace on D.

We take and fix an additive character ψ on k of conductor $\mathfrak{p}$. For any $\ell>0$, we define the character ψ_ℓ of conductor $\mathfrak{p}^\ell$ by $\psi_\ell(x)=\psi(\pi^{-\ell}x)$, and denote by χ_ℓ the induced character on $\mathfrak{o}/\mathfrak{p}^\ell$, which is nontrivial on $\mathfrak{p}^{\ell-1}/\mathfrak{p}^\ell$. (1.16)

By the orthogonal relation of characters, we have the following:

Lemma 1.2 *For $A\in X_m$ and $B\in X_n$ with $m\geq n$, on has*

$$\mu(B,A)=c_0\cdot\int_{V_n}dy\int_{M_{mn}(\mathcal{O})}\psi(\langle A[v]-B,\, y\rangle)dv,$$

where dy and dv are normalized Haar measures on $M_{mn}(L)$ and $V_n(L)$, respectively, and the integral over V_n is understood as

$$\int_{V_n}=\lim_{\ell\to\infty}\int_{V_n(\pi^{-\ell}\mathcal{O})},\quad c_0=\begin{cases}1 & \textit{for (U)}\\ q^{-n(n-1)} & \textit{for (R) and (D)}\end{cases}.$$

For the calculation of local densities for (R) and (D), it is convenient to use the following $N_\ell(B,A)$ and $V_n(\pi,\ell)$:

$$\begin{aligned} \mu(B,A) &= \lim_{\ell\to\infty} \frac{N_\ell(B,A)}{q^{\ell n(4m-2n-1)-n(n+1)}}, \\ N_\ell(B,A) &= \sharp\left\{ \overline{v} \in M_{mn}(\mathcal{O}/\pi^\ell\mathcal{O}) \;\middle|\; A[v]-B \in V_n(\pi,\ell) \right\} \\ V_n(\pi,\ell) &= \left\{ y \in V_n \;\middle|\; y_{ii} \in \mathfrak{p}^\ell,\ y_{ij} \in \mathcal{P}^{2\ell-1}\ ({}^\forall i,j) \right\}. \end{aligned} \tag{1.17}$$

§2 Linear independence of local densities

We consider local densities $\mu(B,A)$ as functions of $B \in X_n^+$ and study their linear independence when $A \in X_m^+$ varies by scaling hyperbolic plains. We have studied in [BHS] the similar problem on local densities of symmetric forms and applied it on global study of Siegel modular forms. Here we concentrate local theory.

In order to state the results, we need some more notation.

$$\Gamma_k^+ \supset \Lambda_{k,\ell}^+ = \left\{ \lambda \in \Gamma_k^+ \;\middle|\; \ell \geq \lambda_1 \right\}, \tag{2.1}$$

for $\lambda \in \Gamma_k$,

$$\begin{aligned} H^\lambda &= h^{\lambda_1} \perp h^{\lambda_2} \perp \cdots \perp h^{\lambda_k} \in X_{2k}^+, \\ h^r &= \begin{cases} \begin{pmatrix} 0 & \pi^r \\ (\pi)^r & 0 \end{pmatrix} & \text{for (U)}, \\ \begin{pmatrix} 0 & \mathit{\Pi}^r \\ (-\mathit{\Pi})^r & 0 \end{pmatrix} & \text{for (R) and (D)} \end{cases} \\ H_k &= H^{\mathbf{0}} \in X_{2k}^+, \text{ if } \lambda = \mathbf{0} \in \Gamma_k. \end{aligned} \tag{2.2}$$

For $\lambda \in \Gamma_{k,\ell}^+$, we define $\widehat{\lambda} \in \Gamma_{\ell,k}^+$ by $\widehat{\lambda}_i = \sharp\left\{ j \mid \lambda_j \geq i \right\}$, ${}^\forall i$. For $\lambda, \mu \in \Gamma_n^+$, we set $\langle \lambda,\, \mu \rangle = \sum_{i=1}^n \lambda_i \mu_i$.

Theorem 2.1 *Assume that $k, \ell, n \in \mathbb{N}$, $r \in \mathbb{Z}, r \geq 0$ satisfy the condition $k \geq n$ and*

$$2k + r \;\; \geq \;\; \begin{cases} 2n & \text{for (U)} \\ 4n-1 & \text{for (R)} \\ 8n-1 & \text{for (D).} \end{cases} \tag{2.3}$$

Take an $S \in X_r^+$ if $r > 0$. Then one has the following.
(1) *For $\lambda \in \Lambda_{k,\ell}^+$, $T \in X_n^+$, it satisfies*

$$\mu(T, H^\lambda \perp S) = \begin{cases} \sum_{\tau\in\Gamma_{n,\ell}} a_\tau q^{\langle \widehat{\lambda},\, \widehat{\tau} \rangle} & \text{for (R)} \\ \sum_{\tau\in\Gamma_{n,\ell}} a_\tau q^{2\langle \widehat{\lambda},\, \widehat{\tau} \rangle} & \text{for (U) and (D)} \end{cases}$$

where, $a_\tau = a_\tau(k,S,T)$ *is a constant independent of* λ.

