Monogenity of certain abelian and non-abelian extension fields

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Our claims of recent research jointed with PhD/Postdoc scholars in Pakistan are as follows.

Claim A *The non-cyclic abelian fields* $K = \boldsymbol{Q}(\zeta_{|p^*|}, \sqrt{\ell^*})$ *are non-monogenic except for the two classes of the fields with the conductors* $3p^* = |-3| \cdot p$ *for* $\ell^* = -3$ *or* $4p^* = |4 \cdot (-1)| \cdot p$ *for* $\ell^* = -4$ *under the conditions* $(p^*, \ell^*) = 1$, *with a prime number* p *and a squarefree odd number* $|\ell| > 3$ *or even* $|\ell^*| \geqq 2^3$ *of the conductor* $p^*\ell^*$ *for the conductor* $p^* = \pm p \equiv 1 \pmod 4$ *of a prime cyclotomic field* $k_{|p^*|}$ *and the conductor* ℓ^* *of a quadratic subfield* $k = \boldsymbol{Q}(\sqrt{l^*}) \subset k_{|\ell^*|}$ *with the odd field discriminat* $d_k = \ell^* = \pm\ell$ *or the even* $d_k = \ell^* = \pm 4\ell$.

The above claim applying an idea of [5] is a generalization of N. S. Khan [6] and M. Sultan [8].

Claim B *Let* K *be a Dihedral quartic field* $\boldsymbol{Q}(\sqrt{a + b\omega})$, *where* $a^2+ab+b^2\frac{1-m}{4}$ *is a squrefree integer and the quadratic subfield* $k = \boldsymbol{Q}(\omega)$ *of* K *has the odd conductor* m *with* $\omega = \frac{1+\sqrt{m}}{2}$. *Then all the integral bases and monogenities of* K *are given in* Table 1 *separated into the twelve families* ${}_m\mathrm{C}^a_{b\_b'}$ *with* $m \equiv 1, 5 \pmod 8$, $a \equiv 1, 3 \pmod 4), b\_b' \equiv 1\_3, 2\_4 \pmod 4)$ *and* $a \equiv 2, 4 \pmod 4), b\_b' \equiv 1\_3 \pmod 4)$. *Here the twenty four being equal to* 32 -8 *empty families can be summarized into twelve types and e.g. the family* ${}_1C^1_{1\_3}$ *is denoted by* $[1, 1, 1\_3] = [m \equiv 1 \pmod 8), a \equiv 1 \pmod 4), b \equiv 1\_3 \pmod 4)]$.

§1 Introduction. On Hasse's problem to determine the monogenity of an algebraic number field, we consider a non-cyclic, but abelian octic field K over the rationals $\boldsymbol{Q}$. This problem is proposed by W. Narkiewicz in general [7]. Let F be an algebraic number field over the rationals $\boldsymbol{Q}$ of finite extension degree $[F : \boldsymbol{Q}] = n$. Z_F and Z denote the ring of integers in F and the ring of rational integers, respectively. If there exist an integer $\xi \in F$ such that $Z_F = \boldsymbol{Z}[\xi] = \boldsymbol{Z}[1, \xi, \cdots, \xi^{n-1}]$ of rank n over the ring $\boldsymbol{Z}$ of rational integers, it is said that a field F is monogenic or the ring Z_F has a power integral basis.

In §2, on Claim A we shall introduce the most difficult, but simplest case of the determination of monogenity on the field $K = k_5 \cdot k$ with conductor $5 \cdot |-7|$ among the octic fields $k_5 \cdot \boldsymbol{Q}(\sqrt{\ell})$ with a squarefree ℓ, where k_5 and k denote the 5th cyclotomic field $\boldsymbol{Q}(\exp(2\pi i/5))$ and an imaginary quadratic field $\boldsymbol{Q}(\sqrt{-7})$, respectively. Then this method and the experiments by GP/PARI shall involve a deep feeling to generalize into Claim A.

In §3, we shall describe the easiest case on the field $K = k_5 \cdot k$ with conductor $5 \cdot (2^2 \cdot 7)$, where k denotes a real quadratic field $\boldsymbol{Q}(\sqrt{7})$.

In §4, on Claim B we shall give a table to classify the families within monogenity of all the Dihedral quartic extension fields K accoding to the quadratic subfields $k = \boldsymbol{Q}(\sqrt{m})$ of K with odd field discriminants m modulo 8 involving a monogenic example of A. C. Kable [3, 9]. Table 1 includes a comparison with a work of K. S. Williams et al [2]. At present the classificatin is incomplete aginst the family of K with quadratic subfields k of even field discriminants [1].

§2 Monogenity of an octic field K with an imaginary quadratic subfield k. The next Lemma is fundamental for the determination of monogenity of non-cyclic but, abeian octic fields K. Namely Z_{k_5} has a relative integral basis $Z_{k_5^+}[1, \zeta]$ over the subring $Z_{k_5^+}$.

Lemma 2.1. *Let η be the Gauß period $\zeta + \zeta^{-1}$ of length 2. Then It holds that*

$$Z_{k_5} = Z_{k_5^+}[1, \zeta] = \boldsymbol{Z}[1, \eta][1, \zeta]$$

as a $\boldsymbol{Z}$-module.

proof. Since $Z_{k_5^+}[1, \zeta] \subsetneqq Z_{k_5}$, we show the converse inclusion. By $\zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0$ $\eta^2 + \eta - 1 = 0$ for $\eta = \zeta + \zeta^{-1} = \frac{-1+\sqrt{5}}{2}$, $Z_{k_5^+} = \boldsymbol{Z}[1, \eta]$ holds. Then we have $1 \cdot \zeta, \eta\zeta = \zeta^2 + 1$, $0 \equiv \eta \cdot 1 = \zeta + \zeta^4 = -1 - \zeta^2 - \zeta^3 = -\eta\zeta - \zeta^3 \equiv \zeta^3 \pmod{\boldsymbol{Z}[1, \eta][1, \zeta]}$,
and hence $\boldsymbol{Z}[1, \zeta, \zeta^2, \zeta^3] \subsetneqq Z_{k_5^+}[1, \zeta]$. □

On the claim A, for the simplicity we choose an octic abelian but non-cyclic field K of conductor $5 \cdot |-7|$, i.e. $p = 5$ and $\ell = -7$. We assume that K is monogenic, namely there exists an integer ξ such that $Z_K = \boldsymbol{Z}[\xi]$. Denote $\xi - \xi^\rho$ by ξ_ρ for $\xi \in K$ and $\rho \in Gal(K/\boldsymbol{Q}) =<\sigma><\tau>$ with $<\sigma>= Gal(k_5/\boldsymbol{Q})$ and $<\tau>= Gal(k/\boldsymbol{Q})$ where $\sigma : \zeta \mapsto \zeta^r$, a primitive root r modulo 5, $\sqrt{-7} \mapsto \sqrt{-7}$, and $\tau : \sqrt{-7} \mapsto -\sqrt{-7}, \zeta \mapsto \zeta$.
Then using Hasse's conductor-discriminant theorem, we have the norms
$N_{K/\boldsymbol{Q}}(\xi_{\sigma^j} = 5 \ (1 \leqq j \leqq 3)$, $N_{k/\boldsymbol{Q}}(\xi_\tau) = -7$ and

the product $N_{K/k_5}(N_{k_5/\boldsymbol{Q}}(\prod_{1\leqq j\leqq 3}\xi_{\sigma^j}))\cdot\ N_{K/k}(N_{k/\boldsymbol{Q}}(\xi_\tau)$ is equal to $(5^3)^2\cdot(-7)^4 = d_K$. Here d_F denotes the field discriminant for a field F. Thus the partial different $\xi_{\sigma^2\tau}$ should be a unit in K. We note that an octic Field K is comosed by linearly disjoint subfields k_5 and k. Let d_F denote the field discriminant of an algebraic number field F.

