AN EXTENSION OF CONVERSE THEOREMS TO THE SELBERG CLASS

ANDREW R. BOOKER, MICHAEL FARMER, AND MIN LEE

1. Modular forms, their L-functions and converse theorems

Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ be the usual Poincaré upper half plane. For an integer k, we define an action of $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ to functions $h : \mathbb{H} \to \mathbb{C}$ as

$$(h|_k\gamma)(z) = (\det \gamma)^{\frac{k}{2}}(cz + d)^{-k}h(\gamma z).$$

For positive integers N and k, a Dirichlet character χ modulo N, let f be a modular form for $\Gamma_0(N)$ of weight k with nebentypus χ. We further assume that f is cuspidal and it has the following Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n)n^{\frac{k-1}{2}}e(nz)$$

where $e(z) = e^{2\pi iz}$. Let $S_k(N, \chi)$ be the space of cuspidal modular forms of level N, weight k and nebentypus character χ. It is well-known that the dimension of the space $S_k(N, \chi)$ is finite.

Each modular form has the associated L-function. For a given cusp form $f \in S_k(N, \chi)$, we consider the completed L-function:

$$\Lambda(s, f) = \Gamma_{\mathbb{C}}\left(s + \frac{k-1}{2}\right)L(s, f),$$

where

$$L(s, f) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$$

and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The series converges absolutely for $\Re(s) > 1$.

Let $w_N = \begin{pmatrix} 0 & -\sqrt{N}^{-1} \\ \sqrt{N} & 0 \end{pmatrix}$ and let

$$\tilde{f}(z) = (f|_k w_N)(z).$$

Then $\tilde{f} \in S_k(N, \bar{\chi})$ since $w_N\Gamma_0(N) = \Gamma_0(N)w_N$. The L-functions for f and $\tilde{f}$ have the following properties:

- $\Lambda(s, f)$ and $\Lambda(s, \tilde{f})$ continue to entire functions of finite order;
- the L-functions satisfy the functional equation

$$\Lambda(s, f) = i^k N^{\frac{1}{2}-s}\Lambda(1 - s, \tilde{f}). \tag{1.1}$$

It is well-known that one can show these properties by using the integral representation of the L-function. The completed L-function $\Lambda(s, f)$ can be written as the Mellin transform of $f(iy)$:

$$\int_0^{\infty} f(iy)y^{s+\frac{k-1}{2}}\frac{dy}{y} = \sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} \int_0^{\infty} e^{-2\pi ny}y^{s+\frac{k-1}{2}}\frac{dy}{y} = \Lambda(s, f).$$

The analytic continuation and functional equation follow by the relation $\tilde{f} = f|_k w_N$.

By reverting the above argument, we can prove Hecke's converse theorem (1936) : for $N \le 4$, the modular forms of level N are characterised by the L-functions, i.e., the analytic properties of them and the functional equation (1.1). Here is a sketch of proof. For given sequences of complex numbers $\{f_n\}_{n=1}^{\infty}$ and $\{\tilde{f}_n\}_{n=1}^{\infty}$, we construct the completed L-functions $\Lambda(s, f)$ and $\Lambda(s, \tilde{f})$. Assume that the completed L-functions continue to entire functions of finite order and satisfy the functional equation (1.1) with the given positive integers k and $N \le 4$. By applying the inverse Mellin transform to the completed L-function $\Lambda(s,f)$ and using the functional equation (1.1), one can prove that the function $f(z)=\sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} e(nz)$, which is constructed by the Fourier expansion with the coefficients of the given L-function, is invariant under the action of $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $W_N=w_N T w_N^{-1}\in\Gamma_0(N)$. When $N\le 4$, the set $\{T, W_N, -W_N, -I_2\}$ can generate the group $\Gamma_0(N)$. Therefore, we can show that the function $f(z)$ is a modular form of weight k for $\Gamma_0(N)$.

When $N>4$, this method does not hold. In fact, the vector space of sequences $\{f_n\}_{n=1}^{\infty}$, $\{\tilde{f}_n\}_{n=1}^{\infty}$ satisfying the above conditions is infinite dimensional [15].

To extend the converse theorem in the line of Hecke's arguments, one needs more relations associating the given sequences (or L-functions) and the elements in $\Gamma_0(N)$. Weil (1967) [17] and Razar (1977) [14] proved converse theorems for arbitrary level by assuming the functional equations of the L-function twisted by Dirichlet characters.