(2) *As functions of* $T \in X_n^+$, *the set* $\left\{ \mu(T, H^\lambda \perp S) \;\middle|\; \lambda \in \Gamma_{k,\ell}^+ \right\}$ *spans a* $\binom{n+\ell}{n}$*-dimensional* $\mathbb{Q}$*-space, and the set* $\left\{ \mu(T, H^\mu \perp H_{k-n} \perp S) \;\middle|\; \mu \in \Gamma_{n,\ell}^+ \right\}$ *forms a basis.*

Recall the character ψ defined in (1.16). For a function f on V_n we define its Fourier transform if the following integral is well-defined:

$$\mathcal{F}(f)(y) = \int_{V_n} f(x)\psi(-\langle x,\, y\rangle)dx. \tag{2.4}$$

For $A \in V_m$ and $C \in V_n$, we define the Gauss sum by

$$\mathcal{G}(A,C) = \int_{M_{mn}(\mathcal{O})} \psi(\langle A[v],\, C\rangle)dv. \tag{2.5}$$

It is easy to see that the value of $\mathcal{G}(A,C)$ is determined by $GL_m(\mathcal{O})$-orbit containing A and $GL_n(\mathcal{O})$-orbit containing B. By Lemma 1.2, we see

$$\begin{aligned} &\mathcal{F}(\mathcal{G}(A,\ ))(B) = c_0\, \mu(B,A), \\ &\text{if } \int_{V_n(D)} \mathcal{G}(A,x)\, \psi(-\langle x,\, B\rangle)dx \ \text{ is well-defined.} \end{aligned} \tag{2.6}$$

For $x \in V_n = V_n(L)$, we define a constant $\nu[x]$ as follows. If $x \in V_n(\mathcal{O})$, we set $\nu[x] = 1$; if x has eigenvalues of negative π_L-exponents and $\pi_L^{-\tau_1}, \ldots, \pi_L^{-\tau_r}$ are the all, we set

$$\nu[x] = q^{|\tau|}, \quad |\tau| = \sum_{i=1}^{r} \tau_i \ (>0). \tag{2.7}$$

It is clear that $\mathcal{G}(A,C)$ decomposes into products if A or C decomposes into orthogonal sums, i.e.

$$\mathcal{G}(A \perp_{i=1}^{r} A_i, \perp_{j=1}^{s} C_j) = \prod_{i=1}^{s} \prod_{j=1}^{s} \mathcal{G}(A_i, C_j). \tag{2.8}$$

Hence the calculation of Gauss sums is reduced to that within size 1 or 2, which is possible to calculate (for details, see [H4], [H7]). We note here the following for convenience.

Lemma 2.2 $\mathcal{G}(A,C)$ *is a product of the following quantities for each case.*

$$\textit{For (U)}, \quad I(a) = \int_{\mathcal{O}} \psi(\pi^a\, \mathrm{N}_{\mathrm{k}'/\mathrm{k}}(x))dx = (-q)^{\min\{0,a\}}, \quad (a \in \mathbb{Z}).$$

$$\textit{For (R)},\quad I(a;\epsilon) = \int_{\mathcal{O}} \psi(\Pi^a \epsilon\, \mathrm{N}_{\mathrm{k}'/\mathrm{k}}(x))dx \quad (a \in 2\mathbb{Z},\ \epsilon \in \mathfrak{o}^{\times})$$

$$= \begin{cases} 1 & \textit{if } a > 0, \\ q^{(a+1)/2}\epsilon_0 \left(\dfrac{(-1)^{a/2+1}\epsilon}{\mathfrak{p}}\right), \quad (\epsilon_0 \in \mathfrak{o},\ \epsilon_0^2 = \left(\dfrac{-1}{\mathfrak{p}}\right)) & \textit{if } a < 0, \end{cases}$$

$$J(b) = \int_{\mathcal{O}\times\mathcal{O}} \psi(\mathrm{Tr}_{k'/k}(\Pi^b xy))dxdy = q^{\min\{0,b+1\}}, \quad (b \in \mathbb{Z}).$$

$$\textit{For (D)},\quad I(a) = \int_{\mathcal{O}} \psi(\Pi^a\, \mathrm{N}_{\mathrm{rd}}(x))dx = q^{\min\{0,a+1\}}, \quad (a \in 2\mathbb{Z}),$$

$$J(b) = \int_{\mathcal{O}\times\mathcal{O}} \psi(\mathrm{T}_{\mathrm{rd}}(\Pi^b xy))dxdy = q^{2\min\{0,b+1\}}, \quad (b \in \mathbb{Z}).$$

Proposition 2.3 *For $A \in X_m$ and $C \in V_n$, it holds*

$$|\mathcal{G}(A,C)| \le c(A) \left\{ \begin{array}{ll} \nu[C]^{-m/2} & \textit{for (R)} \\ \nu[C]^{-m} & \textit{for (U) and (D)} \end{array} \right\}. \tag{2.9}$$

where, $c(A)$ is a constant adjusted for each case.