Lemma 2.2. *For three rings Z_K, Z_{k_5} and Z_k, it holds that*

$$Z_K = Z_{k_5}\cdot Z_k, \text{ and } d_K = d_{k_5}^2\cdot d_k^4.$$

When a field K contains an imaginary quadratic field k, we consider the most difficult, but the simplest field $K=\boldsymbol{Q}(\zeta,\omega)$ for the determination of monogenity using the norm $N_{K/\boldsymbol{Q}}(\xi_{\sigma^2\tau})$ of the partial different $\xi_{\sigma^2\tau}=\xi-\xi^{\sigma^2\tau}$ of a power basis candidate number $\xi\in Z_K$ with a mixed embedding $\sigma^2\tau$. We select a most suitable field tower;

$$K\supset N=\boldsymbol{Q}(\eta_-\cdot\sqrt{-7})\supset k_5^+\supset\boldsymbol{Q}$$

among three quartic subfields

$$k_5,\quad L=\boldsymbol{Q}(\eta,\omega),\quad N=\boldsymbol{Q}(\eta_-\cdot\sqrt{-7}),$$

and three quadratic subfields

$$k_5^+,\quad k_{L_2}=\boldsymbol{Q}(\sqrt{5}\sqrt{-7}),\quad k_{L_3}=\boldsymbol{Q}(\sqrt{-7})$$

where $\eta_-=\zeta-\zeta^{-1}=2i\sin(2\pi/5)$ and $\eta_-\cdot\sqrt{-7}\in\mathbb{R}$. Here $\mathbb{R}$ denotes the field of real numbers. We have $N_{K/N}(\mathfrak{D}_K(\xi_{\sigma^2\tau}))=\mathfrak{D}_K(\xi_{\sigma^2\tau})\cdot\mathfrak{D}_K(\xi_{\sigma^2\tau})^{\sigma^2\tau}$ Then using Lemma 1 and Lemma 2, the partial different $\xi_{\sigma^{\frac{5-1}{2}}\tau}=\xi-\xi^{\sigma^2\tau}$ being equal to
$\alpha_0+\alpha_1\zeta+(\beta_0+\beta_1\zeta)\omega-\{\alpha_0+\alpha_1\zeta^{-1}+(\beta_0+\beta_1\zeta^{-1})\omega^\tau\}$
$=\alpha_1(\zeta-\zeta^{-1})+\beta_0\sqrt{-7}+\beta_1(\zeta\omega-\zeta^{-1}\omega^\tau)$ should be a unit in K. From
$F_{<\sigma^2\tau>}=N=\boldsymbol{Q}(\eta_-\cdot\sqrt{-7})$ with $\eta_-=\zeta-\zeta^{-1}$ we have $N_{K/N}(\xi_{\sigma^2\tau})=\xi_{\sigma^2\tau}\cdot\xi_{\sigma^2\tau}^{\sigma^2\tau}=(\xi-\xi^{\sigma^2\tau})(\xi^{\sigma^2\tau}-\xi)=-{\xi_{\sigma^2\tau}}^2$. Here for a subgroup $H\subset Gal(K/\boldsymbol{Q})$, F_H denotes the fixed subfield $\{\eta\in K;\eta=\eta^\rho\forall\rho\in H\}$ of K. Then it holds that
$-N_{K/N}(\xi_{\sigma^2\tau}))={\xi_{\sigma^2\tau}}^2$
$=((\alpha_1+\frac{1}{2}\beta_1)(\zeta-\zeta^{-1})+(\beta_0+\frac{1}{2}\beta_1)(\zeta+\zeta^{-1}))\sqrt{-7})^2$
$=((\alpha_1+\frac{1}{2}\beta_1)(\zeta-\zeta^{-1}))^2+((\beta_0+\frac{1}{2}\beta_1)(\zeta+\zeta^{-1}))^2(-7)$
$+2((\alpha_1+\frac{1}{2}\beta_1)(\zeta-\zeta^{-1})\cdot(\beta_0+\frac{1}{2}\beta_1)(\zeta+\zeta^{-1}))\sqrt{-7})$
$=((\alpha_1+\frac{1}{2}\beta_1)(2i\sin(\frac{2\pi}{5})))^2+((\beta_0+\frac{1}{2}\beta_1)(\zeta+\zeta^{-1}))^2\cdot(-7)$
$+2(\alpha_1+\frac{1}{2}\beta_1)(2i\sin(\frac{2\pi}{5}))\cdot(\beta_0+\frac{1}{2}\beta_1)(\zeta+\zeta^{-1})\cdot\sqrt{-7})$. Put

$$C=-((\alpha_1+\tfrac{1}{2}\beta_1)(\zeta-\zeta^{-1}))^2-((\beta_0+\tfrac{1}{2}\beta_1)(\zeta+\zeta^{-1}))^2\cdot(-7),$$
$$D=2(\alpha_1+\tfrac{1}{2}\beta_1)(\zeta-\zeta^{-1})\cdot(\beta_0+\tfrac{1}{2}\beta_1)(\zeta+\zeta^{-1})\cdot\sqrt{-7}),$$

where by $((\alpha_1+\frac{1}{2}\beta_1)(\zeta-\zeta^{-1}))^2\leqq 0$ and $((\beta_0+\frac{1}{2}\beta_1)(\zeta+\zeta^{-1}))^2(-7)\leqq 0$ in k_5^+, it follows that $C\geqq 0$ and $C^\sigma\geqq 0$ and $D,D^\sigma\in\mathbb{R}$ with $(\zeta-\zeta^{-1})^\sigma\in i\mathbb{R}$ and $(\zeta+\zeta^{-1})^\sigma\in\mathbb{R}$.

Then we evaluate the next three cases.
(i) $CD \neq 0$,
(ii) $C = 0$, $D \neq 0$,
(iii) $C \neq 0$, $D = 0$.
On (i) from
$|D| = \sqrt{2^2(\alpha_1 + \frac{1}{2}\beta_1)^2(\zeta - \zeta^{-1})^2 \cdot (\beta_0 + \frac{1}{2}\beta_1)^2(\zeta + \zeta^{-1})^2 \cdot |-7|}$,
and $N_{K/k_5^+}(\xi_{\sigma^2\tau}) = (C+D)^2$ we have $N_{K/\boldsymbol{Q}}(\xi_{\sigma^2\tau}) = N_{k_5^+/\boldsymbol{Q}}((C+D)^2)$
$= (C+D)^2(C^\sigma + D^\sigma)^2 = |(C+D)^2| \cdot |(C^\sigma + D^\sigma)^2|$
$\geqq |2CD| \cdot |2C^\sigma D^\sigma| = (2^{1+1})|CC^s| \cdot |DD^s| \geqq 2^2 \cdot |\frac{1}{2^2}c_C| \cdot |\frac{1}{2^2}c_D(-7)| \geqq \frac{7}{4} > 1$
with $1 \leqq c_C, c_D \in \boldsymbol{Z}$.
On (ii), we have
$D = \sqrt{|2^2(\alpha_1 + \frac{1}{2}\beta_1)^2(\zeta - \zeta^{-1})^2 \cdot (\beta_0 + \frac{1}{2}\beta_1)^2(\zeta + \zeta^{-1})^2 \cdot (-7))|}$
Then it follows that

$$DD^\sigma = \sqrt{|2^{2+2}N_{k_5^+}((\alpha_1 + \tfrac{1}{2}\beta_1)^2(\zeta - \zeta^{-1})^2)N_{k_5^+}(\beta_0 + \tfrac{1}{2}\beta_1)^2(\zeta + \zeta^{-1})^2 \cdot 7)|}$$
$$\geqq \sqrt{2^{2+2}\tfrac{1}{2^4} \cdot \tfrac{1}{2^4} \cdot 7^2} \cdot c_2 \geqq \tfrac{7}{2^2} \cdot c_2 \geqq \tfrac{7}{4} \text{ with } 1 \leqq c_2 \in \boldsymbol{Z}.$$

On (iii), put $C_1 = \sqrt{-((\alpha_1 + \frac{1}{2}\beta_1)(\zeta - \zeta^{-1}))^2}$ and $C_2 = \sqrt{-(\beta_0 + \frac{1}{2}\beta_1(\zeta + \zeta^{-1}))^2 \cdot (-7)}$.
Then it holds that $|N_{k_5^+/\boldsymbol{Q}}(C_1^2 + C_2^2))| = |(C_1^2 + C_2^2)((C_1^2)^\sigma + (C_2^2)^\sigma)|$
$\geqq |2^2C_1C_2C_1^\sigma C_2^\sigma| = |2^2C_1C_1^\sigma||C_2C_2^\sigma| \geqq 2^2\frac{1}{2^2}\frac{1}{2^2}|(-7)| \cdot c_3 \geqq \frac{7}{2^2} \cdot c_3 \geqq \frac{7}{4}$ with $1 \leqq c_3 \in \boldsymbol{Z}$.
Thus we have showed that an octic field K is non-monogenic. □

