Finite crystalline height representations and syntomic complexes

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Abstract. Using finite crystalline height representations and their naturally associated invariants, we study local and global syntomic complexes with coefficients. The text is organized as follows. After briefly recalling the p-adic crystalline comparison theorem and importance of syntomic methods in its proof we pose a question on syntomic complex with coefficients. To answer our question, we quickly recount the theory of finite crystalline height representations developed in [Abh21] and show that Galois cohomology of such representations (upto a twist), is essentially computed by (Fontaine-Messing) syntomic complex with coefficients in the associated F-isocrystal. In global applications, for smooth (p-adic formal) schemes, we show a comparison between syntomic complex with coefficient in a locally free Fontaine-Laffaille module and complex of p-adic nearby cycles of the associated étale local system on the (rigid) generic fiber. Proofs of aforementioned results can be found in [Abh22].

1. p-adic comparison theorem

Let p denote a fixed prime, κ a perfect field of characteristic p, K a discrete valuation field of mixed characteristic with ring of integers O_K and residue field κ and $F = W(\kappa)[1/p]$ the fraction field of ring of p-typical Witt vectors with coefficients in κ. Fontaine's *crystalline conjecture* for an O_K-scheme $\mathfrak{X}$ examines the relationship between p-adic étale cohomology of its generic fiber and crystalline cohomology of its special fiber. More precisely,

Theorem 1.1. *Let $\mathfrak{X}$ be a proper and smooth scheme defined over O_K, with $X = \mathfrak{X} \otimes_{O_K} K$ its generic fiber $\mathfrak{X}_\kappa = \mathfrak{X} \otimes_{O_K} \kappa$ its special fiber. Then for each $k \in \mathbb{N}$ there exists a natural isomorphism*

$$H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{cris}}(O_{\overline{K}}) \xrightarrow{\sim} H^k_{\text{cris}}(\mathfrak{X}_\kappa/W(\kappa)) \otimes_{W(\kappa)} \mathbf{B}_{\text{cris}}(O_{\overline{K}}),$$

compatible with filtration, Frobenius and action of G_K on each side.

Here $\mathbf{B}_{\text{cris}}(O_{\overline{K}})$ denotes the crystalline period ring constructed by Fontaine (see [Fon94a]), and it is equipped with a filtration, Frobenius and continuous action of G_K.

In [FM87] Fontaine and Messing initiated a program for proving the statement via *syntomic* methods. By subsequent works of [KM92, Kato-Messing], [Kat94, Kato] and the remarkable work of [Tsu99, Tsuji] this program was concluded with a proof of the crystalline comparison theorem (more generally, the semistable comparison theorem). There have been several other proofs as well as generalizations of crystalline comparison theorem: [Fal89; Fal02, Faltings], [Niz98, Nizioł], [Bei12; Bei13, Beilinson], [Sch13, Scholze], [YY14, Yamashita-Yasuda], [CN17, Colmez-Nizioł], [BMS18, Bhatt-Morrow-Scholze] among others.

Theorem 1.1 also holds for proper and smooth p-adic formal schemes. This was shown by Andreatta and Iovita in [AI13] using Faltings approach of almost étale extensions. The natural variation of Theorem 1.1 for proper semistable p-adic formal schemes was obtained by Colmez and Nizioł in [CN17] using syntomic methods. One of the most important and difficult steps in proofs utilising syntomic methods is establishment of a comparison between syntomic complex and the complex of p-adic nearby cycles.

2. p-ADIC NEARBY CYCLES

Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_K with X its (rigid) generic fiber and $\mathfrak{X}_\kappa$ its special fiber. Let $j : X_{\text{ét}} \to \mathfrak{X}_{\text{ét}}$ and $i : \mathfrak{X}_{\kappa,\text{ét}} \to \mathfrak{X}_{\text{ét}}$ denote natural morphisms of étale sites. For $r \geq 0$, let $\mathscr{S}_n(r)_{\mathfrak{X}}$ denote the syntomic sheaf modulo p^n on $\mathfrak{X}_{\kappa,\text{ét}}$. It can be thought of as a derived Frobenius and filtration eigenspace of crystalline cohomology. In [FM87], Fontaine and Messing constructed a period morphism

$$\alpha_{r,n}^{\text{FM}} : \mathscr{S}_n(r)_{\mathfrak{X}} \longrightarrow i^*\mathrm{R}j_*\mathbb{Z}/p^n(r)'_X, \tag{2.1}$$

from the syntomic complex to the complex of p-adic nearby cycles, where $\mathbb{Z}_p(r)' := \frac{1}{a(r)!p^{a(r)}}\mathbb{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$ with $0 \leq b(r) < p-1$. In the case of schemes, for $0 \leq r \leq p-1$ and after truncating the complexes in (2.1) in degrees $\leq r$ the map $\alpha_{r,n}^{\text{FM}}$ was shown to be a quasi-isomorphism in the work of Kato [Kat87; Kat94], Kurihara [Kur87], and Tsuji [Tsu99]. In [Tsu96], Tsuji generalized the result for schemes to some non-trivial étale local systems arising from Fontaine-Laffaille modules over O_F (see [FL82]).

Colmez and Nizioł have shown that the Fontaine-Messing period map $\alpha_{r,n}^{\text{FM}}$, after a suitable truncation, is essentially a quasi-isomorphism. More precisely,

Theorem 2.1 ([CN17, Theorem 1.1])**.** *For $0 \leq k \leq r$, the map*

$$\alpha_{r,n}^{\text{FM}} : \mathcal{H}^k(\mathscr{S}_n(r)_{\mathfrak{X}}) \longrightarrow i^*\mathrm{R}^k j_*\mathbb{Z}/p^n(r)'_X,$$

is a p^N-isomorphism, i.e. the kernel and cokernel of this map is killed by p^N, where $N = N(e,p,r) \in \mathbb{N}$ depends on the absolute ramification index e of K, the prime p and the twist r.

Theorem 2.1 also holds for base change of proper and smooth (p-adic formal) schemes. In particular, after passing to the limit and inverting p, for $0 \leq k \leq r$ we obtain isomorphisms (see [Tsu99, Theorem 3.3.4])

$$\alpha_r^{\text{FM}} : H^k_{\text{syn}}(\mathfrak{X}_{O_{\overline{K}}}, r)_{\mathbb{Q}} \xrightarrow{\sim} H^k_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)). \tag{2.2}$$

The isomorphism in (2.2) is one of the most important step in proving Theorem 1.1 via syntomic methods. These ideas have been used in [FM87], [KM92], [Kat87], [Kat94], [Tsu99] and [YY14].