It is known that the integral (singular series) $\int_{V_n} \nu[x]^{-s}dx$ is absolutely convergent if

$$\mathrm{Re}(s) > 2n-1 \quad \text{for (U) and (R) (cf. [Shi, Th.13.6].} \tag{2.10}$$

For case (D), we embed $M_n(D)$ into $M_{2n}(k')$ by φ_n where $\{1, \epsilon, \Pi, \epsilon\Pi\}$ is the standard basis of D over k and $k' = k(\epsilon)$ is unramified, and consider a similar integral $\int_{M_{2n}(k')} \nu_{k'}(y)^{-s}dy$. Here for $y \in M_{2n}(k')$, if y has eigenvalues of negative π-exponents and those sum is $c(y)$, we set $\nu_{k'}(y) = q^{|2c(y)|}$; while we set $\nu_{k'}(y) = 1$ if y is integral. Then $\nu[x]^2 = \nu_{k'}(\varphi_n(x))$ for $x \in M_n(D)$. This integral is absolutely convergent if $\mathrm{Re}(s) > 4n-1$ (cf. [Shi, 3.14]). Hence, together with (2.6) and Proposition 2.3, we obtain the following Proposition.

Proposition 2.4 *For $A \in X_m$ and $T \in V_n$, the Fourier transform $\mathcal{F}(\mathcal{G}(A,\ ))(T)$ is absolutely convergent if m, n satisfy the condition*

$$m \ge 2n \textit{ for (U)}, \quad m \ge 4n-1 \textit{ for (R)}, \quad m \ge 8n-1 \textit{ for (D)}.$$

Then, for $B \in X_n$, one has

$$\mathcal{F}(\mathcal{G}(A,\ ))(B) = c_0 \mu(B, A),$$

where $c_0 = 1$ for (U), while $c_0 = q^{-n(n-1)}$ for (R) and (D) (cf. Lemma 1.2).

Proposition 2.5 *Assume m and n satisfy the condition in Proposition 2.4, and let $A_i \in X_m(\mathcal{O}), 1 \le i \le N$. Then the following are equivalent:*
(i) *As functions of T on X_n^+, $\mu(T, A_i)$, $1 \le i \le N$ are linearly independent over $\mathbb{Q}$.*
(ii) *As functions of X on $V_n(k')$, $\mathcal{G}(A_i, X)$, $1 \le i \le N$ are linearly independent over $\mathbb{C}$.*

Proof of Theorem 2.1:
We prepare some more notation. For each $X \in V_n \backslash V_n(\mathcal{O})$, let $\pi_L^{-\tau_1}, \dots, \pi_L^{-\tau_r}$ be those elementary divisors of X which are of negative powers of π_L and $\tau_1 \geq \cdots \tau_r \geq 1$, where π_L is the prime element of $L = k'$ or D given in (1.2). Set

$$\tau = \tau(X) = \begin{cases} 0 \in \Lambda_n^+ & \text{if } X \in V_n(\mathcal{O}) \\ (\tau_1, \dots, \tau_r) \in \Lambda_r^+ (\subset \Lambda_n^+) & \text{if } X \notin V_n(\mathcal{O}), \end{cases} \tag{2.11}$$

and

$$\sigma = \sigma(X) = \begin{cases} 0 \in \Lambda_n^+ & \text{if } X \in V_n(\mathcal{O}) \\ (\tau_1 - 1, \dots, \tau_r - 1) \in \Lambda_r^+ (\subset \Lambda_n^+) & \text{if } X \notin V_n(\mathcal{O}), \end{cases} \tag{2.12}$$

We note that $\nu[X] = q^{|\tau|}$ (cf. (2.7)).

Assume $\lambda \in \Gamma_k^+$ and $X \in V_n \backslash V_n(\mathcal{O})$ with $\tau(X)$ and $\sigma(X)$ (cf. (2.11), (2.12)). Then

$$\mathcal{G}(H^\lambda, X) = q^{c(\lambda, X)},$$

where

$$c(\lambda, X) = \begin{cases} 2 \sum_{t \geq 0} m_\lambda(t) \sum\limits_{\substack{1 \leq j \leq r, \\ \tau_j > t}} (t - \tau_i) & \text{for (U)} \\ \sum_{t \geq 0} m_\lambda(t) \sum\limits_{\substack{1 \leq i \leq r, \\ \sigma_i > t}} (t - \sigma_i) & \text{for (R)} \\ 2 \sum_{t \geq 0} m_\lambda(t) \sum\limits_{\substack{1 \leq i \leq r, \\ \sigma_i > t}} (t - \sigma_i) & \text{for (D),} \end{cases}$$

and $m_\lambda(t) = \sharp \{ i \mid \lambda_i = t \}$. When $X \in V(\mathcal{O})$, $c(\lambda, X) = 0$ by definition of $\tau(X)$ and $\sigma(X)$, which is consistent with $\mathcal{G}(H^\lambda, X) = 1$. On the other hand, by a combinatorial calculation of the right hand of $c(\lambda, X)$, which is the same as in [BHS, p.58], we obtain

$$\mathcal{G}(H^\lambda, X) = q^{c(\lambda, X)}, \quad c(\lambda, X) = \begin{cases} 2 \left\langle \widetilde{\lambda}, \widetilde{\tau} \right\rangle - 2k\, |\tau| & \text{for (U)} \\ \left\langle \widetilde{\lambda}, \widetilde{\sigma} \right\rangle - k\, |\sigma| & \text{for (R)} \\ 2 \left\langle \widetilde{\lambda}, \widetilde{\sigma} \right\rangle - 2k\, |\sigma| & \text{for (D)} \end{cases} \tag{2.13}$$

In order to prove Theorem 2.1 (1), we tale $\lambda \in \Gamma_{k,\ell}$ and $T \in X_n^+$. Under the given condition on k, n and r, the assumption of Proposition 2.4 is satisfied as $m = 2k + r$, hence local densities $\mu(T, H^\lambda \perp S)$ are expressed as the Fourier transform of Gauss sums (cf. (2.6)). Here, we write down only for the quaternion hermitian case (D), since the other case will be done quite similarly.