After a simple succeeded non-monogenic octic polynomial $f(x)$ for

$$(A_0)\ \xi = \zeta \cdot \omega \in K$$

with $\zeta = \exp(2\pi i/5)$ and $\omega = \frac{1+\sqrt{-7}}{2}$, we found a hard example $g(x)$ for

$$(A_1)\ \xi = \zeta + \omega \in K.$$

(A_1) The last irreducible polynomial $g(x)$ for an octic field $K = \boldsymbol{Q}(\xi)$, $\xi = \zeta_5 + \sqrt{-7}$;

```
g(x)=(x^4+17*x^2+43)^2
+(x^4+17*x^2+43)*(2*x^3+x^2+16*x+7)-(2*x^3+x^2+16*x+7)^2)
\x=\z_{5}+\sqrt{-7}
gp > nfdisc(g(x))
= 37515625=[5 6][7 4]=d_{K}
gp > poldisc(g(x))
= 1571179133492004569764000000
=[      2 8][      5 6][      7 4][     11 2][     59 2][623221 2]
=Ind_{K}(\x)^{2}\cdot d_{K}, wherer Ind_{K}(\x)=(Z_{K}:\Z[\x])
gp > nfbasis(g(x))
```

```
= [1, x, x^2, x^3, 1/2*x^4 - 1/2*x^3 - 1/2,
1/2*x^5 - 1/2*x^3 - 1/2*x - 1/2,
1/2*x^6 - 1/2*x^3 - 1/2*x^2 - 1/2*x - 1/2,
1/808940858*x^7 - 13969231/404470429*x^6 + 164687045/808940858*x^5
+ 3359403/73540078*x^4 - 31176033/404470429*x^3
+ 384609529/808940858*x^2 - 117297925/404470429*x - 21177539/73540078]
gp > factor(808940858)=[     2 1][    11 1][    59 1][623221 1]
gp > factor(404470429)=[    11 1][    59 1][623221 1]
gp > factor(73540078)=[     2 1][    59 1][623221 1]
```

(A_0) The first irreducible polynomial $f(x)$ for an octic field $K = \boldsymbol{Q}(\xi)$, $\xi = \zeta_5 \cdot \sqrt{-7}$;

```
f(x)=(x^4-2*x^3+8*x^2-7*x+4)^2
     -(x^4-2*x^3+8*x^2-7*x+4)*(-2*x^3+2*x^2-6*x+1)
     -(-2*x^3+2*x^2-6*x+1)^2
     =N_{k^{+}/\Q}(N_{k_{5}/k^{+}}(N_{K/k_{5}}(x-\x)))
gp > nfdisc(f(x))= 37515625=[5 6][|-7| 4]=d_{K}
=\prod_{\r\ne\identity \in Char(Gal{K})}d_{\r}, Char(Gal{K})=<\x><\psi>
=d_{\x}d_{\x^2}d_{\x^3}d_{\psi}d_{\x\psi}d_{\x^2\psi}d_{\x^3\psi}
=5\cdot5\cdot5\cdot(-7)\cdot5(-7)\cdot5(-7)\cdot5(-7),#<\x>=4,#<\psi>=2.
gp > poldisc(f(x))
= 129394971153765625=[  5 6][  7 4][ 11 2][ 19 2][281 2]
=Ind_{K}(\x)^{2}\cdot d_{K}
>gp > nfbasis((f(x))
%6 = [1, x, x^2, x^3, x^4, x^5, x^6,
      1/58729*x^7 - 2973/58729*x^6 + 23443/58729*x^5 + 3429/58729*x^4
      - 27413/58729*x^3 - 13128/58729*x^2 + 7247/58729*x + 2063/5339]


factor(58729)=[ 11 1][ 19 1][281 1].
```

The other example of 5th cyclotomic field k_5. Choose $x = \zeta_5 + \sqrt{5} \in k_5$,

```
gp > nfdisc((x^2-2*x+8)^2+(x^2-2*x+8)*(5*x+1)-(5*x+1)^2)
= 125 =[5 3]=d_{k_{5}}
gp > factor(poldisc((x^2-2*x+8)^2+(x^2-2*x+8)*(5*x+1)-(5*x+1)^2))
=5565125=[  5 3][211 2]
gp > nfbasis((x^2-2*x+8)^2+(x^2-2*x+8)*(5*x+1)-(5*x+1)^2)
= [1, x, x^2, 1/211*x^3 + 43/211*x^2 + 104/211*x - 67/211].
```