The proof of Colmez and Nizioł is different from earlier approaches. They prove Theorem 2.1 first and deduce the comparison in (2.2) via base change in proper and smooth case. To prove their claim, they reduce the problem to local setting and construct another local period map $\alpha_r^{\mathcal{L}\text{az}}$, employing techniques from the theory of (φ, Γ)-modules and a version of integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology. They show that $\alpha_r^{\mathcal{L}\text{az}}$ is a quasi-isomorphism and coincides with local Fontaine-Messing period map up to some fixed power of p.

Remark 2.2. The results of [CN17] have been worked out in the setting of semistable (p-adic formal) schemes. So to obtain the claim for $0 \leq k \leq r$ as in Theorem 2.1, one should work with log-crystalline cohomology. Working without log structures, one would obtain the p-power isomorphism in Theorem 2.1 for $0 \leq k \leq r-1$ (also see Remark 4.5 (1) below).

2.1. Local comparison. Most of the work done for the proof of Theorem 1.1 in [CN17] involves computations in the local setting, i.e. over an étale algebra over a (formal) torus. More precisely, a smooth (p-adic formal) scheme $\mathfrak{X}$ defined over O_K can be covered by affine schemes given as (formal) spectrum of (p-adic completion of an) étale algebra over $O_K[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ for some indeterminates $X_1, \ldots, X_d$. In the local setting, Colmez and Nizioł also show that it is enough to work with p-adic completions, i.e. formal schemes and deduce results for schemes by invoking Elkik's approximation theorem and a form of rigid GAGA (see [CN17, §5.1]).

For simplicity, we will take R to be the p-adic completion of $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ and let $S := O_K \otimes_{O_F} R$. Let $G_S = \pi_1^{\text{ét}}(S[1/p], \overline{\eta})$ for a fixed geometric generic point of $\text{Sp}\,(S[1/p])$. Let R_ϖ^+ denote the (p, X_0)-adic completion of $W(\kappa)[X_0, X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$, and let R_ϖ^{PD} denote the p-adic completion of the divided power envelope with respect to the kernel of the map $R_\varpi^+ \twoheadrightarrow S$ sending X_0 to ϖ (a uniformizer of K). Further, let $\Omega^1_{R_\varpi^{\text{PD}}}$ denote the p-adic completion of the module of differentials of R_ϖ^{PD} relative to $\mathbb{Z}$ and one can extend the Frobenius operator φ to $\Omega^1_{R_\varpi^{\text{PD}}}$. The syntomic cohomology of S is computed by the complex

$$\text{Syn}(S, r) := \text{Cone}(F^r \Omega^\bullet_{R_\varpi^{\text{PD}}} \xrightarrow{p^r - p^\bullet \varphi} \Omega^\bullet_{R_\varpi^{\text{PD}}})[-1].$$

Theorem 2.3 ([CN17, Theorem 1.6])**.** *If K contains enough roots of unity, then the maps*

$$\alpha_r^{\mathcal{L}az} : \tau_{\leq r}\text{Syn}(S, r) \longrightarrow \tau_{\leq r}\text{R}\Gamma_{\text{cont}}(G_S, \mathbb{Z}_p(r)),$$
$$\alpha_{r,n}^{\mathcal{L}az} : \tau_{\leq r}\text{Syn}(S, r)_n \longrightarrow \tau_{\leq r}\text{R}\Gamma_{\text{cont}}(G_S, \mathbb{Z}/p^n(r)) \longrightarrow \tau_{\leq r}\text{R}\Gamma\big((\text{Sp } S[\tfrac{1}{p}])_{\text{ét}}, \mathbb{Z}/p^n(r)\big),$$

are p^{Nr}-quasi-isomorphisms for a universal constant N.

Note that the truncation here denotes canonical truncation in literature. Having enough roots of unity in K is a technical condition (see [CN17, §2.2.1]) and if one fixes K then $K(\zeta_{p^m})$ has enough roots of unity for $m \geq c(K) + 3$, where $c(K)$ is the conductor of K.

In general, if K does not contain enough roots of unity (for example $K = F$), then one passes to an extension $K(\zeta_{p^m})$ for m large enough and then using Galois descent one obtains an alaogous statement over K with constant N depending on the absolute ramification index $e = [K : F]$, p and r (see [CN17, Theorem 5.4]). The proof of Colmez and Nizioł relies on comparing syntomic complexes with the relative version of Fontaine-Herr complex of (φ, Γ)-modules computing the continuous G_S-cohomology of $\mathbb{Z}_p(r)$ (see [Her98] and [AI08]).

Remark 2.4. Similar to Remark 2.2 let us note that in Theorem 2.3 Colmez and Nizioł work with semistable affinoids and log-syntomic complex. Without log structures one should truncate in degree $\leq r - 1$ (see Remark 4.5 (1) below).

Our goal is to generalize Theorem 2.1 to non-trivial coefficients. Clearly, one needs to restrict themselves to a "friendly" category of coefficients, i.e. objects for which local computations similar to [CN17] could be carried out. In the local setting, by techniques employed in the proof of Theorem 2.3 (and applying $K(\pi, 1)$-Lemma of Scholze for p-coefficients, see [Sch13, Theorem 4.9]), the problem could be formulated as

Question 2.5. Is it possible to obtain a statement similar to Theorem 2.3 for non-trivial $\mathbb{Z}_p$-representations of G_R?

Our goal in this article is to give a positive answer to the question posed above. A natural object to consider for a local result of this nature is a G_R-stable $\mathbb{Z}_p$-lattice T inside a crystalline representation V of G_R (in the sense of [Bri08, Chapitre 8]). However, as local computations involve complexes of (φ, Γ)-modules, we should further restrict ourselves to a representation whose corresponding étale (φ, Γ)-module is of "finite height" and "crystalline". Representations capturing these ideas are referred to as *finite crystalline height representations.*

Remark 2.6. Imposing finite height assumption on the (φ, Γ)-module attached to a crystalline representation of G_R is not at all restrictive since all crystalline representations are of finite height (see [Abh23, Theorem 1.7]). However, in the present article we only use results of [Abh21] and [Abh22] so we motivate the objects of interest informally as above.