$$c_0 \mu(T, H^\lambda \perp S) \;\; = \;\; \int_{V_n} q^{2 \left\langle \widetilde{\lambda}, \widetilde{\sigma(X)} \right\rangle} q^{-2k|\sigma(X)|} \mathcal{G}(S, X) \psi(- \langle T, X \rangle) dX,$$

where $\left\langle \widetilde{\lambda}, \widetilde{\sigma(X)} \right\rangle$ is determined by $\widetilde{\sigma(X)}_i$, $1 \le i \le \ell$ with ℓ coming from $\lambda \in \Gamma_{k,\ell}$. Hence we have

$$\begin{aligned} &\mu(T, H^\lambda \perp S) \\ &= c_0^{-1} \sum_{\tau \in \Gamma_{n,\ell}} q^{2\langle \widetilde{\lambda}, \widetilde{\tau} \rangle} \underbrace{\int_{\left\{ X \in V_n \,\middle|\, \widehat{\sigma(X)}_i = \tau_i,\ 1 \le i \le \ell \right\}} q^{-2k|\sigma(X)|} \mathcal{G}(S, X) \psi(-\langle T, X \rangle) dX}_{(*)} . \end{aligned}$$

The value $(*)$ is independent of λ and only dependent on k, τ, S, T. Putting $c_0^{-1} \cdot (*)$ as a_τ we have the required formulation.

Now we prove Theorem 2.1 (2). Denote by W the $\mathbb{Q}$-space spanned by functions $\left\{ \mu(T, H^\lambda \perp S) \,\middle|\, \lambda \in \Gamma_{k,\ell} \right\}$ on $T \in X_n$. Then, by Proposition 2.5, $W \otimes \mathbb{C}$ is isomorphic to the $\mathbb{C}$-space spanned by functions $\left\{ \mathcal{G}(H^\lambda \perp S, X) \,\middle|\, \lambda \in \Gamma_{k,\ell} \right\}$ on $X \in V_n$. The space W is isomorphic to the $\mathbb{Q}$-space W_0 spanned by functions $\left\{ q^{\langle \widehat{\lambda}, \widehat{\tau} \rangle} \,\middle|\, \lambda \in \Gamma_{k,\ell} \right\}$ on $\tau \in \Gamma_{n,\ell}$ by (1). The space W_0 coincides with that considered in [BHS, p.59], where q^2 should be replaced by q, and it is proved there that $\dim(W_0) = \sharp(\Gamma_{n,\ell}) = {}_{n+\ell}C_n$. ∎

§3 Kitaoka series

3.1. Let us consider the following formal power series $P(B, A; X)$ for $A \in X_m$ and $B \in X_n$ with $m \ge n$.

$$P(B, A; X) = \sum_{r \ge 0} \mu(\pi^r B, A) X^r. \tag{3.1}$$

Kitaoka introduce the similar power series for symmetric forms ([Ki]), and conjectured it is rational and proved for some special case, hence we call $P(B, A; X)$ as Kitaoka series. Then Böcherer and Sato proved the rationality by using Denif's theory and calculated its denominator for certain case ([BS]). And the author determined the denominators in [H2] and [H5] by an elementary method, where the rationality is also assured. Following the same method, we prove the rationality and determine the denominators for hertmitian cases. We note here all the results for convenience.

Theorem 3.1 *Assume that $A \in X_m$ and $B \in X_n$ with $m \ge n$. Then the Kitaoka series $P(B, A; X)$ becomes a polynomial of X if it is multiplied by the following polynomial corresponding to each case.*

For case (S) *and even m,* $\prod_{i=0}^{n} (1 - (\epsilon_A q^{\frac{1}{2}(n+i-m+1)})^{n-i} X)$, *where $\epsilon_A = \pm 1$ is given explicitly by A, (for details, see* [H5] *).*

For case (S) *in general,* $(1-X)\prod_{i=0}^{n-1}(1-q^{(n-i)(n+i-m+1)}X^2)$.

For case (U), $\prod_{i=0}^{n}(1-(-1)^{m(n-i)}q^{(n-i)(n+i-m)}X)$.

For case (R) *and* -1 *is square modulo* $\mathfrak{p}$ *or* m *is even,* $\prod_{i=0}^{n}(1-q^{(n-i)(n+i-m-1)}X)$.

For case (R) *in general,* $(1-X)\prod_{i=0}^{n-1}(1-q^{2(n-i)(n+i-m-1)}X^2)$.

For case (D), $\prod_{i=0}^{n}(1-q^{(n-i)(2n+2i-2m-1)}X)$.

3.2. Because of (1.12), it suffices to consider integral forms. From now on, we concentrate the case (D), and give the outline of its proof (for details see [H7, §4, §5]).

When we consider Kitaoka series $P(B,A;X)$, $A\in X_m^+$ and $B\in X_n^+$ are fixed, so we take and fix ℓ to be big enough for B to satisfy the condition (1.11), and recall the character χ_ℓ on $\mathfrak{o}/\mathfrak{p}^\ell$ in (1.16).