The above experiment shows an integral basis of the 5th cyclotomic field k_5 under a *choice* $\zeta_5 + \sqrt{5}$ of x. However we could *not* observe that k_5 is monogenic or not. On the other hand, it is well known that for $x = \zeta_5 \in k_5$;

```
gp > factor(poldisc((x^{5}-1)/(x-1)))=[5 3] and
gp > nfbasis((x^{5}-1)/(x-1))= [1, x, x^2, x^3], i.e. $k_{5}$ is monogenic.
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§3 Monogenity of an octic field K with a real quadratic subfield k. In this section for the simplicity, we may asssume that a real subfield k of K has an even field discrimiant. The case of a real subfield k with an odd field discriminant, we may pursue the evaluation for the monogenity of an octic field K as in §2. we consider an octic abelian field $K = \boldsymbol{Q}(\zeta_5, \sqrt{7})$ with conductor $5 \cdot 2^2 7$ and the field discriminant $d_K = (5^3)^2 \cdot (2^2 7)^4$, which will generalize a work of Noor Saeed Khan [03].

Assime that a number $\xi = \alpha + \beta\omega \in Z_K$ with $\alpha, \beta \in Z_{k_5}$, $\omega = \sqrt{7}$ would generate a power integral basis of the field K. For the Galois group $Gal(K/\boldsymbol{Q}) =< \sigma >< \tau >$, let $\sigma : \zeta \mapsto \zeta^r$, with a primitive root r modulo 5, and $\tau : \sqrt{7} \mapsto -\sqrt{7}, \zeta \mapsto \zeta$. Then we evaluate the norm of a partial different $\xi_{\sigma^2\tau} = \xi - \xi^{\sigma^2\tau}$ with respect to $K/\boldsymbol{Q}$. For any integer $\xi = \alpha_0 + \alpha_1\zeta + (\beta_0 + \beta_1\zeta)\omega \in K$ with $\alpha_j, \beta_j \in Z_k$ and $\omega = \sqrt{7}$, we prove that the partial different $\xi_{\sigma^2\tau}$ would become an 'obstacle factor', namely it could not be a unit in K. We choose a distinct field tower

$$K \supset k_5 \supset k_5^+ \supset \boldsymbol{Q}$$

from the case of an imaginary quadratic subfield k of K in §2. We have $N_{K/k_5}(\xi_{\sigma^2\tau})$
$= (\xi_{\sigma^2\tau}) \cdot (\xi_{\sigma^2\tau})^{\tau}$
$= (\alpha_1(\zeta - \zeta^{-1}) + (2\beta_0 + \beta_1)(\zeta + \zeta^{-1}))\sqrt{7}) \cdot (\alpha_1(\zeta - \zeta^{-1}) + (2b_0 + \beta_1)(\zeta + \zeta^{-1}))(-\sqrt{7}))$
$= (\alpha_1(\zeta - \zeta^{-1})^2 - ((2\beta_0 + \beta_1)(\zeta + \zeta^{-1}))^2 \cdot 7$. Then it follows that $N_{K/k_5^+}(\xi_{\sigma^2\tau})$
$= N_{k_5/k_5^+}(N_{K/k_5}(\xi_{\sigma^2\tau})) = ((\alpha_1(\zeta - \zeta^{-1})^2 - ((2\beta_0 + \beta_1)(\zeta + \zeta^{-1}))^2 \cdot 7)^2$
with $(\alpha_1(\zeta - \zeta^{-1})^2 \leqq 0$ and $-((2\beta_0 + \beta_1)(\zeta + \zeta^{-1}))^2 \cdot 7)^2 \leqq 0$.
Put $A = \sqrt{-(\alpha_1(\zeta - \zeta^{-1})^2}$ and $B = \sqrt{((2\beta_0 + \beta_1)(\zeta + \zeta^{-1}))^2 \cdot 7)^2}$. Then if $AB \neq 0$, we obtain $N_{K/\boldsymbol{Q}}(\xi_{\sigma^2\tau}) \geqq (2AB \cdot 2A^{\sigma}B^{\sigma})^2 \geqq 2^4 c_A c_B > 1$ with $1 \leqq c_A, c_B \in \boldsymbol{Z}$. If $A = 0$, $N_{K/\boldsymbol{Q}}(\xi_{\sigma^2\tau}) = (BB^{\sigma})^2 \geqq 7^2 c_B > 1$ with $1 \leqq c_B \in \boldsymbol{Z}$. If $B = 0$, ξ is not a primitive element of K by $\xi \in k_5$. Therefore an octic field K can not be monogenic. □

The claim A of a general case shall be proved in [4].

§4 Integral bases and Monogenity of Dihedral quartic fields K. Let K be a Dihedral quartic field with a quadratic subfield k of an odd field discriminant over the rationals $\boldsymbol{Q}$. We shall show a table of monogenity comparing a work of K. S. Williams eta al [2]. However,in this note we describe a part of the table, whose complete version is written in [1]. The next lemma is basic to determine an integral basis of a Dihedral quartic field.

Lemma 4.1. Being the same notation as above, it holds that

$$Z_K \cap k = Z_k.$$

Proof. Let γ be any element of the ring Z_k. Then $\gamma \in k$ and γ satisfies a monic polynomial $g(x) \in \boldsymbol{Z}[x]$ such that $g(\gamma) = 0$. Then $\gamma \in Z_K$ by $k \subset K$. Here for any algebraic number field F, the ring Z_F of integers in F is defined by the set

$\{\alpha \in F; \alpha$ satifies a monic polynomial $f(x) \in \boldsymbol{Z}[x]$ such that $f(\alpha) = 0\}$.

Conversely assume that any number $\xi \in Z_K \cap k$. Then by $\xi \in Z_K$ there exists a monic polynomial $h(x) \in \boldsymbol{Z}[x]$ such that $h(\xi) = 0$. Then by $\xi \in k$, $\xi \in Z_k$ follows. □

$({}_{1(16)}1_1)$ Assume that $m \equiv 1 \pmod{16}, \theta^2 = \alpha \equiv 1 + \omega \pmod{4Z_k}$. Then we have $\omega^2 = -4+\omega \equiv \omega \pmod 4$. Let $\xi = s+t\omega+u\theta+v\omega\theta$ be any integer in Z_K with $s, t, u, v \in \boldsymbol{Q}$. Then we have $T_{K/k}\xi = 2s + 2t\omega \in Z_k$ and $4N_{K/k}\xi = (2s + 2t\omega)^2 - (2u + 2v\omega)^2\alpha \in 4Z_k \subset Z_k$. Then it holds that $(2u + 2v\omega)^2\alpha = \gamma \in Z_k$. Put $2u + 2v\omega \cong \frac{\mathfrak{A}}{\mathfrak{B}}$ with $(\mathfrak{A}, \mathfrak{B}) = 1$ for integral ideals $\mathfrak{A}, \mathfrak{B}$. Here $\xi \cong \mathfrak{X}$ for a number ξ and a fractional ideal $\mathfrak{X}$ means that both sides are equal to each other as ideals. Assume that the denominator $\mathfrak{B} \not\cong 1$. Then there exists a prime ideal $\mathfrak{P}|\mathfrak{B}$ and we deduce that $\mathfrak{A}^2|\alpha$ from $\mathfrak{A}^2\alpha = \gamma\mathfrak{B}^2$. Then it holds that $N_k\alpha \equiv 0 \pmod{p^2}$ for a prime number $p \in \mathfrak{P}$, which is a contradiction. Then we have $2u + 2v\omega \in Z_k$. Put $2s = s_1, 2t = t_1, 2u = u_1$ and $2v = v_1$ with $s_1, t_1, u_1, v_1 \in \boldsymbol{Z}$. Then for a half integer $x = \frac{s_1}{2} + \frac{t_1}{2}\omega + \frac{u_1}{2}\theta + \frac{v_1}{2}\omega\theta$, choose $\xi_0 = \frac{1}{2} + \frac{1}{2}\omega + (\frac{1}{2} + \frac{1}{2}\omega)\theta$. we take a relative norm of ξ_0 with respect to K/k. Then it follows that $N_{K/k}\xi_0 = (\frac{1}{2} + \frac{1}{2}\omega)^2 - (\frac{1}{2} + \frac{1}{2}\omega)^2\alpha$ $= \frac{1}{4}(1 + 2\omega + \omega^2) - (1 + 2\omega + \omega^2)(1 + \omega) \equiv \frac{1}{4}((1 + 2\omega + \omega) - (1 + 2\omega + \omega)(1 + \omega))$ $\equiv \frac{1}{4}((1 + 3\omega) - (1 + 3\omega + \omega + 3\omega)) \pmod{Z_k} = \frac{1}{4}(-4\omega) \in Z_k$. Then we have $Z_K \subseteqq \boldsymbol{Z}\left[1, \omega, \theta, (1 + \omega)\frac{1+\theta}{2}, \frac{s_1}{2} + \frac{t_1}{2}\omega + \frac{u_1}{2}\theta + \frac{v_1}{2}\omega\theta\right]$. We identify (s_1, t_1, u_1, v_1) and a number $\frac{s_1}{2} + \frac{t_1}{2}\omega + \frac{u_1}{2}\theta + \frac{v_1}{2}\omega\theta$. $(0, 1, 1, 1) = \frac{1}{2}\omega + (\frac{1}{2} + \frac{1}{2}\omega)\theta$ is not an integer, because of $(1, 0, 0, 0) = \frac{1}{2} \notin Z_K$ and $(1, 1, 1, 1) \in Z_K$ from $T_{K/k}(1 + \omega)\frac{1+\theta}{2} \in Z_k$ and $N_{K/k}(1 + \omega)\frac{1+\theta}{2} \in Z_k$. In the same way we have $(0, 1, 0, 0) \notin Z_K$ and hence $(1, 0, 1, 1) \notin Z_K$, $(0, 0, 1, 0) \notin Z_K$, and hence $(1, 1, 0, 1) \notin Z_K$. On $(0, 0, 0, 1) = \omega\frac{\theta}{2}$, $N_{K/k}(\omega\frac{\theta}{2}) = \frac{1}{4}(\omega^2\alpha) \equiv \frac{1}{4}(\omega(1 + \omega)) \equiv \frac{1}{4}(\omega + \omega) \pmod{Z_k}$, which is impossible. Then we have $(0, 0, 0, 1) \notin Z_K$ and $(1, 1, 1, 0) \notin Z_K$. Since $(1, 1, 0, 0) \notin Z_K$ it holds that $(0, 0, 1, 1) \notin Z_K$. For $(1, 0, 1, 0) = \frac{1+\theta}{2}$ we get $N_{K/k}(\frac{1+\theta}{2}) = \frac{1}{4}(1 - \alpha) = -\frac{1}{4}\omega \notin$