Let us define L-functions of cusp forms twisted by characters: for a cuspidal modular form f and a Dirichlet character ψ,

$$\Lambda(s,f,\psi)=\Gamma_{\mathbb{C}}\left(s+\frac{k-1}{2}\right)L(s,f,\psi)$$

where

$$L(s,f,\psi)=\sum_{n=1}^{\infty}\frac{a_f(n)\psi(n)}{n^s}.$$

Here $a_f(n)n^{\frac{k-1}{2}}$ is the Fourier coefficient of f at n. Then $\Lambda(s,f,\psi)$ continues to an entire function of finite order and satisfies the functional equation similar to (1.1).

For given sequences $\{f_n\}_{n=1}^{\infty}$ and $\{\tilde{f}_n\}_{n=1}^{\infty}$, for Weil's converse theorem we assume analytic continuation and functional equation for all L-functions twisted by primitive Dirichlet characters of conductor q, where q is a prime which does not divide N. We need to consider infinite set of such primes. See, for example, [13, Theorem 4.3.15]. Razar's converse theorem uses different types of Dirichlet characters: Dirichlet characters modulo q where q is divisible by N and $N\le q\le N^2$. Razar's converse theorem requires finitely many twisted L-functions.

Afterwards, the converse theorems for modular forms and generally converse theorems for automorphic representations and their applications have been studied actively. One can find more about converse theorems for GL_n automorphic forms in [12], [9], [6], [2], [4], [5] and others.

1.1. **Main theorem.** In 2002, in his thesis, Venkatesh [16] proved a converse theorem for modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$ allowing a finite set of poles for L-functions, satisfying infinitely many functional equations (twisted by additive characters), with a completely new method - using the Petersson trace formula.

In [1], with Booker and Farmer, we follow Venkatesh's idea in the context of Langlands' Beyond Endoscopy, and prove converse theorems for modular forms of arbitrary level N with gamma factors of Selberg type. Kaczorowski and Perelli [11] recently classified the elements of the Selberg class of degree 2 of conductor 1 without the need of any twists. But very little is known for higher conductor. Since our converse theorem admits a generalization to gamma factors of Selberg type, it can be viewed as a converse theorem for degree 2 elements of the Selberg class, albeit with infinitely many functional equations. Our result is the first that we are aware of to consider both arbitrary level and degree 2 gamma factor with infinitely many functional equations.

Assume that the sequences of complex numbers $\{f_n\}_{n=1}^{\infty}$ and $\{\tilde{f}_n\}_{n=1}^{\infty}$ and $\gamma(s)$ satisfies the following properties:

(1) The series

$$L_f(s):=\sum_{n=1}^{\infty} f_n n^{-s} \quad\text{and}\quad L_{\tilde{f}}(s):=\sum_{n=1}^{\infty}\tilde{f}_n n^{-s}$$

converges absolutely for $\Re(s)>1$;

(2) $Q, \lambda_j \in \mathbb{R}$ and $\mu_j \in \mathbb{C}$ with $Q, \lambda_j > 0$,

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma\left(\lambda_j s + \mu_j\right).$$

Here $\Re(\mu_j) > -\frac{1}{2}\lambda_j$ and $\sum_{j=1}^{r} \lambda_j = 1$.

Given $\alpha \in \mathbb{Q}$, we define the twisted L-functions:

$$L_f(s,\alpha) := \sum_{n=1}^{\infty} f_n e(n\alpha) n^{-s} \quad \text{and} \quad L_{\tilde{f}}(s,\alpha) := \sum_{n=1}^{\infty} \tilde{f}_n e(n\alpha) n^{-s}$$

and the completed L-functions with $\gamma(s)$:

$$\Lambda_f(s,\alpha) := \gamma(s) L_f(s,\alpha) \quad \text{and} \quad \Lambda_{\tilde{f}}(s,\alpha) := \gamma(s) L_{\tilde{f}}(s,\alpha).$$

With these notations and conditions we state our converse theorems.