For a locally constant compactly supported function f on V_n (i.e. for $f\in\mathcal{S}(V_n)$), we define its Fourier transform with respect to ψ_ℓ by

$$(f)_\ell^\wedge(z)=\int_{V_n}f(y)\psi_\ell(-\langle y,\,z\rangle)dy, \tag{3.2}$$

where dy is the Haar measure on V_n normalized by $vol(V_n(\mathcal{O}))=1$. For $A\in X_m^+$ and $C\in V_n(\mathcal{O})$, we define (finite Gauss sum)

$$\mathcal{S}_\ell(A,C)=\sum_{\overline{v}\in M_{mn}(\mathcal{O}/\mathcal{P}^{2\ell})}\chi_\ell(\langle A[v],\,C\rangle). \tag{3.3}$$

Since $\psi_\ell(\langle A[v],\,C\rangle)=\psi(\pi^{-\ell}\langle A[v],\,C\rangle)$, we see for $A\in X_m^+,\ C\in V_n(\mathcal{O})$

$$\mathcal{S}_\ell(A,C)=q^{4\ell mn}\,\mathcal{G}(\pi^{-\ell}A,C). \tag{3.4}$$

We write here the correspondence of Λ_n^+ and $GL_n(\mathcal{O})\backslash X_n^+$ explicitly. Any $\gamma=(\gamma_1,\ldots,\gamma_n)\in\Lambda_n^+$ (cf. (1.14)) is writen as

$$\gamma=r_1^{e_1}\cdots r_t^{e_t}\in\mathbb{Z}^n;\ r_1>\cdots>r_t\geq 0,\ e_i>0,\ e_i\text{ is even if }r_i\text{ is odd,}$$

and we associate the element $\pi^\gamma\in X_n^+$ such as

$$\pi^\gamma=\pi^{r_1^{e_1}}\perp\cdots\perp\pi^{r_t^{e_t}}\in X_n^+, \tag{3.5}$$

$$\pi^{r^e}=\begin{cases}\langle\pi^u\rangle\perp\cdots\langle\pi^u\rangle\in X_e^+ & \text{if } r=2u,\\ \langle\begin{pmatrix}0&\pi^u\Pi\\-\pi^u\Pi&0\end{pmatrix}\rangle\perp\cdots\perp\langle\begin{pmatrix}0&\pi^u\Pi\\-\pi^u\Pi&0\end{pmatrix}\rangle\in X_e^+ & \text{if } r=2u+1.\end{cases}$$

By Lemma 2.2, we obtain

Proposition 3.2 *For $\alpha \in \Lambda_m^+$ and $\beta \in \Lambda_n^+$, it holds*

$$\mathcal{S}_\ell(\pi^\alpha, \pi^\beta) = q^{2\ell mn} \cdot (-1)^c \prod_{i=1}^{m} \prod_{j=1}^{n} q^{\min\{2\ell, \alpha_i+\beta_j+1\}},$$
$$c = \sharp \{ (i,j) \mid \alpha_i \text{ and } \beta_j \text{ are even, and } 2\ell > \alpha_i + \beta_j + 1 \}.$$

In particular, if $2\ell > \alpha_1 + \beta_1 + 1$, then

$$\mathcal{S}_\ell(\pi^\alpha, \pi^\beta) = q^{2\ell mn} \cdot (-1)^c q^{n|\alpha|+m|\beta|+mn}.$$

The group $GL_n(\mathcal{O}/\mathcal{P}^{2\ell})$ acts on $V_n(\mathcal{O}/\mathcal{P}^{2\ell})$, and the coset space bijectively corresponds to the set

$$\Lambda_{n,2\ell}^+ = \left\{ \gamma \in \Lambda_n^+ \;\middle|\; \gamma_1 \leq 2\ell \right\}. \tag{3.6}$$

We recall $N_\ell(B, A)$ defined in (1.17). Further we define for $\gamma \in \Lambda_{n,2\ell}^+$

$$N_\ell^{pr}(\pi^\gamma, \pi^\gamma) = \sharp \left\{ g \in GL_n(\mathcal{O}/\mathcal{P}^{2\ell}) \;\middle|\; \pi^\gamma[g] - \pi^\gamma \in V_n(\pi, \ell) \right\}. \tag{3.7}$$

We note here that $N_\ell^{pr}(\pi^\gamma, \pi^\gamma) = N_\ell(\pi^\gamma, \pi^\gamma)$ if $\gamma_1 < \ell$.

By Lemma 1.2 and (1.17), we have the following expression of local densities.

Proposition 3.3 *Assume that $A \in X_m^+$ and $B \in X_n^+$ with $m \geq n$ and that ℓ satisfies the condition* (1.11). *Then one has*

$$\mu(B, A) \;=\; q^{-\ell n(4m-2n+1)-n(n-1)} N_\ell(B, B) \sum_{\gamma \in \Lambda_{n,2\ell}^+} \frac{\mathcal{S}_\ell(A, \pi^\gamma)}{N_\ell^{pr}(\pi^\gamma, \pi^\gamma)} \, (ch_B)_\ell^\wedge(\pi^\gamma),$$

where ch_B is the characteristic function of $K_n \cdot B$.

Lemma 3.4 *Assume $2\ell \geq \alpha + 2$, where α is the maximal Π-exponent of all the elementary divisors of $A \in X_m^+$. Then*

$$\frac{S_\ell(A, 1_n)}{N_\ell(1_n, 1_n)} = c_0 \, q^{\ell n(2m-2n-1)}, \qquad \frac{S_\ell(A, H_{n/2})}{N_\ell(H_{n/2}, H_{n/2})} = c_1 \, q^{\ell n(2m-2n-1)},$$

where c_0 and c_1 are constants determined by A and n, and independent of ℓ.