Z_K. Then $(0,1,0,1) \notin Z_K$. Finally for $(1,0,0,1) = \frac{1+\omega\theta}{2}$ we have by $N_{K/k}(\frac{1+\omega\theta}{2}) = \frac{1}{4}(1-\omega^2\alpha)$, $0 \equiv \frac{1}{4}(1-\omega(1+\omega)) \equiv \frac{1}{4}(1-(\omega+\omega)) \pmod{Z_k}$. However $\frac{1}{4}(1-2\omega)$ is not an integer, it holds that $(0,1,1,0) \notin Z_K$. Then except for $(1,1,1,1)$ and $(0,0,0,0)$ the other 14 cases are not integers of the field K. Therefore we obtain

$$Z_K \subseteqq \boldsymbol{Z}\left[1, \omega, \theta, (1+\omega)\frac{1+\theta}{2}\right].$$

Since $Z_K \supseteqq \boldsymbol{Z}\left[1,\omega,\theta,(1+\omega)\frac{1+\theta}{2}\right]$ it is deduced that $Z_K = \boldsymbol{Z}\left[1,\omega,\theta,(1+\omega)\frac{1+\theta}{2}\right]$. □

To prove a non-monogenity of an algebraic numbe Fields F, it is enough to show the divisible fact that for a prime factor p of the field discriminant d_F such that $p^e \| d_F$, $p^{e+1} | d_F \xi$ follows for any integer ξ in F. The next proof includes the *exact* divisibility $2^2 \| N_K(\xi_{\sigma^2})$ for the second partial different $\xi_{\sigma^2} = \xi - \xi^{\sigma^2}$ of any generator ξ of a power basis in K. To avoid the possibility of $Z_K = \boldsymbol{Z}[\xi]$, it is necessary for us to deduce $N_K(x_\sigma) \equiv 0 \pmod{2^1}$ for the first partial different ξ_σ, namely $d_K\xi \equiv \pmod{2^{1+2+1}}$. After the *strict* obsevation within the restriction on the quadratic subfield $k = \boldsymbol{Q}(\sqrt{m}), m < 0$ of K, we show a *moderate* proof of non-monogenity, i.e. $d_K\xi \equiv \pmod{2^{1+1+1}}$ for any integer ξ in a Dihedral quartic field K, whose quadratic subfield k is real or imaginary.

Strict observation of non-monogenity. Assume that $Z_K = \boldsymbol{Z}[\xi]$ for an integer $\xi = s + t\omega + u\theta + v(1+\omega)\frac{1+\theta}{2}$, $s,t,u,v \in \boldsymbol{Z}$. We evaluate the intermediate partial factor $\xi_{\sigma^2} = \xi - \xi^{\sigma^2}$ $= 2u\theta + v(1+\omega)\theta$. By $N_{K/k}\xi_{\sigma^2} = \xi_{\sigma^2} \cdot (\xi_{\sigma^2})^{\sigma^2} = (2u+v+v\omega)^2\theta(-\theta)$ $= ((2u+v)^2 + (2u+v)v + v^2(-\frac{1-m}{4}+\omega))(-\alpha)$, it holds that $N_K\xi_{\sigma^2}$ $= ((2u+v)^2+(2u+v)v+v^2(-\frac{1-m}{4}))^2 \cdot N_k\alpha$. Put $U = 2u+v$ and $V = U^2+Uv+v^2\frac{1-m}{4}$ for $m \equiv 1 \pmod{16}, m \leqq -15$. Then there exists $(u,v) = (1,-1)$ or (even, 1) such that $2^2 \| V$ and $V \geqq 2^2$, where on Hasse's symbol $\|$, $a \| b^e$ means $a \equiv 0 \pmod{b^e}$, but $a \not\equiv 0 \pmod{b^{e+1}}$. Then we are obliged to evaluate the norm of first partial factor $\xi_\sigma = \xi - \xi^\sigma$ of the different $\mathfrak{D}_K(\xi)$ of a number ξ. Let $(t,u,v) = (t,0,1) = t\omega + (1+\omega)\frac{1+\theta}{2}$. Put $\mu_4 = (1+\omega)\frac{1+\theta}{2}$. Then it holds that for

$\xi_\sigma = t\omega + (1+\omega)\frac{1+\theta}{2} - (t\omega^\sigma + (1+\omega^\sigma)\frac{1+\theta^\sigma}{2}) = t\sqrt{m} + (1+\omega)\frac{\theta}{2} - (1+\omega^\sigma)\frac{\theta^\sigma}{2}$
$= 2t\omega - t + \mu_4 - \frac{1+\omega}{2} + \mu_4^\sigma - \frac{1+\omega^\sigma}{2}$ we have $\xi_\sigma \equiv t + \mu_4 + \mu_4^\sigma - 1 \pmod{2Z_k}$.

Let t be odd and $m = 1 + 16m_1$. Then it follows that $N_{K/k}\xi_\sigma = \xi \cdot \xi^{\sigma^2} = \mu_4 \cdot \mu_4^{\sigma^2}$
$\equiv ((1+\omega)\frac{1+\theta}{2} + (1+\omega^\sigma)\frac{1+\theta^\sigma}{2}) \cdot ((1+\omega)\frac{1-\theta}{2} + (1+\omega^\sigma)\frac{1-\theta^\sigma}{2})$
$\equiv (1+\omega)^2\frac{1-\alpha}{4} + ((1+\omega)^2\frac{1-\alpha}{4})^\sigma + N_k(1+\omega)\frac{1+\theta-\theta^\sigma-\theta\theta^\sigma}{4} + N_k(1+\omega)\frac{1+\theta^\sigma-\theta-\theta^\sigma\theta}{4}$
$\equiv (1+2\omega+\omega-4m_1)\frac{\omega}{4} + ((1+2\omega+\omega-4m_1)\frac{\omega}{4})^\sigma + N_k(1+\omega)\frac{2-2\theta\theta^\sigma}{4}$
$\equiv \frac{\omega+3\omega-4m_1\omega}{4} + (\frac{\omega+3\omega-4m_1\omega}{4})^\sigma + (1+1+\frac{-16m_1}{4})\frac{1-\theta\theta^\sigma}{2} \equiv (1-2m_1)(\frac{1}{2}+\frac{1}{2}) + (1-$

$\theta\theta^{\sigma})\,(\mathrm{mod}\ 2Z_k)$. $N_K\xi_\sigma \equiv -\alpha\alpha^{\sigma} \equiv 0\,(\mathrm{mod}\ 2)$. Therefore it deduces that $N_K\mathfrak{D}_K(\xi) \equiv 0\,(\mathrm{mod}\ 2^{1+2+1})$, which is a contradiction.
Next t be even. Then using the case of an odd t, it follows that $N_{K/k}\xi_\sigma$
$\equiv (1+(1+\omega)\frac{1+\theta}{2}+(1+\omega^{\sigma})\frac{1+\theta^{\sigma}}{2})\cdot(1+(1+\omega)\frac{1-\theta}{2}+(1+\omega^{\sigma})\frac{1-\theta^{\sigma}}{2})$
$\equiv 1+(1+\omega)+(1+\omega)^{\sigma}$
$+\{(1+\omega)^2\frac{1-\alpha}{4}+((1+\omega)^2\frac{1-\alpha}{4})^{\sigma}+N_k(1+\omega)\frac{1+\theta-\theta^{\sigma}-\theta\theta^{\sigma}}{4}+N_k(1+\omega)\frac{1+\theta^{\sigma}-\theta-\theta^{\sigma}\theta}{4}\}$
$\equiv 1+2(1+\frac{1}{2})+\{-\alpha\alpha\} \equiv -\alpha\alpha\,(\mathrm{mod}\ 2Z_k)$. Then we get $N_K\xi_\sigma \equiv 0\,(\mathrm{mod}\ 2)$.
In the same way, for $(t,u,v)=(t,1,-1)=t\omega+\theta-(1+\omega)\frac{1+\theta}{2}=\xi$ it follows that $N_K\xi_\sigma \equiv 0\,(\mathrm{mod}\ 2)$. Then we have deduced that any Dihedral field K in the family $({}_{1(16)}1_1), m<0$ is non-monogenic. □

Moderate observation of non-monogenity.

By way of Lemma 4.2, we show a moderate proof of non-monogenity for three families ${}_{m\equiv1(16)}C_1^1$, ${}_{m\equiv1(8)}C_2^1$ and ${}_{m\equiv9(16)}C_1^3$ without the restriction under the imaginary quadratic field

$x=\theta=\sqrt{\alpha},\ \alpha=1+\omega,\ \omega=\frac{1+\sqrt{-15}}{2},\ N_k\alpha=6.$