3. Finite height representations

In the classical case, i.e. for a mixed characteristic local field K, in [Fon90] Fontaine established an equivalence of catgeories between $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_K and étale (φ, Γ)-modules over a certain period ring $\mathbf{A}_K$ (resp. $\mathbf{B}_K$). Moreover, in [Fon79; Fon82; Fon94a; Fon94b] Fontaine described crystalline representations of G_K in terms of weakly admissible filtered φ-modules over F. For $K = F$, by the works of [Wac96; Wac97, Wach], [Col99, Colmez] and [Ber04, Berger] it is known that crystalline representations of G_F can also be described in terms of finite height (φ, Γ)-modules (closely related to the étale (φ, Γ)-module of Fontaine).

In the relative case, let us now fix $p \geq 3$ (see Remark 4.5), an absolutely unramified extension F over $\mathbb{Q}_p$, $K = F(\zeta_{p^m})$ for a fixed $m \geq 1$ and let $\varpi = \zeta_{p^m} - 1$. Let R denote the p-adic completion of an étale algebra over $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ with non-empty geometrically integral special fiber and let $S := O_K \otimes_{O_F} R$. We also fix lift $\varphi : R \to R$ of the absolute Frobenius $R/pR \to R/pR$ given as $x \mapsto x^p$ (see [Abh22, §2.1]). Denote by Ω^1_R the p-adic completion of module of differentials of R with respect to $\mathbb{Z}$.

3.1. (φ, Γ)-modules. Let us fix an algebraically closed field $\overline{\text{Fr }(R)}$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite R-subalgebras $R' \subset \overline{\text{Fr }(R)}$ such that $R'[1/p]$ is étale over $R[1/p]$. We write $\mathbb{C}^+(\overline{R}) = \widehat{\overline{R}}$ as the p-adic completion, $\mathbb{C}(\overline{R}) = \mathbb{C}^+(\overline{R})[1/p]$ and $G_R = \text{Gal}(\overline{R}[1/p]/R[1/p])$. For $n \in \mathbb{N}$, let $F_n = F(\zeta_{p^n})$ with ring of integers O_{F_n} and let R_n denote the integral closure of $R \otimes_{O_F} O_{F_n}[X_1^{1/p^n}, \ldots X_d^{1/p^n}]$ inside $\overline{R}[1/p]$ and let $R_\infty := \cup_n R_n$. We set $\Gamma_R := \text{Gal}(R_\infty[1/p]/R[1/p])$, $H_R := \text{Ker }(G_R \to \Gamma_R)$ and we have an exact sequence

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_F \longrightarrow 1,$$

where $\Gamma'_R = \text{Gal}(R_\infty[1/p]/F_\infty R[1/p]) \simeq \mathbb{Z}_p^d$, and $\Gamma_F = \text{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^\times$ (see [Abh21, §3.1.1]). Similarly, one can define corresponding groups for S, i.e. groups G_S and Γ_S.

Let $\mathbb{C}^+(\overline{R})^\flat = \lim_{x \mapsto x^p} \mathbb{C}^+(\overline{R})/p$ denote the tilt of perfectoid algebra $\mathbb{C}^+(\overline{R})$ and let $\mathbb{C}(\overline{R})^\flat = \mathbb{C}^+(\overline{R})^\flat[1/p^\flat]$ where $p^\flat = (p, p^{1/p}, p^{1/p^2}, \ldots) \in \mathbb{C}^+(\overline{R})^\flat$, both algebras equipped with a natural action of G_R. Let $W(\mathbb{C}(\overline{R})^\flat)$ denote the ring of p-typical Witt vectors equipped with Witt vector Frobenius and natural G_R-action. Using a certain period ring $\mathbf{A} \subset W(\mathbb{C}(\overline{R})^\flat)$, stable under induced Frobenius and G_R-action, in [And06] Andreatta generalized Fontaine's results to $\mathbb{Z}_p$-representations (resp. p-adic representations) of G_R. To any $\mathbb{Z}_p$-representation T of G_R, Andreatta functorially attaches an étale (φ, Γ_R)-module $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_R}$ over the period ring $\mathbf{A}_R = \mathbf{A}^{H_R}$. This induces an equivalence of categories between $\mathbb{Z}_p$-representations and étale (φ, Γ_R)-modules over $\mathbf{A}_R$. Similarly, to any p-adic representation V of G_R, using the period ring $\mathbf{B} = \mathbf{A}[1/p]$, one can attach an étale (φ, Γ_R)-module $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_R}$ over $\mathbf{B}_R = \mathbf{B}^{H_R} = \mathbf{A}_R[1/p]$. Again, this induces an equivalence of categories between p-adic representations and étale (φ, Γ_R)-modules over $\mathbf{B}_R$. Now let $\mathbf{A}_{\text{inf}}(\overline{R}) = W(\mathbb{C}^+(\overline{R})^\flat)$, $\mathbf{A}^+ = \mathbf{A} \cap \mathbf{A}_{\text{inf}}(\overline{R}) \subset W(\mathbb{C}(\overline{R})^\flat)$ and set $\mathbf{D}^+(T) = (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_R} \subset \mathbf{D}(T)$ a module over $\mathbf{A}_R^+ = (\mathbf{A}^+)^{H_R}$ equipped with induced (φ, Γ_R)-action. Let $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathbb{C}^+(\overline{R})^\flat$ where ζ_{p^n} is a primitive p^n-th root of unity. Set $\pi = [\varepsilon] - 1$, $q = \varphi(\pi)/\pi \in \mathbf{A}_{\text{inf}}(\overline{R})$.

In [Abh21], we studied the notion of finite q-height representations, motivated by classical definition of finite crystalline height representations [Fon90; Wac96; Wac97; Col99; Ber04] (see [Abh21, Remark 1.4]). Moreover, finite q-height representations are closely related to crystalline representations of G_R (see Theorem 3.3 below). We introduce the following definition:

Definition 3.1 ([Abh21, Definition 1.3]). A $\mathbb{Z}_p$-representation T of G_R is *positive* and of *finite q-height* if there exists a finite projective $\mathbf{A}_R^+$-submodule $\mathbf{N}(T) \subset \mathbf{D}^+(T)$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ such that:

(1) $\mathbf{N}(T)$ is stable under the action of φ and Γ_R, and $\mathbf{A}_R \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \xrightarrow{\sim} \mathbf{D}(T)$;

(2) The $\mathbf{A}_R^+$-module $\mathbf{N}(T)/\varphi^*(\mathbf{N}(T))$ is killed by q^s for some $s \in \mathbb{N}$;

(3) The action of Γ_R is trivial on $\mathbf{N}(T)/\pi\mathbf{N}(T)$;

(4) There exists $R' \subset \overline{R}$ finite étale over R such that $\mathbf{A}_{R'}^+ \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ is free over $\mathbf{A}_{R'}^+$.