Proposition 3.5 *Assume that ℓ satisfies the condition* (1.11). *Then*

$$P(B, A; X) \;=\; q^{-\ell n(4m-2n+1)-n(n-1)} N_\ell(B, B) \sum_{\widetilde{\gamma} \in \Lambda_{n,2\ell}^+} (ch_B)_\ell^\wedge(\pi^{\widetilde{\gamma}}) \, P_{\widetilde{\gamma}}(A; X),$$

where

$$P_{\widetilde{\gamma}}(A; X) \;=\; \sum_{r=0}^{\infty} (q^{4n(n-m)} X)^r \sum_{\gamma \in \Lambda(\widetilde{\gamma}, r)} \frac{\mathcal{S}_{\ell+r}(A, \pi^\gamma)}{N_{\ell+r}^{pr}(\pi^\gamma, \pi^\gamma)}, \tag{3.8}$$
$$\Lambda(\widetilde{\gamma}, r) \;=\; \left\{ \gamma \in \Lambda_{n,2(\ell+r)}^+ \;\middle|\; \gamma \equiv \widetilde{\gamma} \pmod{2\ell} \right\}.$$

Fix a $\widetilde{\gamma} \in \Lambda^+_{n,2\ell}$ and set $i = m_{\widetilde{\gamma}}(2\ell)\,(= \sharp\{j \mid 1 \le j \le n,\ \widetilde{\gamma}_j = 2\ell\})$. Then we may write $\widetilde{\gamma} = (2\ell)^i v$ with some $v \in \Lambda^+_{n-i,2\ell-1}$, and

$$\Lambda(\widetilde{\gamma}, r) \;=\; \begin{cases} \{\widetilde{\gamma}\} = \{v\} & \text{if } i = 0 \\ \left\{ ((2\ell)^i + \rho)v \in \Lambda^+_{n,2(\ell+r)} \;\middle|\; \rho \in \Lambda^+_{i,2r} \right\} & \text{if } i \ge 1, \end{cases} \tag{3.9}$$

We calculate the main quotient term of $P_{\widetilde{\gamma}}(A;X)$((3.8)) as follows.

Lemma 3.6 *Assume $\widetilde{\gamma} \in \Lambda^+_{n,2\ell}$ and $i = m_{\widetilde{\gamma}}(2\ell)$. Take any $\gamma \in \Lambda(\widetilde{\gamma}, r)$ written as in* (3.9)*. Then, for sufficiently large r with respect to A, one has*

$$\frac{\mathcal{S}_{\ell+r}(A, \pi^\gamma)}{N^{pr}_{\ell+r}(\pi^\gamma, \pi^\gamma)} = \begin{cases} a_0\, q^{rn(2m-2n-1)} & \text{if } i = 0 \\ a_i\, q^{r(n-i)(2m-2n-2i-1)} \dfrac{\mathcal{S}_r(A, \pi^\rho)}{N^{pr}_r(\pi^\rho, \pi^\rho)} & \text{if } 1 \le i \le n \end{cases}$$

where a_i is a constant determined by ℓ, A and $\widetilde{\gamma}$, and independent of r.

For formal power series $R_1(X)$ and $R_2(X)$ of X, we write $R_1(X) \sim R_2(X)$ if $R_1(X) - R_2(X)$ is a polynomial of X.

Proposition 3.7 *Assume $\widetilde{\gamma} \in \Lambda^+_{n,2\ell}$ and set $i = m_{\widetilde{\gamma}}(2\ell)$. Then it holds*

$$P_{\widetilde{\gamma}}(A;X) \;\sim\; \begin{cases} a_0 \cdot \sum_{r=0}^\infty \left(q^{n(2n-2m-1)}X\right)^r & \text{if } i = 0 \\ a_i \cdot \sum_{r=0}^\infty \left(q^{n(2n-2m-1)+i(2i-2m+1)}X\right)^r \displaystyle\sum_{\rho \in \Lambda^+_{i,2r}} \frac{\mathcal{S}_r(A, \pi^\rho)}{N^{pr}_r(\pi^\rho, \pi^\rho)} & \text{if } i \ge 1, \end{cases}$$

where a_i is the same constant as in Lemma 3.6.
Especially, if $i = 0$, then $P_{\widetilde{\gamma}}(A;X) \cdot (1 - q^{n(2n-2m-1)}X)$ is a polynomial of X.