```
gp > nfdisc((x^{2}-1)^{2}-(x^{2}-1)+4)
= 5400=[2 3][3 3][5 2]=2^{2}\cdot d_{k}^{2}\cdot N_{k}\a
gp > poldisc((x^{2}-1)^{2}-(x^{2}-1)+4)
= 21600=2^{2}\cdot d_{K}
gp > nfbasis((x^{2}-1)^{2}-(x^{2}-1)+4)
= [1, x, x^2=\o+1, 1/2*x^3 - 1/2*x^2=(1+\o)\frac{\x-1}{2}]
```

On a real quadratic subfield k of K, with $x=\theta=\sqrt{\alpha},\ \alpha=1+\omega,\ \omega=\frac{1+\sqrt{17}}{2},\ N_k\alpha=-2$ GP/PARI shows that

```
gp > nfdisc((x^{2}-1)^{2}-(x^{2}-1)-4)
= -2312=[-1 1][ 2 3][17 2]=2^{2}\cdot d_{k}\cdot N_{k}\a
gp > poldisc((x^{2}-1)^{2}-(x^{2}-1)-4)
= -9248=[-1 1][ 2 5][17 2]=2^{2}\cdot d{K}
gp > nfbasis((x^{2}-1)^{2}-(x^{2}-1)-4)
= [1, x, x^2=1+\o, 1/2*x^3 - 1/2*x^2=\o\frac{\tt}{2}]
```

$({}_{9(16)}1_1)$ On the subfamily $m\equiv 9\,(\mathrm{mod}\ 16)$ in ${}_1\mathrm{C}_1^1$ it follows that ${}_{9(16)}\mathrm{C}_1^1=\emptyset$ because of $N_k\alpha \equiv 0\,(\mathrm{mod}\ 2^2)$.

$({}_{1(16)}1_2)$ On a family ${}_{1(16)}\mathrm{C}_2^1$ with $m<0$ and $m>0$, referring

```
gp > nfdisc((x^{2}-1)^{2}-2*(x^{2}-1)-16),
     \a=1+2\o, \o=\frac{1+\sqrt{-15}}{2}
= -15028=[-1 1][ 2 2][13 1][17 2]=2^{2}\cdot d_{k}\dot N_{k}\a
gp > poldisc((x^{2}-1)^{2}-2*(x^{2}-1)-16)
= -961792= (2^{3})^{2}\cdot d_{K}
gp > nfbasis((x^{2}-1)^{2}-2*(x^{2}-1)-16)
= [1, x, 1/2*x^2 - 1/2, 1/4*x^3 - 1/4*x^2 + 1/4*x - 1/4]
```

it holds that $Z_K = \boldsymbol{Z}[1, \omega, \theta, (1+\omega)\frac{1+\theta}{2}]$. Then it is shown that this field is non-monogenic by way of the moderate observation.

Lemma 4.2. *Let $\{1, \omega, \theta, (1+\omega)\frac{1+\theta}{2}\}$ be an integral basis of a Dihedral quartic field $K = \boldsymbol{Q}(\theta)$ in the families ${}_{m\equiv 1(16)}C_1^1$, ${}_{m\equiv 1(8)}C_2^1$ or ${}_{m\equiv 9(16)}C_1^3$ with $\theta = \sqrt{\alpha}$, $\alpha \equiv 1+\omega, 1+2\omega$ or $3+\omega \pmod 4 \in {}_{1(16)}C_1^1 \cup {}_{1(8)}C_2^1 \cup_{9(16)} C_1^3$ and $\omega = \frac{1+\sqrt{m}}{2}$. Then it holds that for any $\xi \in Z_K$*

(1) $\quad N_{K/k}\xi_{\sigma^2} \equiv 0 \pmod{2Z_k}$,

and

(2) $\qquad N_{K/k}\xi_{\sigma} \equiv 0 \pmod{2Z_k}$ *except for $\alpha = \xi^2 \in {}_{m\equiv 9(16)}C_2^1$.*

Sketch of a proof. Assume that $Z_K = \boldsymbol{Z}[\xi]$ for an integer $\xi = t\omega + u\theta + v(1+\omega)\frac{1+\theta}{2}, t, u, v \in \boldsymbol{Z}$. We evaluate the intermidiate partial factor $\xi_{\sigma^2} = \xi - \xi^{\sigma^2} = 2u\theta + v(1+\omega)\theta \pmod{2Z_k}$. For $\alpha \in_{1(16)} C_1^1 \cup \cup_{9(16)} C_1^3$ by $N_{K/k}\xi_{\sigma^2} = \xi_{\sigma^2}\cdot(\xi_{\sigma^2})^{\sigma^2} \equiv v^2(1+\omega)^2(\theta(-\theta)$ $\equiv v^2(1+0+\omega)(1-\alpha) \equiv v(1+\omega)(1+\omega) \equiv v(\omega+\omega) \equiv 0 \pmod{2Z_k}$. On the other hand, let $\alpha \in_{1(16)} C_2^1$ and $\xi = t\omega + u\theta + v(1+\omega)\frac{1+\theta}{2}$. On the partial differents $\xi_{\sigma^j} = \xi - \xi^{\sigma^j}$, $j = 1, 2$, we have $N_{K/k}\xi_{\sigma^2} \equiv 0 \pmod 2$ and $N_{K/k}\xi_{\sigma} \equiv 0 \pmod 2$ for $(t, u, v) = (0, u, v)_{u,v \pmod 2}$ and $(1, u, v)_{u,v \pmod 2}$. For instance, on a concrete evaluation of $N_K\xi_{\sigma}$ of $\xi = (0, 1, 1) = \theta + (1+\omega)\frac{1+\theta}{2}$ out of eight cases, it is deduced that $N_{K/k}\xi_{\sigma} \equiv 2(1-\theta\theta^{\sigma}) \pmod{4Z_k}$, and hence $N_K\xi_{\sigma} \equiv 0 \pmod 4$, whose 2th power is sufficient. Thus $N_K\mathfrak{D}_K(\xi) \equiv 0 \pmod{2^{1+1+1}}$, which contradicts to $2^2 \| \, d_K$. □

Remark 4.1. From Lemma 4.2, the condition $m < 0$ can be removed for the three families. On the excluded family ${}_{m\equiv 9(16)}C_2^1 \ni \boldsymbol{Q}(\xi)$ in (2), it is deduced that $N_K\xi_{\sigma}$ is odd. But we have $N_K\xi_{\sigma} \equiv \pmod{2^4}$ in (1).

```
gp > ((x^{2}-5)^{2}-(x^{2}-5)*(2*x+1)+4*(2*x+1)^{2})
= x^4 - 2*x^3 + 5*x^2 + 26*x + 34, m=-15
 \x=\o+\tt=\frac{1+\sqrt{-15}}{2}+\tt, \tt=\sqrt{\a}, \a=1+2*\o
gp > nfdisc((x^{2}-5)^{2}-(x^{2}-5)*(2*x+1)+4*(2*x+1)^{2})
= 17100=[ 2 2][ 3 2][ 5 2][19 1]=2^{2}\cdot d_{k}^{2}\cdot N_{k}\a
```

```
gp > poldisc((x^{2}-5)^{2}-(x^{2}-5)*(2*x+1)+4*(2*x+1)^{2})
= 24692400=[ 2 4][ 3 2][ 5 2][19 3]=(2*19)^{2}d_{K}
gp > nfbasis((x^{2}-5)^{2}-(x^{2}-5)*(2*x+1)+4*(2*x+1)^{2})
= [1, x, x^2, 1/38*x^3 + 7/38*x^2 - 4/19*x - 4/19]
```

$({}_{9}1_2)$ Along the same process of the proof for the family ${}_{1(16)}\mathrm{C}_1^1$, it is deduced that an explicit integral basis $Z_K = \boldsymbol{Z}[1, \omega, \theta, (1+\omega)\frac{1+\theta}{2}]$ and that by Lemma 4.2 the family ${}_{1(8)}\mathrm{C}_2^1$ is non-monogenic. $\square$

An example of $m = -7$ for ${}_{9(16)}\mathrm{C}_2^1$ is shown that