The *height* of T is defined to be the smallest $s \in \mathbb{N}$ satisfying (2) above. Furthermore, a positive finite q-height p-adic representation V of G_R is a representation admitting a positive finite q-height $\mathbb{Z}_p$-lattice $T \subset V$ and we set $\mathbf{N}(V) := \mathbf{N}(T)[1/p]$ satisfying properties analogous to (1)-(4) above. The height of V is defined to be the height of T.

3.2. Crystalline representations. Akin to Fontaine's methods to study p-adic representations of G_F in [Fon82], Brinon studied p-adic representations of G_R in [Bri08]. To classify relative crystalline representations, one uses the crystalline period rings $\mathbf{A}_{\mathrm{cris}}(\overline{R})$ and $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ which are p-adically complete R-algebras equipped with a continuous G_R-action, a Frobenius endomorphism and a filtration. Moreover, on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ we have an $\mathbf{A}_{\mathrm{cris}}(\overline{R})$-linear connection ∂ satisfying Griffiths transversality i.e. $\partial(\mathrm{Fil}^r \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})) \subset \mathrm{Fil}^{r-1}\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_R \Omega^1_R$ for $r \in \mathbb{Z}$ (see [Bri08, Chapitre 6] and [Abh21, §2.2] for details). For a p-adic representation V of G_R let

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) := (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V)^{G_R}.$$

This construction is functorial in V and takes values in the category of filtered (φ, ∂)-modules over $R[1/p]$. The representation V is said to be *crystalline* if the natural map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \rightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ is an isomorphism. Restricting the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$ to the subcategory of crystalline representations of G_R establishes an equivalence with the essential image of the restriction.

Remark 3.2. Let $\pi_1 = \varphi^{-1}(\pi) \in \mathbf{A}_{\mathrm{inf}}(\overline{R})$ and take $\mathbf{A}_{R,\varpi}^+ = \mathbf{A}_R^+[\pi_1]$. Consider the algebra $R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+$ and the natural surjective map $R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+ \twoheadrightarrow S = R[\varpi]$. The kernel of the preceding surjection is given by the ideal $I = \big(\pi/\pi_1, X_1 \otimes 1 - 1 \otimes [X_1^\flat], \ldots, X_d \otimes 1 - 1 \otimes [X_d^\flat]\big) \subset R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+$. Set $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of $(R \otimes_{\mathbb{Z}} \mathbf{A}_{R,\varpi}^+)[x^k/k!, x \in I]$. We have $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R})$ equipped with induced filtration, Frobenius, G_R-action and a connection satisfying Griffiths transversality with respect to filtration (see [Abh21, §4.3.1] for details).

Finite q-height representations of G_R introduced in Definition 3.1 are related to p-adic crystalline representations of G_R using the period ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$.

Theorem 3.3 ([Abh21, Theorem 1.6]). *Let V be a positive and finite q-height representation of G_R, then*

(1) *V is a positive crystalline representation, i.e. all its Hodge-Tate weights are ≤ 0.*

(2) *We have an isomorphism of $R[1/p]$-modules*

$$\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \big(\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)\big)^{\Gamma_R}[\tfrac{1}{p}],$$

compatible with Frobenius, filtration, and connection on each side.

(3) *Over $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}}$ we have a natural isomorphism*

$$\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xleftarrow{\sim} \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(V),$$

compatible with Frobenius, filtration, connection and the action of Γ_R on each side.

The preceding result helps us in constructing an R-submodule inside $\mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ satisfying certain key properties helpful in establishing our main local result (see Theorem 4.2).

4. Syntomic coefficients and (φ, Γ)-modules

In this section, let us consider the following class of representations: Let V be a positive finite q-height representation of G_R with $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice as in Definition 3.1 such that the $\mathbf{A}_R^+$-module is free of rank $= \dim_{\mathbb{Q}_p} V$. Assume that the Wach module $\mathbf{N}(T)$ is free of rank $= \dim_{\mathbb{Q}_p} V$ over $\mathbf{A}_R^+$ and let $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ be a free R-submodule of rank $= \dim_{\mathbb{Q}_p} V$ such that $M[1/p] = \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ and the induced connection over M is p-adically quasi-nilpotent, integrable and satisfies Griffiths transversality with respect to the induced filtration. Furthermore, assume that $p^s M \subset \varphi^*(M)$ and there exists a p^N-isomorphism $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_R M \simeq \mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T)$ with $N = n(T, e) \in \mathbb{N}$ for $e = [K : F] = p^{m-1}(p-1)$ and compatible with Frobenius, filtration, connection and Γ_R-action. See [Abh22, Example 5.2] on obtaining M as above from $\mathbf{N}(T)$.

4.1. Main results. Our objective is to relate the relative Fontaine-Herr complex (see [AI08]) computing continuous G_R-cohomology of $T(r)$ to syntomic complex with coefficients in the R-module $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Let us first consider the case of cyclotomic extension $S = R[\varpi]$. Let $R_\varpi^+ = R[\![X_0]\!]$ equipped with a Frobenius endomorphism extending the Frobenius on R by setting $\varphi(X_0) = (1+X_0)^p - 1$. We have a surjective map $f : R_\varpi^+ \twoheadrightarrow S = R[\varpi]$ sending $X_0 \mapsto \varpi$. Let R_ϖ^{PD} denote the p-adic completion of divided power envelope of R_ϖ^+ with respect to Ker f. The Frobenius on R_ϖ^+ extends naturally to an endomorphism of R_ϖ^{PD} (see [Abh21, §3.2- §3.3]).