3.3. Now we consider for $\widetilde{\gamma} \in \Lambda_{n,2\ell}$ with $i = m_{\widetilde{\gamma}}(2\ell) \ge 1$. By Proposition 3.7, we have to calculate the sum for $\rho \in \Lambda^+_{i,2r}$. When $r = 0$, we have

$$\Lambda(\widetilde{\lambda}, 0) = \{\widetilde{\lambda}\}, \quad \Lambda^+_{i,0} = \{\mathbf{0}\},$$

which gives the constant term of $P_{\widetilde{\gamma}}(A;X)$. Hence it suffices to consider the sum for $r \ge 1$ in (3.8). Hereafter we assume $r \ge 1$ and we decompose $\Lambda^+_{i,2r}$ as follows.

$$\Lambda^+_{i,2r} = \bigsqcup_{j=0}^{i-1} \Lambda^{(j)}, \quad \Lambda^{(j)} = \Lambda^{(j+)} \sqcup \Lambda^{(j-)}, \quad \Lambda^{(j-)} = \emptyset \ \text{ unless } i \equiv j \pmod 2, \tag{3.10}$$

$$\Lambda^{(0+)} = \left\{ (2b)^i \;\middle|\; 0 \le b \le r \right\},$$

$$\Lambda^{(0-)} = \left\{ (2b+1)^i \;\middle|\; 0 \le b \le r-1 \right\} \ \text{if } i \equiv 0 \pmod 2,$$

$$\Lambda^{(j+)} = \left\{ (2b)^i + (\beta, 0^{i-j}) \in \Lambda^+_{i,2r} \;\middle|\; 0 \le b \le r-1,\ \beta \in \Lambda^+_{j,2(r-b)},\ \beta_j \ge 1 \right\} \ \text{if } j \ge 1,$$

$$\Lambda^{(j-)} = \left\{ ((2b)^j + \beta)(2b-1)^{i-j} \in \Lambda^+_{i,2r} \;\middle|\; 1 \le b \le r,\ \beta \in \Lambda^+_{j,2(r-b)} \right\} \ \text{if } j \ge 1,\ i \equiv j \pmod 2,$$

We set for $1 \le i \le n$, according to the summation range

$$\begin{aligned} Q_i(X) &:= \sum_{r\ge1} X^r \sum_{\rho\in\Lambda^+_{i,2r}} \frac{S_r(A,\pi^\rho)}{N^{pr}_r(\pi^\rho,\pi^\rho)} = \sum_{j=0}^{i-1} Q_i^{(j)}(X), \\ Q_i^{(j)}(X) &:= \sum_{r\ge1} X^r \sum_{\rho\in\Lambda^{(j)}} \frac{S_r(A,\pi^\rho)}{N^{pr}_r(\pi^\rho,\pi^\rho)} = \sum_{k=+,-} Q_i^{(jk)}(X), \\ Q_i^{(jk)}(X) &:= \sum_{r\ge1} X^r \sum_{\rho\in\Lambda^{(jk)}} \frac{S_r(A,\pi^\rho)}{N^{pr}_r(\pi^\rho,\pi^\rho)}, \quad (0\le j\le i-1,\ k=\pm). \end{aligned} \tag{3.11}$$

Then, the assertion for the case $i = m_{\widetilde{\gamma}}(2\ell) \ge 1$ in Proposition 3.7 can be written as follows.

$$P_{\widetilde{\gamma}}(A,X) \sim a_i Q_i(q^{n(2n-2m-1)+i(2i-2m+1)}X), \tag{3.12}$$

where a_i is the same constant as in Proposition 3.7. The assertion (3.12) is valid also for $i=0$ by putting $Q_0(X) = \sum_{r\ge1} X^r$ (cf. Proposition 3.7).

Hereafter we calculate $Q_i^{(j)}$ according to the decomposition of $\Lambda^+_{i,2r}$ (for details, see [H8, §5]).
For $j=0$, one has

$$\begin{aligned} (1-q^{4i(m-i)}X)\cdot Q_i^{(0-)}(X) &\sim c\sum_{r\ge1}(q^{i(2m-2i-1)}X)^r, \\ (1-q^{4i(m-i)}X)\cdot Q_i^{(0+)}(X) &\sim c'\sum_{r\ge1}(q^{i(2m-2i-1)}X)^r, \end{aligned}$$

where c and c' are constants independent of r, actually $c=c_1,\ c'=c_0$ in Lemma 3.4. Since $Q_i^{(0)}(X) = Q_i^{(0+)}(X) + Q_i^{(0-)}(X)$, we have

$$(1-q^{4i(m-i)}X)(1-q^{i(2m-2i-1)}X)\cdot Q_i^{(0)}(X) \text{ is a polynomial in } X. \tag{3.13}$$

Recalling the definition of $Q_i(X)$ and adjusting the variable (cf. (3.12)), we have

Proposition 3.8 *Assume $\widetilde{\gamma}\in\Lambda_{n,2\ell}$ and $m_{\widetilde{\gamma}}(2\ell)=1$. Then*

$(1-q^{4(m-1)}X)(1-q^{2m-3}X)\cdot Q_1(X)$ *is a polynomial in X,*
$(1-q^{n(2n-2m-1)+2m-1}X)(1-q^{n(2n-2m-1)}X)\cdot P_{\widetilde{\gamma}}(A,X)$ *is a polynomial in X.*

Next we consider the case $j\ge1$. We have, with a suitable constant c,

$$(1-q^{4i(m-i)}X)\cdot Q_i^{(j-)}(X) \sim c\,X\cdot Q_j(q^{(i-j)(2m-2i-2j-1)}X) \quad (j\ge1). \tag{3.14}$$