```
gp > nfbasis((x^{2}-1)^{2}-2*(x^{2}-1)+8), \x=\sqrt{\a}, \a=1+2\o
= [1, x, 1/2*x^2 - 1/2=\o, 1/4*x^3 - 1/4*x^2 + 1/4*x - 1/4]
= [1, x, 1/2*x^2 + 1/2, 1/4*x^3 - 1/4*x^2 + 1/4*x - 1/4
= [1, x, \o,   (1/2*x^2 + 1/2)*(1/2*x - 1/2)
  \equiv (1+\o)\frac{-1+\x}{2} \equiv (1+\o)\frac{1+\x}{2}]
= [1, x, \o,(1+\o)\frac{1+\x}{2}] with \a=1+2\o, m=-7.
gp > nfdisc((x^{2}-1)^{2}-2*(x^{2}-1)+8)
= 2156=[ 2 2][(-7) 2][11 1]=2^{2}\cdot d_{k}^{2}\cdot N_{k}\a.
```

$({}_{1(16)}3_1)$ For the family ${}_{1(16)}\mathrm{C}_1^3$ we have

$$Z_K = \boldsymbol{Z}\left[1, \omega, \theta, \omega\frac{\theta}{2}\right].$$

Proof of an integral basis for ${}_{1(16)}C_1^3$.
For $\xi_0 = (0,0,0,1) = \omega\frac{\theta}{2}$ it holds that
$N_{K/k}\xi_0 = \omega^2\frac{1}{4}(-\alpha) \equiv \frac{-1}{4}\omega(3+\omega) \equiv \frac{-1}{4}(3\omega+\omega) \equiv 0 \,(\mathrm{mod}\ Z_k)$. Then we have $\boldsymbol{Z}\left[1, \omega, \theta, \omega\frac{\theta}{2}\right] \subseteqq Z_K$ and $Z_K \subseteqq \boldsymbol{Z}\left[1, \omega, \theta, \omega\frac{\theta}{2}, \frac{s}{2} + \frac{t}{2}\omega + \frac{u}{2}\theta + \frac{v}{2}\omega\theta\right]_{s,t,u,v\in\boldsymbol{Z}}$.
Since $(1,0,0,0), (0,1,0,0), (0,0,1,0) \notin Z_K$, we have $(1,0,0,0)+(0,0,0,1) = (1,0,0,1), (0,1,0,1), (0,0,1,1) \notin Z_K$. By $(1,1,0,0) = \frac{1+\omega}{2} \notin Z_K$ $(1,1,0,0)+(0,0,0,1) = (1,1,0,1) \notin Z_K$ holds. On $\xi = (1,0,1,0)$ it follows that $N_{K/k}\xi = \frac{1}{4}(1-\alpha) = \frac{1}{4}(-2-\omega) \notin Z_K$ and hence $(1,0,1,0)+(0,0,0,1) = (1,0,1,1) \notin Z_K$. We have $\xi = (0,1,1,0) = \frac{\omega+\theta}{2} \notin Z_K$ by $N_{K/k}\xi = \frac{1}{4}(\omega^2-\alpha) = \frac{1}{4}(-2+\omega-(3+\omega)) \notin Z_K$ and $(0,1,1,0)+(0,0,0,1) = (0,1,1,1) \notin Z_K$. From $(0,1,0,1)+(0,0,0,1) = (0,1,0,0) \notin Z_K$ it follows that $(0,1,0,1) \notin Z_K$. Then 14 representatives (s,t,u,v) are not integers in K except for $(0,0,0,1), (0,0,0,0)$. Therefore we have deduced that $({}_{9(16)}3_1)$

$$Z_K \subseteqq \boldsymbol{Z}\left[1, \omega, \theta, \omega\frac{\theta}{2}\right]. \qquad \square$$

Proof of non-monogenity for ${}_{1(16)}C_1^3$. Let ξ be an integer $t\omega + u\theta + v\omega\frac{T}{2}$, which would generate a power integral basis and ξ_{σ^2} the second partial factor $\xi - \xi^{\sigma^2}$ of the different $\mathfrak{D}_K(\xi)$ of a number ξ. On the relatie norm $N_{K/k}\xi_{\sigma^2} = (2u+v\omega)^2)N_{K/k}\theta$ put $2u + v\omega$

by U. Then it holds that $N_kU = 4u^2+4uv+v^2\frac{1-m}{4} \equiv 0 \pmod{2^2}$, and hence we get $N_kU^2 \equiv 0 \pmod{2^4}$, which contradicts against $2^2\| \, d_K$. Then any number in Z_K can not generate a power integral basis. $\Box$

$({}_{9(16)}3_1)$ On the family ${}_{9(16)}\mathrm{C}_1^3$, $m \equiv 9 \pmod{16}$,
we have $(0,0,0,1) = \omega\frac{\theta}{2} \notin Z_K$. But $(1,1,1,1) = (1+\omega)\frac{1+\theta}{2} \in Z_K$. Then it deduces that $Z_K = \boldsymbol{Z}\left[1,\omega,\theta,(1+\omega)\frac{1+\theta}{2}\right]$. Let $\alpha = 3+\omega, \omega = \frac{1+\sqrt{-7}}{2}, N_k\alpha = \frac{7^2-(-7)}{4} = 14, N_k\omega = 2.$