Set $M_\varpi^{\mathrm{PD}} := R_\varpi^{\mathrm{PD}} \otimes_R M$ equipped with a tensor product Frobenius endomorphism, tensor product filtration and a connection defined using the Leibniz rule on the differential on R_ϖ^{PD} and connection on M. The connection on M_ϖ^{PD} further satistfies Griffiths transversality with respect to the filtration. In particular, we have a filtered de Rham complex

$$\mathrm{Fil}^r \mathscr{D}_{S,M}^\bullet : \mathrm{Fil}^r M_\varpi^{\mathrm{PD}} \longrightarrow \mathrm{Fil}^{r-1} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^1_{R_\varpi^{\mathrm{PD}}} \longrightarrow \mathrm{Fil}^{r-2} M_\varpi^{\mathrm{PD}} \otimes_{R_\varpi^{\mathrm{PD}}} \Omega^2_{R_\varpi^{\mathrm{PD}}} \longrightarrow \cdots.$$

To describe the action of Frobenius on $\Omega^1_{R_\varpi^{\mathrm{PD}}}$ we fix a basis of $\Omega^1_{R_\varpi^{\mathrm{PD}}}$ as $\{\frac{dX_0}{1+X_0}, \frac{dX_1}{X_1}, \ldots, \frac{dX_d}{X_d}\}$. For $j \in \mathbb{N}$, let $I_j = \{0 \leq i_1 < \cdots < i_j \leq d\}$ and for $\mathbf{i} = (i_1, \ldots, i_j) \in I_j$, set

$$\omega_{\mathbf{i}} = \begin{cases} \frac{dX_0}{1+X_0} \wedge \frac{dX_{i_2}}{X_{i_2}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} & \text{if } i_1 = 0, \\ \frac{dX_{i_1}}{X_{i_1}} \wedge \cdots \wedge \frac{dX_{i_j}}{X_{i_j}} & \text{otherwise.} \end{cases}$$

Define the Frobenius operator φ on $\Omega^j_{R_\varpi^{\mathrm{PD}}}$ by $\varphi\big(\sum_{\mathbf{i} \in I_j} x_{\mathbf{i}} \omega_{\mathbf{i}}\big) = \sum_{\mathbf{i} \in I_j} \varphi(x_{\mathbf{i}}) \omega_{\mathbf{i}}$. Note that this is not the usual definition of Frobenius on $\Omega^j_{R_\varpi^{\mathrm{PD}}}$. But in order to define a useful operator ψ integrally, we need to divide the usual Frobenius on $\Omega^1_{R^\star_\varpi}$ by powers of p (see [Abh22, Remark 5.3]). Furthermore, with the usual definition of Frobenius we have $\varphi\partial = \partial\varphi$ over $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. However, using a definition similar to above for φ on Ω^1_R as well, we note that for $f \in M$, we now have $\partial_M(\varphi(f)) = \sum_{i=1}^d \partial_i(\varphi(f))\omega_i = \sum p\varphi(\partial_i(f))\omega_i = p\varphi(\partial_M(f))$.

Definition 4.1. Let $r \in \mathbb{N}$ and define the *syntomic complex* of S with coefficients in M as

$$\mathrm{Syn}(S, M, r) := [\ \mathrm{Fil}^r\mathscr{D}^\bullet_{S,M} \xrightarrow{p^r - p^\bullet\varphi} \mathscr{D}^\bullet_{S,M}\ ],$$
$$\mathrm{Syn}(S, M, r)_n := \mathrm{Syn}(S, M, r) \otimes \mathbb{Z}/p^n.$$

Our main local result is as follows:

Theorem 4.2 ([Abh22, Theorem 1.5])**.** *Let V be a postive finite q-height representation of G_R of height s with $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice as above and let $r \in \mathbb{N}$ such that $r \geq s+1$. Then there exists p^N-quasi-isomorphisms*[1]

$$\alpha_r^{\mathcal{L}az} : \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r) \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T(r)),$$
$$\alpha_{r,n}^{\mathcal{L}az} : \tau_{\leq r-s-1}\mathrm{Syn}(S, M, r)_n \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_S, T/p^n(r)),$$

where $N = N(T, e, r) \in \mathbb{N}$ depending on the representation T, the absolute ramification index e of K and the twist r.

The proof of Theorem 4.2 proceeds in two main steps: First, we modify the syntomic complex with coefficients in M to relate it to a "differential" Koszul complex with coefficients in $\mathbf{N}(T)$ (see [Abh22, Proposition 5.30]). Next, in the second step we modify Koszul complex from the first step to obtain Koszul complex computing continuous G_S-cohomology of $T(r)$ (see [Abh22, Theorem 5.5 and Proposition 6.1]). The key idea behind relating these two steps is provided by the comparison isomorphism in Theorem 3.3 and a version of Poincaré Lemma (see [Abh22, §5.6]). The idea for the proof is inspired by the work of Colmez and Nizioł [CN17], however our setting demands several non-trivial technical refinements. See [Abh22] for details.

We can descend the quasi-isomorphism in Theorem 4.2 to R. Note that we have a filtered de Rham complex over R with coeffcients in M as

$$\mathrm{Fil}^r\mathscr{D}^\bullet_{R,M} : \mathrm{Fil}^r M \longrightarrow \mathrm{Fil}^{r-1}M \otimes_R \Omega^1_R \longrightarrow \mathrm{Fil}^{r-2}M \otimes_R \Omega^2_R \longrightarrow \cdots.$$

Similar to above, one can define the Frobenius operator φ on Ω^1_R.

Definition 4.3. Let $r \in \mathbb{N}$ and define the *syntomic complex* of R with coefficients in M as

$$\mathrm{Syn}(R, M, r) := [\ \mathrm{Fil}^r\mathscr{D}^\bullet_{R,M} \xrightarrow{p^r - p^\bullet\varphi} \mathscr{D}^\bullet_{R,M}\ ];$$
$$\mathrm{Syn}(R, M, r)_n := \mathrm{Syn}(R, M, r) \otimes \mathbb{Z}/p^n.$$

Using Theorem 4.2 for $\varpi = \zeta_{p^2} - 1$ and Galois descent from [Abh22, Lemma 6.24], we obtain

Corollary 4.4 ([Abh22, Corollary 1.6])**.** *Let V be a positive finite q-height representation of G_R of height s with $T \subset V$ a G_R-stable $\mathbb{Z}_p$-lattice as above and let $r \in \mathbb{N}$ such that $r \geq s+1$. Then there exists p^N-quasi-isomorphisms*

$$\tau_{\leq r-s-1}\mathrm{Syn}(R, M, r) \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T(r)),$$
$$\tau_{\leq r-s-1}\mathrm{Syn}(R, M, r)_n \simeq \tau_{\leq r-s-1}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R, T/p^n(r)),$$

where $N = N(p, r, s) \in \mathbb{N}$ depending on the prime p, the twist r and the q-height s of V.