3.4. In order to calculate $Q_i^{(j+)}(X)$ for $j \geq 1$, we need to decompose $\Lambda^{(j+)}(\subset \Lambda_{i,2r}^{+})$ as follows ($r \geq 1$ is assumed):

$$\Lambda^{(j+)} = \bigsqcup_{k=0}^{[j/2]} \Lambda^{(j+,k)}$$

for $0 \leq k \leq [j/2]$ and $j - 2k > 0$,

$$\Lambda^{(j+,k)} = \left\{ ((2b+2)^{j-2k} + \gamma)(2b+1)^{2k}(2b)^{i-j} \in \Lambda_{i,2r} \;\middle|\; 0 \leq b \leq r-1,\ \gamma \in \Lambda^{+}_{j-2k,2(r-b-1)} \right\},$$

for even j and $k = j/2$,

$$\Lambda^{(j+,j/2)} = \left\{ (2b+1)^{j}(2b)^{i-j} \in \Lambda_{i,2r} \;\middle|\; 0 \leq b \leq r-1 \right\}, \tag{3.15}$$

and define $Q_i^{(j+,k)}(X)$ as before.
Then, for $0 \leq k \leq [\frac{j}{2}]$, $j - 2k > 0$, together with a suitable constant a_k, which is new and independent of r, we obtain, ,

$$(1 - q^{4i(m-i)}X) \cdot Q_i^{(j+,k)}(X) \sim a_k\, X \cdot Q_{j-2k}(q^{(i-j+2k)(2m-2i-2j+4k-1)}X). \tag{3.16}$$

Finally, for even j, we obtain, with a suitable constant a_0,

$$(1 - q^{4i(m-i)}X)(1 - q^{i(2m-2i-1)}X) \cdot Q_i^{(j+,j/2)}(X) \text{ is a polynomial,} \tag{3.17}$$

which is the same type as the case $Q_i^{(0)}(X)$ (cf. (3.13)).

3.5. We recall that

$$Q_i(X) = \sum_{j=0}^{i-1} Q_i^{(j)}(X) = Q_i^{(0)}(X) + \sum_{j=1}^{i-1} \left(Q_i^{(j-)}(X) + \sum_{k=0}^{[j/2]} Q_i^{(j+,k)}(X) \right).$$

Then, by (3.13), (3.14), (3.16), and (3.17), we obtain

$$\begin{aligned}
&(1 - q^{4i(m-i)}X)(1 - q^{i(2m-2i-1)}X) \cdot Q_i(X) \\
&\sim\ (1 - q^{4i(m-i)}X)(1 - q^{i(2m-2i-1)}X) \left\{ \sum_{1 \leq j \leq i-1} \left(Q_i^{(j-)}(X) + \sum_{\substack{0 \leq k \leq [j/2] \\ j-2k>0}} Q_i^{(j+,k)}(X) \right) \right\} \\
&\sim\ (1 - q^{i(2m-2i-1)}X) \left(\sum_{1 \leq j \leq i-1} \sum_{\substack{0 \leq k \leq [j/2] \\ j-2k>0}} c_{jk}\, X \cdot Q_{j-2k}(q^{(i-j+2k)(2m-2i-2j+4k-1)}X) \right) \\
&\sim\ (1 - q^{i(2m-2i-1)}X) \left(\sum_{1 \leq j \leq i-1} c_j X \cdot Q_j(q^{(i-j)(2m-2i-2j-1)}X) \right),
\end{aligned} \tag{3.18}$$

where c_{jk} and c_j are constants. We adjust the variable by (3.12), i.e. we substitute $q^{n(2n-2m-1)+i(2i-2m+1)}X$ for X in (3.18). Then we obtain, with suitable constants c'_j,

$$\begin{aligned}(1-q^{(n-i)(2n+2i-2m-1)}X)(1-q^{n(2n-2m-1)}X)\cdot Q_i(q^{n(2n-2m-1)+i(2i-2m+1)}X)\\ \sim\ (1-q^{n(2n-2m-1)}X)\left(\sum_{j=1}^{i-1} c'_j X\cdot Q_j(q^{n(2n-2m-1)+j(2j-2m+1)}X)\right).\end{aligned} \tag{3.19}$$

Thus, by induction, we obtain for $1\le i\le n$,

$$\prod_{j=0}^{i}(1-q^{(n-j)(2n+2j-2m-1)}X)\times Q_i(q^{n(2n-2m-1)+i(2i-2m+1)}X) \text{ is a polynomial.}$$

Hence, by (3.12), we obtain for any $\widetilde{\gamma}\in\Lambda^+_{n,2\ell}$ (even if $i=m_{\widetilde{\gamma}}(2\ell)=0$, cf. Proposition 3.7).

$$\prod_{j=0}^{n}(1-q^{(n-j)(2n+2j-2m-1)}X)\times P_{\widetilde{\gamma}}(A,X) \text{ is a polynomial.}$$

Since $\Lambda^+_{n,2\ell}$ is a finite set in Proposition 3.5, we complete the proof of Theorem 3.1, i.e. we have proved that

$$\prod_{i=0}^{n}(1-q^{(n-i)(2n+2i-2m-1)}X)\times P(B,A;X) \text{ is a polynomial.} \qquad \blacksquare$$

Remark 3.9 *It is easy to see that the polynomials of X appearing in Theorem 3.1 have rational coefficients.*
It is difficult to calculate local densities in general. By a direct calculation, we give a small example consistent with the assertion of Theorem 3.1.

$$P(1,1_2;X)=(1-q^{-2})\left(1+\frac{1+q^{-1}}{1-q^{-3}}\cdot\frac{X}{1-X}-\frac{1+q^{-2}}{1-q^{-3}}\cdot\frac{q^{-4}X}{1-q^{-3}X}\right).$$

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