Theorem 1.1. *Let N be a positive integer, χ be a Dirichlet character modulo N and ω be a non-zero complex number. For every $q \in N\mathbb{Z}_{\geq 1}$ and every pair $u, v \in \mathbb{Z}$ with $uv \equiv 1 \bmod q$, assume that $\Lambda_f\left(s, \frac{u}{q}\right)$ and $\Lambda_{\tilde{f}}\left(s, -\frac{v}{q}\right)$ continue to entire functions of finite order and satisfy the functional equation*

$$\Lambda_f\left(s, \tfrac{u}{q}\right) = \omega \chi(v) q^{1-2s} \Lambda_{\tilde{f}}\left(1-s, -\tfrac{v}{q}\right). \tag{1.2}$$

Then there exists a positive integer k such that $f(z) = \sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} e(nz)$ is a modular form of weight k, level N and nebentypus character χ.

Theorem 1.2. *Let N be a positive integer, χ be a Dirichlet character modulo N and ω be a non-zero complex number. For every $q \in \mathbb{Z}_{\geq 1}$ with $\gcd(q, N) = 1$, and every pair $u, v \in \mathbb{Z}$ with $uNv \equiv 1 \bmod q$, assume that $\Lambda_f\left(s, \frac{u}{q}\right)$ and $\Lambda_{\tilde{f}}\left(1-s, -\frac{v}{q}\right)$ continue to entire functions of finite order and satisfy the functional equation*

$$\Lambda_f\left(s, \tfrac{u}{q}\right) = \omega \chi(q) (Nq^2)^{\frac{1}{2}-s} \Lambda_{\tilde{f}}\left(1-s, -\tfrac{v}{q}\right).$$

Then there exists a positive integer k such that $f(z) = \sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} e(nz)$ is a modular form of weight k, level N and nebentypus character χ.

Theorem 1.1 appears in [1, Theorem 1.1]. Theorem 1.2 and converse theorems with twisted L-functions associated with the cusps for $\Gamma_0(N)$ will appear in a follow-up paper.

Remark 1.3.
- Theorem 1.2 can be considered analogous to Weil's converse theorem. In a follow-up paper we will use functional equation of additively twisted L-functions with $\alpha = \frac{u}{q}$, where $q \in M\mathbb{Z}_{\geq 1}$ for $M \mid N$ with $\gcd(M, N/M) = 1$ and $u \in \mathbb{Z}$ with $\gcd(u, q) = 1$. Theorem 1.2 is the case when $M = 1$.
- Using the Bruggeman-Kuznetsov trace formula and the method of [7], one can prove a similar converse theorem for Maass forms. This is a work in preparation.
- If we suppose $L_f(s)$ (without twist) lies in the Selberg class (so it has Euler product) then we can combine methods used in [10] and [3] to constrain the possible poles of the completed twisted L-functions. Then it would be possible to relax the analyticity conditions for the completed twisted L-functions and allow them to have arbitrary poles inside the critical strip.

2. Idea of proof

2.1. **Petersson formula.** Let $H_k(N, \chi)$ be the orthonormal basis for $S_k(N, \chi)$. Each $g \in H_k(N, \chi)$ has a Fourier expansion of the form:

$$g(z) = \sum_{n=1}^{\infty} \rho_g(n) n^{\frac{k-1}{2}} e(nz) \quad \text{for some } \rho_g(n) \in \mathbb{C}.$$

The Petersson formula [8, Corollary 14.23] gives, for $k \geq 2$,

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g \in H_k(N,\chi)} \rho_g(n)\overline{\rho_g(m)} = \delta_{n=m} + 2\pi i^{-k} \sum_{q \in N\mathbb{Z}_{\geq 1}} \frac{S_\chi(m,n;q)}{q} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q}\right),$$

where

$$S_\chi(m,n;q) = \sum_{\substack{a,\bar{a} \bmod q,\\ a\bar{a} \equiv 1 \bmod q}} \chi(a) e\left(\frac{ma+n\bar{a}}{q}\right)$$

is the twisted Kloosterman sum and $J_{k-1}(y)$ is the classical J-Bessel function.