[1] A homomorphism $f : M \to N$ of $\mathbb{Z}_p$-modules is said to be a p^n-isomorphism for some $n \in \mathbb{N}$, if the kernel and cokernel of f are killed by p^n.

Remark 4.5. (1) Taking $T = \mathbb{Z}_p$ in Theorem 4.2 we obtain a statement similar to Theorem 2.1. However, note that we truncate in degree $\leq r-1$. This is because we do not work with log-structures unlike [CN17]. Working with log-syntomic complex, where we consider log-structure over R_ϖ^+ with respect to the arithmetic variable X_0 and Kummer Frobenius (see [CN17, §2.2]) would enable us to show a p-power quasi-isomorphism also in degree r.

(2) To obtain the statement over $\overline{F}$ one could proceed as in [Abh22, Remark 1.8]. Alternatively, one could directly work over $\mathbb{C}_p = \widehat{\overline{F}}$ as in [Gil21] to avoid complications arising from Frobenius on arithmetic variable X_0.

(3) The case $p = 2$ is different from $p \geq 3$ as for $p = 2$ the constant N in Theorem 4.2 depends on the relative dimension of R (see [CN17, Lemma 3.11].

To conclude this section, let us note that for S as in Theorem 4.2, using the fundamental exact sequence in p-adic Hodge theory, one can define the local version of Fontaine-Messing period map (see [Abh22, §6.7]) for T as in Theorem 4.2. Then we are able to show:

Theorem 4.6 ([Abh22, Theorem 1.10])**.** *The Fonatine-Messing period map is* $p^{N(T,e,r)}$*-equal to* $\alpha_{r,n}^{\mathcal{L}az}$ *from Theorem 4.2.*

4.2. Proof of Theorem 4.2.

The idea for the proof of Theorem 4.2 can be captured in the following commutative diagram of complexes. Note that we have $\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M_\varpi^{\mathrm{PD}}) = \mathrm{Syn}(S, M, r)$ and the map $\alpha_r^{\mathcal{L}\mathrm{az}}$ is obtained by composing the maps in lower boundary where we note that $C_G(T(r)) \xrightarrow{\sim} \mathrm{R}\Gamma(G_S, T(r))$. The isomorphisms in the diagram indicate a p-power quasi-isomorphism between complexes. Notations are explained after the diagram.

$$
\begin{array}{ccccccc}
\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M_\varpi^{\mathrm{PD}}) & \longrightarrow & C_G(\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r \Delta^{\mathrm{PD}})) & \xleftarrow[\sim]{\mathrm{PL}} & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r \Delta^{\mathrm{PD},\partial})) & \longrightarrow & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r TA_{\mathrm{cris}})) \\
\wr\downarrow \tau_{\leq r} & & \downarrow & & \downarrow & & \wr\uparrow \mathrm{FES} \\
\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r M_\varpi^{[u,v]}) & \longrightarrow & C_G(\mathrm{K}_{\partial,\varphi}(\mathrm{F}^r \Delta^{[u,v]})) & \xleftarrow[\sim]{\mathrm{PL}} & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r \Delta^{[u,v],\partial})) & & C_G(T(r)) \\
\wr\downarrow \mathrm{PL} & \nearrow & & & \downarrow & \swarrow^{\sim}_{\mathrm{FES}} & \wr\uparrow \mathrm{AS} \\
\mathrm{K}_{\partial,\varphi,\partial_A}(\mathrm{F}^r \Delta_\varpi^{[u,v]}) & & & & C_G(\mathrm{K}_{\varphi}(\mathrm{F}^r TA^{[u,v]})) & & C_G(\mathrm{K}_{\varphi}(TA_{\overline{R}}(r))) \\
\wr\uparrow \mathrm{PL} & & & & \uparrow & & \wr\uparrow \\
\mathrm{K}_{\varphi,\partial_A}(\mathrm{F}^r N_\varpi^{[u,v]}) & & & & & & C_\Gamma(\mathrm{K}_{\varphi}(D_{R_\infty}(r))) \\
\tau_{\leq r}\wr\downarrow t^\bullet & & & \nearrow & & & \wr\uparrow \\
\mathcal{K}_{\varphi,\mathrm{Lie}\,\Gamma}(\mathrm{F}^r N_\varpi^{[u,v]}) & \xleftarrow[\mathcal{L}\mathrm{az}]{\sim} & \mathcal{K}_{\varphi,\Gamma}(\mathrm{F}^r N_\varpi^{[u,v]}) & & & & C_\Gamma(\mathrm{K}_{\varphi}(D_\varpi(r))) \\
\wr\uparrow t^r & & t^r\uparrow & & & & \wr\uparrow \\
\mathcal{K}_{\varphi,\mathrm{Lie}\,\Gamma}(N_\varpi^{[u,v]}(r)) & \xleftarrow[\mathcal{L}\mathrm{az}]{\sim} & \mathcal{K}_{\varphi,\Gamma}(N_\varpi^{[u,v]}(r)) & \xleftarrow[\mathrm{can}]{\sim} & \mathcal{K}_{\varphi,\Gamma}(N_\varpi^{(0,v]+}(r)) & \xrightarrow{\sim} & \mathrm{K}_{\varphi,\Gamma}(D_\varpi(r)).
\end{array}
$$

In [Abh22, §2.5-§2.6] we define successively larger rings $R_\varpi^{\mathrm{PD}} \subset R_\varpi^{[u]} \subset R_\varpi^{[u,v]}$ equipped with compatible filtration and Frobenius operator where one can take $u = (p-1)/p$ and $v = p-1$. The ring $R_\varpi^{[u]}$ can be thought of as analytic functions convergent on the disk $v_p(X_0) \geq u/e$ and similarly $R_\varpi^{[u,v]}$ can be thought of as analytic functions convergent on the annulus $u/e \leq v_p(X_0) \leq v/e$ (see [CN17, Remark 2.4]). Furthermore, in [Abh22, Definition 5.22, Lemma 5.23] we define $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset E_{R,\varpi}^{\mathrm{PD}} \subset E_{R,\varpi}^{[u]} \subset E_{R,\varpi}^{[u,v]}$ equipped with compatible Frobenius, filtration Γ_S-action and connection satisfying Griffiths transversality. Moreover, in [Abh22, Definition

5.22, Lemma 5.23] we define $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(\overline{R}) \subset E_{\overline{S}}^{\mathrm{PD}} \subset E_{\overline{S}}^{[u,v]}$ equipped with compatible Frobenius, filtration G_R-action and connection satisfying Griffiths transversality.