```
gp > nfdisc((x^{2}-3)^{2}-(x^{2}-3)+2)
= 2744=[2 3][7 3]=2^{2}\cdot d_{k}^{2}\cdot N_{k}\a
gp > poldisc((x^{2}-3)^{2}-(x^{2}-3)+2)
= 10976=2^{2}\cdot d_{K}.
```

$({}_{1(16)}3_2)$ On the familly ${}_{1(16)}\mathrm{C}_2^3$, $m \equiv 1 \pmod{16}$ and $\omega^2 \equiv \omega \pmod 2$, We find $\xi = (0,1,0,1) = \omega\frac{1+\theta}{2}$. In fact, it holds that $T_{K/k}\xi = \omega, N_{K/k}\xi = \omega^2\frac{1-\alpha}{4} \equiv \frac{-2}{4}\omega(1+\omega) \equiv \frac{-1}{2}(\omega+\omega) \equiv 0 \pmod{Z_k}$. Then $\xi \in Z_K$, which deduces that $\boldsymbol{Z}\left[1,\omega,\theta,\omega\frac{1+\theta}{2}\right] \subseteqq Z_K$. Conversely, let ξ be any half integer $x = \frac{s_1}{2}+\frac{t_1}{2}\omega+\frac{u_1}{2}\theta+\frac{v_1}{2}\omega\theta$ with $s_1,t_1,u_1,v_1 \in \boldsymbol{Z}$. Since $(s_1,t_1,u_1,v_1) = (0,0,0,1),(1,1,1,1) \notin Z_K$ it deduces that $(0,0,0,1)+(0,1,0,1) = (0,1,1,0) = \frac{\omega+\theta}{2} \notin Z_K$ and $(1,1,1,1)+(0,1,0,1) = (1,0,1,0) = \frac{1+\theta}{2} \notin Z_K$. Here $(a,b,c,d) = (s,t,u,v)$ means that $(a,b,c,d) \equiv (s,t,u,v) \pmod{Z_K}$. By
$(1,0,0,0),(0,1,0,0),(0,0,1,0) \notin Z_K$, it holds that $(1,1,0,1),(0,0,0,1),(0,1,1,1) \notin Z_K$. For $\xi = (1,1,1,0) = \frac{1+\omega+\theta}{2}$, $N_{K/k}\xi = \frac{1}{4}((1+\omega)^2-\alpha) \equiv \frac{1}{4}((1+2\omega+\omega)-(3+2\omega))$
$\equiv \frac{1}{4}(-2+\omega) \pmod{Z_k,}$ which is not an integer. Then $(1,1,1,0)+(0,1,0,1) = (1,0,1,1) \notin Z_k$ holds. Since $(1,1,0,0) \notin Z_k$, it follows that $(1,0,0,1) \notin Z_k$. For $\xi = (0,1,1,0) = \frac{\omega+\theta}{2}$, from $N_{K/k}\xi = \frac{1}{4}(\omega^2-\alpha) \equiv \frac{1}{4}(\omega-(3+2\omega)) \equiv \frac{1}{4}(-3-\omega) \not\equiv 0 \pmod{Z_k}$, we have $(0,1,1,0)+(0,1,0,1) = (0,0,1,1) \notin Z_k$. Thus, 14 representatives $(s_1,t_1,u_1,v_1)_{(\mathrm{mod}\ 2)}$ are not integers in K except for $(0,1,0,1),(0,0,0,0)$. Therfore it is deduced that for ${}_{1(16)}\mathrm{C}_2^3$,

$$Z_K \subseteqq \boldsymbol{Z}\left[1,\omega,\theta,\omega\frac{1+\theta}{2}\right]. \qquad \Box$$

Proof of non-monogenity for ${}_{1(16)}C_2^3$. Assume that $Z_K = \boldsymbol{Z}[\xi]$ for a suitable integer ξ $= t\omega+u\theta+v\omega\frac{1+\theta}{2} \in K$. On the second partial factor $\xi_{\sigma^2} = \xi - \xi^{\sigma^2} = 2u\theta+v\omega\theta$ of the different $\mathfrak{D}_K(\xi)$ of a number ξ, we have $N_{K/k}\xi_{\sigma^2} = (2u+v\omega)^2(-\alpha)$. Put $U = 2u+v\omega$. Then it follows that $N_kU^2 \equiv (4u^2+4uv+v^2\frac{1-m}{4})^2 \equiv 0 \pmod{2^4}$, which is impossible by $2^2\| \, d_K$. Then there exit infinitely many non-monogenic Dihedral octic fields in the subfamily ${}_{1(16)}\mathrm{C}_2^3 \subset {}_{1(8)}\mathrm{C}_2^3$. $\Box$

($_5 4_1$) The monogenic familly ${}_5\mathrm{C}_1^4$ of $m \equiv 5 \pmod 8$ includes a Dihedral quintic field $K = \boldsymbol{Q}(\theta)$ with $\omega = \frac{1+\sqrt{5}}{2}, \theta = \sqrt{\omega}$ of A. C. Kable [3].
Proof. Let $\alpha \equiv 0 + \omega \pmod 4$ and $\theta = \sqrt{\alpha}$ with $m \equiv 5 \pmod{16}$. We note that $\omega^2 \equiv -1 + \omega \pmod 4$. Then we have $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1) = \xi \notin Z_K$. In fact, $N_{K/k}\xi = \frac{1}{4}\omega^2(-\alpha) \equiv \frac{-1}{4}(-1+\omega)(\omega) = \frac{-1}{4}(-\omega + (-1+\omega)) \notin Z_k$. It follows that $(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1) = \frac{\theta}{2} + \frac{\omega\theta}{2} = \xi \notin Z_k$. In fact, $N_{K/k}\xi = \frac{1}{4}(1+\omega)^2(-\alpha) \equiv \frac{-1}{4}(1+2\omega+(-1+\omega))(\omega) \not\equiv \frac{-1}{4}(3\omega)(\omega) \equiv \frac{-1}{4}(-3+3\omega)$ $\equiv 0 \pmod{Z_k}$. On $(1,1,1,1) = (1+\omega)\frac{1+\theta}{2} = \xi$, it holds that $N_{K/k}\xi = \frac{1}{4}(1+\omega^2)(1-\alpha)$ $\equiv \frac{1}{4}(1+2\omega+(-1+\omega))(-3-\omega) \equiv \frac{-1}{4}(3\omega)(3+\omega) \equiv \frac{-1}{4}(9\omega+3(-1+\omega)) \not\equiv 0 \pmod{Z_k}$. THen all the 15 cases $(s,t,u,v)_{s,t,u,v \pmod 2}$ can not be integers except for $(0,0,0,0)$. Therefore we deduced that $Z_K = \boldsymbol{Z}[1,\omega,\theta,\theta\omega] = \boldsymbol{Z}\left[1,\theta^2,\theta,\theta^3\right]$. □

```
gp > nfdisc(x^{4}-x^{2}-1)= -400  \o=\frac{1+\sqrt{5}}{2}, \a=0+\o
gp > factor(-400)=[-1 1][ 2 4][ 5 2] \x=\sqrt{\a}, N_{k}\a=-1
gp > poldisc(x^{4}-x^{2}-1)= -400
gp > nfbasis(x^{4}-x^{2}-1)=[1, x, x^2, x^3]
> which coincides with the example of A. C. Kable [3].

gp > nfdisc((x^{2}-4)^{2}-(x^{2}-4)-1)= 7600 \x=\sqrt{\a}, \a=4+\o
gp > poldisc((x^{2}-4)^{2}-(x^{2}-4)-1)= 7600
gp > factor(7600)=[ 2 4][ 5 2][19 1]     N_{k}\a=(81-5)/4=19
gp > nfbasis((x^{2}-4)^{2}-(x^{2}-4)-1)= [1, x, x^2, x^3]
```

($_5 4_3$) The last familly ${}_5\mathrm{C}_3^4$, $m \equiv 5 \pmod 8$ is a disjoint monogenic one against ($_5 4_1$). We have $Z_K = \boldsymbol{Z}[1,\omega,\theta,\theta\omega] = \boldsymbol{Z}\left[1,\theta,\theta^2,\theta^3\right]$ as in ($_5 4_1$).
For $\omega = \frac{1+\sqrt{5}}{2}, \alpha = 4 - \omega, N_k\alpha = (7^2-5)/4 = 11$ and $x = \theta = \sqrt{\alpha}$, GP/PARI shows that

```
gp > nfdisc((x^{2}-4)^{2}+(x^{2}-4)-1)
= 4400=[ 2 4][ 5 2][11 1]=2^{4}d_{k}^{2}N_{k}\a
gp > nfbasis((x^{2}-4)^{2}+(x^{2}-4)-1)
= [1, x, x^2, x^3].
```

A complete classification of Table 1 for 18 families shall be written in [1] comparing [2].

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Table 1: Integral bases of Dihedral quartic fields of the odd field discriminants $d_k = m$

m	a	b	family	$\boldsymbol{Z}_K$	d_K	[2]
1(8)	1(4)	1(4)	${}_{1(16)}\mathrm{C}_1^1$	$\boldsymbol{Z}[1, \omega, \theta, (1+\omega)\frac{1+\theta}{2}]$ **Non**	$m \equiv 1(16), 2^2m^2N_k\alpha$	$D_{14},\ m=-15$
			${}_{9(16)}\mathrm{C}_1^1 = \emptyset$		$N_k\alpha \equiv 0 \pmod{2^2}$	
1(8)	1(4)	2(4)	${}_{1}\mathrm{C}_2^1$	$\boldsymbol{Z}[1, \omega, \theta, (1+\omega)\frac{1+\theta}{2}]$ **Non**	$2^2m^2N_k\alpha$	$D_4,\ m=-15$
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
1(8)	3(4)	1(4)	${}_{1(16)}\mathrm{C}_1^3$	$\boldsymbol{Z}[1, \omega, \theta, \omega\frac{\theta}{2}]$ **Non**	$m \equiv 1(16),\ 2^2m^2N_k\alpha$	D_{26}, even though $2^2 \parallel N_k\alpha$
1(8)	3(4)	1(4)	${}_{9(16)}\mathrm{C}_1^3$	$\boldsymbol{Z}[1, \omega, \theta, (1+\omega)\frac{1+\theta}{2}]$ **Non**	$m \equiv 9(16),\ 2^2m^2N_k\alpha$	$D_{14},\ m=-7$
1(8)	3(4)	2(4)	${}_{1(16)}\mathrm{C}_2^3$	$\boldsymbol{Z}[1, \omega, \theta, \omega\frac{1+\theta}{2}]$ **Non**	$m \equiv 1(16),\ 2^2m^2N_k\alpha$	$D_1,\ m=17$
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
5(8)	4(4)	1(4)	${}_{5}\mathrm{C}_1^4$	$\boldsymbol{Z}[1, \omega=\theta^2, \theta, \omega\theta=\theta^3]$	$2^4m^2N_k\alpha$	$C_6,\ m=5$
5(8)	4(4)	2, 4(4)	${}_{5}\mathrm{C}_{2,4}^4 = \emptyset$		$N_k\alpha \equiv 0 \pmod{2^2}$	
5(8)	4(4)	3(4)	${}_{5}\mathrm{C}_3^4$	$\boldsymbol{Z}[1, \omega=\theta^2, \theta, \omega\theta=\theta^3]$	$2^4m^2N_k\alpha$	$C_7,\ m=5$

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