Our goal is to isolate $g \in H_k(N,\chi)$ which is associated with the given sequences $\{f_n\}_{n=1}^\infty$ and $\{\tilde{f}_n\}_{n=1}^\infty$ based on the given functional equations. We insert the Rankin-Selberg convolution of the given $\{f_n\}_{n=1}^\infty$ and $g \in H_k(N,\chi)$ on the spectral side of the Petersson trace formula:

$$\begin{aligned}(2.1)\quad \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g \in H_k(N,\chi)} \rho_g(n) L(s, f \times \bar{g}) &= \delta_{n=m}\zeta^{(N)}(2s) f_n n^{-s} \\ &+ 2\pi i^{-k}\zeta^{(N)}(2s) \sum_{q \in N\mathbb{Z}_{\geq 1}} \sum_{m=1}^\infty \frac{f_m}{m^s} \frac{S_\chi(n,m;q)}{q} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q}\right).\end{aligned}$$

Here

$$L(s, f \times \bar{g}) = \zeta^{(N)}(2s) \sum_{m=1}^\infty \frac{f_m \overline{\rho_g(m)}}{m^s}.$$

To study (2.1), we use the analytic properties and functional equations of L-functions twisted by additive characters.

2.2. **Sketch of proof of Theorem 1.1.** We continue from (2.1). We will show that the left-hand side of (2.1) continues analytically to $\Re(s) > \frac{1}{2}$ except a possible simple pole at $s = 1$ by applying the analytic properties of the twisted L-functions. Then the right-hand side of (2.1) also continues analytically to $\Re(s) > \frac{1}{2}$ except a possible simple pole at $s = 1$ and we can isolate $g \in H_k(N,\chi)$ which is associated with $\{f_n\}_{n=1}^\infty$.

We open up the Kloosterman sums $S_\chi(n,m;q)$ and the J-Bessel function with the Mellin-Barnes integral representation,

$$2\pi J_{k-1}(4\pi y) = \frac{1}{2\pi i} \int_{\Re(u)=\sigma_0} \frac{\Gamma_\mathbb{C}\left(u + \frac{k-1}{2}\right)}{\Gamma_\mathbb{C}\left(-u + \frac{k+1}{2}\right)} y^{-2u}\, du$$

with $\sigma_0 \in (\frac{1-k}{2}, 0)$. The series and integrals converge absolutely when $\Re(s) > \frac{5}{4}$ and we are allowed to change the order of the sums and integrals. Then infinitely many additively twisted L-functions associated with f appear in the geometric side:

$$\begin{aligned}\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g \in H_k(N,\chi)} \rho_g(n) L(s, f \times \bar{g}) &= \delta_{n=m}\zeta^{(N)}(2s) f_n n^{-s} \\ &+ i^{-k}\zeta^{(N)}(2s) \sum_{q \in N\mathbb{Z}_{\geq 1}} \frac{1}{2\pi i} \int_{\Re(u)=\sigma_0} \frac{\Gamma_\mathbb{C}\left(u + \frac{k-1}{2}\right)}{\Gamma_\mathbb{C}\left(-u + \frac{k+1}{2}\right)} \\ &\times (4\pi)^{-2u} n^{-u} q^{2u-1} \sum_{\substack{a,\bar{a} \bmod q,\\ a\bar{a} \equiv 1 \bmod q}} \chi(a) e\left(\frac{n\bar{a}}{q}\right) L_f(s+u, a/q)\, du.\end{aligned}$$

We apply the functional equations (1.2) for the additively twisted L-functions $L_f(s+u, a/q)$ and get

$$\text{(2.2)} \quad \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g\in H_k(N,\chi)} \rho_g(n) L(s, f\times \bar{g})$$
$$= \zeta^{(N)}(2s) f_n n^{-s} + i^{-k} f_n n^{-1+s} N^{2-2s} \frac{\varphi(N)}{N} \zeta(2s-1) F(k) + P_f(s)$$

where $P_f(s)$ is a function which is analytic for $\Re(s) > \frac{1}{2}$ (see [1, (3.4)]). Here

$$F(k) = \frac{1}{2\pi i} \int_{\Re(u)=-\frac{5}{2}} \frac{\Gamma_{\mathbb{C}}\left(\frac{k-1+u}{2}\right)\gamma(-\frac{u}{2})}{\Gamma_{\mathbb{C}}\left(\frac{k+1-u}{2}\right)\gamma(1+\frac{u}{2})}\, du.$$

Note that the right-hand side of (2.2) continues analytically to $\Re(s) > \frac{1}{2}$, except a simple at $s=1$ for $\zeta(2s-1)$. Since $S_k(N,\chi)$ is a finite dimensional space, $H_k(N,\chi)$ is a finite set. By considering different positive integers n, we can solve a system of linear equations and prove that each Rankin-Selberg convolution $L(s, f\times\bar{g})$ for $g\in H_k(N,\chi)$ on the left-hand side of (2.2) also has analytic continuation to $\Re(s) > \frac{1}{2}$ except a possible pole at $s=1$.