In the diagram, we take $\Delta^{\mathrm{PD}} = E_{\overline{S}}^{\mathrm{PD}} \otimes_R M$, $\Delta^{\mathrm{PD},\partial} = (\Delta^{\mathrm{PD}})^{\partial=0}$, $TA_{\mathrm{cris}} = \mathbf{A}_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Z}_p} T$, $\Delta^{[u,v]} = E_{\overline{S}}^{[u,v]} \otimes_R M$, $\Delta^{[u,v],\partial} = (\Delta^{[u,v]})^{\partial=0}$, $TA^{[u,v]} = \mathbf{A}_{\overline{R}}^{[u,v]} \otimes_{\mathbb{Z}_p} T$, $\Delta_{\varpi}^{[u,v]} = E_{R,\varpi}^{[u,v]} \otimes_R M$, $TA_{\overline{R}}(r) = W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T(r)$, $D_{\varpi}(r) = \mathbf{A}_{R,\varpi}^{+} \otimes_{\mathbf{A}_R^+} \mathbf{D}(T(r))$, $N_{\varpi}^{\star}(r) = \mathbf{A}_{R,\varpi}^{\star} \otimes_{\mathbf{A}_R^+} \mathbf{N}(T(r))$ and $D_{R_\infty}(r) = \mathbf{A}_{S_\infty} \otimes_{\mathbf{A}_{R,\varpi}} D_{\varpi}(r)$ (see [Abh22, §2] for period rings). Moreover, $G = G_S$, $\Gamma = \Gamma_S$ with C_G and C_Γ denoting the complex of continuous cochains of G and Γ, respectively. The letter "K" denotes the Koszul complex (see [Abh22, §4]) with subscripts: ∂ denotes the operators $((1+X_0)\frac{\partial}{\partial X_0}, \ldots, X_d\frac{\partial}{\partial X_d})$ for a choice of coordinates $(X_0, X_1, \ldots, X_d)$ on R_{ϖ}^{+}, Γ denotes the operators $(\gamma_0 - 1, \ldots, \gamma_d - 1)$ for our choice of topological generators of Γ, Lie Γ denotes the operators $(\nabla_0, \ldots, \nabla_d)$ with $\nabla_i = \log \gamma_i$ and ∂_A denotes $((1+X_0)\frac{\partial}{\partial X_0}, X_1\frac{\partial}{\partial X_1}, \ldots, X_d\frac{\partial}{\partial X_d})$ as operators on $\mathbf{A}_{\overline{R}}^{[u,v]}$ and $E_{\overline{R}}^{[u,v]}$ via the isomorphism $\iota_{\mathrm{cycl}} : \mathbf{A}_{\overline{R}}^{[u,v]} \xrightarrow{\sim} R_{\varpi}^{[u,v]}$ (see [Abh22, §2.7]). The letter "$\mathcal{K}$" denotes a subcomplex of the Koszul complex (see [Abh22, §6.2-§6.5]).

Next, we describe maps between rows. FES denotes a map originating from fundamental exact sequences in [Abh22, §2.2.1 & §2.4.4]. AS denotes a map coming from the Artin-Schreier theory in [Abh22, §2.4.4]. PL denotes maps originating from filtered Poincaré Lemma of [Abh22, §2.8]. Going from first row to the second row is induced by the inclusion $R_{\varpi}^{\mathrm{PD}} \subset R_{\varpi}^{[u,v]}$. The leftmost slanted vertical map from third to second row is induced by the inclusion $E_{R,\varpi}^{[u,v]} \subset E_{\overline{S}}^{[u,v]}$. The vertical map from second to third row is induced by taking horizontal sections [Abh22, §6.7]. The rightmost vertical map from fourth to third row is the inflation map from Γ_R to G_R, using the inclusion $R_\infty \subset \overline{R}$ (one could use almost étale descent to obtain the quasi-isomorphism) and the rightmost vertical map from the fifth to fourth row uses the inclusion $R \subset R_\infty$ (the quasi-isomorphism is obtained by decompletion techniques). The leftmost vertical arrow from fourth to fifth row is given by multplication by suitable powers of t as in [Abh22, Lemma 6.2] and the rightmost vertical arrow from sixth to fifth row is comparison between complex computing continuous cohomology of Γ_R and Koszul complex as in [Abh22, §4.2]. The inclusions $\mathbf{A}_{R,\varpi}^{+} \subset \mathbf{A}_{\mathrm{inf}}(\overline{R}) \subset \mathbf{A}_{\overline{R}}^{[u,v]}$ and $\mathbf{A}_{\mathrm{inf}}(\overline{R}) \otimes_{\mathbf{A}_R^+} \mathbf{N}(T) \subset \mathbf{A}_{\mathrm{inf}}(\overline{R}) \otimes_{\mathbb{Z}_p} T$ induce the slanted vertical arrow from fifth to third row.

Finally, we describe maps between columns. Top two maps from first to second column are induced by inclusions $R_{\varpi}^{\mathrm{PD}} \subset E_{\overline{S}}^{\mathrm{PD}}$ and $R_{\varpi}^{[u,v]} \subset E_{\overline{S}}^{[u,v]}$. The bottom two maps $\mathcal{L}$az between first and second column are Lazard isomorphisms discussed in [Abh22, §6.2]. Bottom map from third to second column is induced canonically by $\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \mathbf{A}_{R,\varpi}^{[u,v]}$ (see [Abh22, §2.7] for definitions). The horizontal map from third to fourth column is induced by taking horizontal sections [Abh22, §6.7]. The bottom horizontal map from fifth to fourth column is obtained by the inclusion $\mathbf{A}_{R,\varpi}^{(0,v]+} \subset \mathbf{A}_{R,\varpi}$ (see [Abh22, §6.5-§6.6]).

5. Global Applications: Fontaine-Laffaille modules

We finally come to global applications of results described in the previous section. In this section we will consider locally free Fontaine-Laffaille modules introduced by Faltings in [Fal89, §II]. These objects are obtained by gluing together local data. Let $s \in \mathbb{N}$ such that $s \leq p-2$.