The right-hand side of (2.2) has a pole at $s=1$ with a non-zero residue if $F(k)\neq 0$. So if $F(k)\neq 0$ for some $k\geq 4$, then at least one of the Rankin-Selberg convolutions in the spectral side (left-hand side) of (2.2) should have a pole at $s=1$ with a non-zero residue. This implies that, after taking the residue at $s=1$ for both sides, we have

$$f_n = \frac{1}{F(k)} \frac{2i^k N}{\varphi(N)} \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g\in H_k(N,\chi)} \rho_g(n) \mathrm{Res}_{s=1} L(s, f\times\bar{g}),$$

for any $n\in\mathbb{Z}_{\geq 1}$. Since $S_k(N,\chi)$ is a vector space, we have $f(z) = \sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} e(nz) \in S_k(N,\chi)$.

If $F(k)=0$ for all $k\geq 4$, then we can show that $\gamma(s) = cH^s\Gamma_{\mathbb{C}}\left(s+\frac{\ell-1}{2}\right)$ for $\ell\in\{1,2,3\}$ and $\chi(-1) = (-1)^\ell$, for some $c, H\in\mathbb{R}_{>0}$ [1, Proposition 3.2]. The next step is proving that H should be less than or equal 1 for these cases. Then, we use the methods of the proof of the classical converse theorem to prove the results in Theorem 1.1. With $H\leq 1$, this requires more work, but we will not explain further here. See [1] for the details.

We can prove Theorem 1.2 similarly going through the Petersson trace formula at the cusp 0.

□

References

[1] Andrew R. Booker, Michael Farmer, and Min Lee. An extension of Venkatesh's converse theorem to the Selberg class. *Forum Math. Sigma*, 11:Paper No. e26, 10, 2023.
[2] Andrew R. Booker and M. Krishnamurthy. A strengthening of the GL(2) converse theorem. *Compos. Math.*, 147(3):669–715, 2011.
[3] Andrew R. Booker and M. Krishnamurthy. Weil's converse theorem with poles. *Int. Math. Res. Not. IMRN*, (19):5328–5339, 2014.
[4] J. W. Cogdell and I. I. Piatetski-Shapiro. Converse theorems for GL_n. *Inst. Hautes Études Sci. Publ. Math.*, (79):157–214, 1994.
[5] J. W. Cogdell and I. I. Piatetski-Shapiro. Converse theorems for GL_n. II. *J. Reine Angew. Math.*, 507:165–188, 1999.
[6] Stephen Gelbart and Hervé Jacquet. A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978.
[7] Jeff Hoffstein, Min Lee, and Maria Nastasescu. First moments of Rankin-Selberg convolutions of automorphic forms on GL(2). *Res. Number Theory*, 7(4):Paper No. 60, 44, 2021.
[8] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
[9] H. Jacquet and R. P. Langlands. *Automorphic forms on* GL(2), volume Vol. 114 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.

[10] Jerzy Kaczorowski and Alberto Perelli. Twists, Euler products and a converse theorem for L-functions of degree 2. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 14(2):441–480, 2015.

[11] Jerzy Kaczorowski and Alberto Perelli. Classification of L-functions of degree 2 and conductor 1. *Adv. Math.*, 408(part A):Paper No. 108569, 46, 2022.

[12] Wen Ch'ing Winnie Li. On converse theorems for GL(2) and GL(1). *Amer. J. Math.*, 103(5):851–885, 1981.

[13] Toshitsune Miyake. *Modular forms*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda.

[14] Michael J. Razar. Modular forms for $G_0(N)$ and Dirichlet series. *Trans. Amer. Math. Soc.*, 231(2):489–495, 1977.

[15] Raphael S. Steiner. Near counterexamples to Weil's converse theorem. *J. Théor. Nombres Bordeaux*, 31(2):313–321, 2019.

[16] Akshay Venkatesh. *Limiting forms of the trace formula*. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–Princeton University.

[17] André Weil. Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Math. Ann.*, 168:149–156, 1967.

E-mail address: andrew.booker@bristol.ac.uk

E-mail address: xy20887@bristol.ac.uk

E-mail address: min.lee@bristol.ac.uk

School of Mathematics, University of Bristol, Woodland Road, Bristol, BS8 1UG, United Kingdom