Definition 5.1. Define the category of *free relative Fontaine-Laffaille* modules of level $[0, s]$, denoted by $\mathrm{MF}_{[0,s],\mathrm{free}}(R, \Phi, \partial)$, as follows:
An object with weights/level in the interval $[0, s]$ is a quadruple $(M, \mathrm{Fil}^\bullet M, \partial, \Phi)$ such that,

(1) M is a free R-module of finite rank. It is equipped with a decreasing filtration $\{\mathrm{Fil}^k M\}_{k\in\mathbb{Z}}$ by finite R-submodules with $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{s+1} M = 0$ such that $\mathrm{gr}_{\mathrm{Fil}}^k M$ is a finite

free R-module for $k \in \mathbb{Z}$.

(2) The connection $\partial : M \to M \otimes_R \Omega^1_R$ is quasi-nilpotent and integrable, and satisfies Griffiths transversality with respect to the filtration, i.e. $\partial(\mathrm{Fil}^k M) \subset \mathrm{Fil}^{k-1} M \otimes_R \Omega^1_R$ for $k \in \mathbb{Z}$.

(3) Let $(\varphi^*(M), \varphi^*(\partial))$ denote the pullback of (M, ∂) by $\varphi : R \to R$, and equip it with a decreasing filtration $\mathrm{Fil}^k_p(\varphi^*(M)) = \sum_{i \in \mathbb{N}} p^{[i]} \varphi^*(\mathrm{Fil}^{k-i} M)$ for $k \in \mathbb{Z}$. We suppose that there is an R-linear morphism $\Phi : \varphi^*(M) \to M$ such that Φ is compatible with connections, $\Phi(\mathrm{Fil}^k_p(\varphi^*(M))) \subset p^k M$ for $0 \leq k \leq s$, and $\sum_{k=0}^s p^{-k} \Phi(\mathrm{Fil}^k_p(\varphi^*(M))) = M$. We denote the composition $M \to \varphi^*(M) \xrightarrow{\Phi} M$ by φ.

A morphism between two objects of the category $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ is a continuous R-linear map compatible with the homomorphism Φ and the connection ∂ on each side.

The category $\mathrm{MF}_{[0,s],\,\mathrm{free}}(R, \Phi, \partial)$ is a full subcategory of the abelian category $\mathfrak{MF}^{\nabla}_{[0,s]}(R)$ of [Fal89, §II]. One can functorially attach to such a module, a free $\mathbb{Z}_p$-module $T_{\mathrm{cris}}(M)$ equipped with a continuous G_R-action such that $V_{\mathrm{cris}}(M) = T_{\mathrm{cris}}(M)[1/p]$ is crystalline and s equals the maximum among the absolute value of Hodge-Tate weights of $V_{\mathrm{cris}}(M)$. Moreover, in [Abh21, Theorem 5.4] it has been shown that $V_{\mathrm{cris}}(M)$ is a finite q-height representation of height s. Furthermore, $V_{\mathrm{cris}}(M)$ satisfies assumptions of Theorem 3.3 and Theorem 4.2 (with very precise bounds on the constant $N(p, r, s)$, see [Abh22, Example 5.2 (iii)]).

The category of free relative Fontaine-Laffaille modules globalizes well. Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme defined over O_F with X as its (rigid) generic fiber and $\mathfrak{X}_\kappa$ as its special fiber. Cover $\mathfrak{X}$ by affine (formal) schemes $\{\mathfrak{U}_i\}_{i \in I}$ where $\mathfrak{U}_i = \mathrm{Spec}\, A_i$ (resp. $\mathfrak{U}_i = \mathrm{Spf}\, A_i$) such that p-adic completions $\widehat{A}_i$ satisfy assumptions for R above and fix Frobenius lifts $\varphi_i : \widehat{A}_i \to \widehat{A}_i$.

Definition 5.2. Define $\mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ as the category of finite locally free filtered $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{M}$ equipped with a p-adically quasi-nilpotent integrable connection satisfying Griffiths transverality with respect to the filtration and such that there exists a covering $\{\mathfrak{U}_i\}_{i \in I}$ of $\mathfrak{X}$ as above with $\mathcal{M}_{\mathfrak{U}_i} \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(\widehat{A}_i, \Phi, \partial)$ for all $i \in I$ and on $\mathfrak{U}_{ij}$ the two structures glue well for different Frobenii (see [Abh22, §8.1]).

By [Fal89, Theorem 2.6*], the functor T_{cris} associates to any object of $\mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ a compatible system of étale sheaves on $\mathrm{Spec}\,(\widehat{A}_i[\frac{1}{p}])$. Again, these sheaves glue well to give us an étale sheaf on the (rigid) generic fiber X of $\mathfrak{X}$. The étale $\mathbb{Z}_p$-local system on the generic fiber associated to $\mathcal{M}$ will be denoted as $\mathbb{L}$. Our global result is as follows:

Theorem 5.3 ([Abh22, Theorem 1.11])**.** *Let $\mathfrak{X}$ be a smooth (p-adic formal) scheme over O_F, $\mathcal{M} \in \mathrm{MF}_{[0,s],\,\mathrm{free}}(\mathfrak{X}, \Phi, \partial)$ a Fontaine-Laffaille module of level $[0, s]$ for $0 \leq s \leq p-2$ and let $\mathbb{L}$ be the associated $\mathbb{Z}_p$-local system on the (rigid) generic fiber X of $\mathfrak{X}$. Then for $0 \leq k \leq r-s-1$ the Fontaine-Messing period map*

$$\alpha^{\mathrm{FM}}_{r,n,\mathfrak{X}} : \mathcal{H}^k(\mathscr{S}_n(\mathcal{M}, r)_{\mathfrak{X}}) \longrightarrow i^* \mathrm{R}^k j_* \mathbb{L}/p^n(r)'_X,$$

is a p^N-isomorphism for an integer $N = N(p, r, s)$, which depends only on p, r and s.

The theorem is proved by reducing it to the local setting, where we can directly apply Theorem 4.2.

Remark 5.4. In light of Remark 4.5 (2), it should be possible to base change the isomorphism of Theorem 5.3 to $\overline{F}$.

Remark 5.5. In [BMS19, §10] Bhatt, Morrow and Scholze have refined the definition of syntomic complex (using prismatic cohomology) and showed that it computes p-adic nearby cycles for trivial coefficients. By the work of Morrow and Tsuji on coefficients in integral p-adic Hodge theory and prismatic cohomology [MT20], we should be able to refine our results and obtain an integral result for coefficients (in the geometric case). Furthermore, by recent introduction of completed/analytic prismatic F-crystals on the absolute prismatic site [DLMS22; GR22], we should be able to further refine these results, thus including the arithmetic case. We will report on these ideas in future.

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