Asymptotic behaviors of thermodynamic quantities in perturbed graph directed Markov systems

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Abstract

We study asymptotic perturbations of graph directed Markov systems (GDMSs) which are firstly introduced by [Maudlin and Urbński, 2003]. As main results, we give asymptotic expansions of thermodynamic quantities (the Hausdorff dimensions of the limit sets, the Gibbs measures associated with the dimensions and the measure-theoretic entropies of these measures) of asymptotically perturbed GDMSs. This is an extended version of the finite graph case in [T. 2016]. As a concrete example, we apply this result to perturbed infinite linear IFSs.

1 Introduction

Graph directed Markov system (GDMS for short) consists of a family of contraction maps attached to each edge of a countable directed graph. It is a generalization of usual iterated function system (IFS) and is firstly introduced by Mauldin and Urbański [1]. As examples, self-similar sets, continued fraction transformations, Schottky groups, non-uniform expanding maps with countable Markov partition are included. The figure 1 shows a concrete example of GDMS.

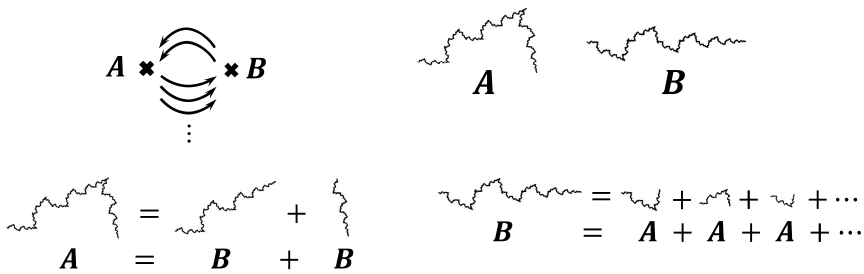

Fig 1: The limit set of a certain GDMS

Let E be a countable edge set of a directed graph. We introduce an n-order asymptotic expansion $(T_e(\epsilon,\cdot))_{e\in E}$ of contraction maps $(T_e)_{e\in E}$

$$T_e(\epsilon,\cdot) = T_e + T_{e,1}\epsilon + \cdots + T_{e,n}\epsilon^n + o(\epsilon^n) \qquad (e \in E)$$

as $\epsilon \to 0$ and we are interested in the following studies:

(1) Asymptotic behaviours of thermodynamic quantities (the Hausdorff dimensions of the limit sets, the Gibbs measures associated with these dimensions, and the measure-theoretic entropies of these measures) in perturbed GDMSs.

(2) Perturbed GDMS with degeneration.

(3) Dimension estimate in nonconformal mapping via asymptotic perturbation from conformal mapping.

One of our motivation of the above (1) is to obtain the effect of maps on the quantities precisely. For example, by perturbing one of the maps that make up an iterated function system and by examining the change in the Hausdorff dimension of the limit set, the degree of influence on the dimension of one map can be studied. In (2), we treat the case that the directed graph associated with the perturbed GDMS is different from the directed graph associated with the unperturbed GDMS (e.g. [5, Example 4.3] and [6, Section 5.2]). In view of dynamical systems, such a situation is sometimes called 'singular perturbation of symbolic dynamics' or 'perturbed system with holes' (e.g. [2, 7]). The study (3) is an another application of our asymptotic analysis. In this report, we treat the study (1).

In our previous result, we gave asymptotic behaviours of the Hausdorff dimensions of the limit sets of asymptotically perturbed GDMS in finite graph case [5, Theorem 1.1]. This result was extended from the finite graph case to the infinite graph case in [8, Theorem 2.2]. Secondly, we obtained an asymptotic expansion of Gibbs measures associated with the dimension and an asymptotic expansion of the measure-theoretic entropy of this measure in perturbed GDMS with finite graph [5, Theorem 1.3].

In this present report, we extend these asymptotic expansion results of the Gibbs measures and of the entropy from the finite graph case to the infinite graph case (Theorem 4.1 and Theorem 4.2).

In the next section 2.1, we recall the notion of thermodynamic formalism and some results of Ruelle transfer operators. In Section 2.2, we give definitions of conformal graph directed Markov systems. We formulate an asymptotic perturbation of GDMS in Section 3. Our main results and the outline of the proofs are treated in Section 4. In the final section 5, we apply our results to a perturbed infinite linear IFS as a concrete example.

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2 Preliminaries

2.1 Thermodynamic formalism and Ruelle operators

In this section, we recall the notion of thermodynamic formalism in symbolic dynamics and some results of Ruelle operators which are mainly given in [1]. We begin with one-sided shift space with countable states.

Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed graph with finite vertices V, countable edges E, and two maps $i(\cdot)$ and $t(\cdot)$ from E to V. For each $e \in E$, $i(e)$ is called the initial vertex of e and $t(e)$ called the terminal vertex of e. Denoted by E^∞ the one-sided shift space

$$E^\infty = \{\omega \in \prod_{k=0}^{\infty} E \,:\, t(\omega_n) = i(\omega_{n+1}) \text{ for any } n \geq 0\}.$$

An element ω of E^∞ denotes $\omega = \omega_0\omega_1\omega_2\cdots$ with $\omega_0, \omega_1, \omega_2, \cdots \in E$. We endow the set E^∞ with the shift transformation $\sigma : E^\infty \to E^\infty$ defined as $(\sigma\omega)_n = \omega_{n+1}$ for any $n \geq 0$. A word $w = w_1w_2\cdots w_n \in E^n$ is *admissible* if $t(w_j) = i(w_{j+1})$ for all $1 \leq j < n$. For admissible word $w \in E^n$, the *cylinder set* of w is defined by $[w] = \{\omega \in E^\infty : \omega_0 \cdots \omega_{n-1} = w\}$. For $\theta \in (0, 1)$, a metric $d_\theta : E^\infty \times E^\infty \to \mathbb{R}$ is defined by $d_\theta(\omega, \upsilon) = \theta^{\min\{n\geq 0\,:\,\omega_n \neq \upsilon_n\}}$ if $\omega \neq \upsilon$ and $d_\theta(\omega, \upsilon) = 0$ if $\omega = \upsilon$. Then the metric topology induced by d_θ coincides with the product topology on E^∞ induced by the discrete topology on S. Remark that (E^∞, d_θ) is a complete and separable metric space. Note also that if $\{a \in S : [a] \neq \emptyset\}$ is an infinite set, then E^∞ is not compact. The *incidence* matrix A of E^∞ is defined by $A = (A(ee'))_{E\times E}$ with $A(ee') = 1$ if $t(e) = i(e')$ and $A(ee') = 0$ if $t(e) \neq i(e')$. The matrix A is *finitely irreducible* if there exists a finite subset F of $\bigcup_{n=1}^{\infty} E^n$ such that for any $e, e' \in E$, ewe' is a path on the graph G for some $w \in F$. A function $f : E^\infty \to \mathbb{K}$ is called *locally d_θ-Lipschitz continuous* if the number

$$[f]_\theta := \sup_{e\in E} \sup_{\omega,\upsilon\in[e]\,:\,\omega\neq\upsilon} \frac{|f(\omega) - f(\upsilon)|}{d_\theta(\omega, \upsilon)}$$

is finite. Denoted by $\|\cdot\|_\infty$ the supremum norm defined as $\|f\|_\infty := \sup_{\omega\in E^\infty} |f(\omega)|$. Put $\|f\|_\theta := \|f\|_\infty + [f]_\theta$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Denoted by $C(E^\infty, \mathbb{K})$ the set of all $\mathbb{K}$-valued continuous functions on E^∞ and by $C_b(E^\infty, \mathbb{K})$ the set of all functions f in $C(E^\infty, \mathbb{K})$ with $\|f\|_\infty < +\infty$. Let $F_\theta(E^\infty, \mathbb{K})$ be the set of all $\mathbb{K}$-valued locally d_θ-Lipschitz functions on E^∞, and $F_{\theta,b}(E^\infty, \mathbb{K})$ the set of all functions f in $F_\theta(E^\infty, \mathbb{K})$ with $\|f\|_\infty < +\infty$. For simplicity, if $\mathbb{K} = \mathbb{C}$ then we may omit '$\mathbb{K}$' from the notation of those functions.

For function $\varphi : E^\infty \to \mathbb{R}$, the *topological pressure* $P(\varphi)$ of the potential φ is formality given by

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in S^n\,:\,[w]\neq\emptyset} \exp(\sup_{\omega\in[w]} \sum_{k=0}^{n-1} \varphi(\sigma^k\omega)). \tag{2.1}$$

If the incidence matrix of G is finitely irreducible and φ is in $F_\theta(E^\infty, \mathbb{R})$, then $P(\varphi)$ is well-defined in $(-\infty, +\infty]$.

A (σ-invariant) Borel probability measure μ on E^∞ is said to be a *Gibbs measure* of the potential $\varphi : E^\infty \to \mathbb{R}$ if there exist $c \geq 1$ and $P \in \mathbb{R}$ such that for any $\omega \in E^\infty$ and $n \geq 1$

$$c^{-1} \leq \frac{\mu([\omega_0\omega_1 \ldots \omega_{n-1}])}{\exp(-nP + \sum_{k=0}^{n-1} \varphi(\sigma^k \omega))} \leq c.$$

In what follows, we assume that Gibbs measures treated as follows are σ-invariant. If A is finitely irreducible and φ is in $F_\theta(E^\infty, \mathbb{R})$ with $P(\varphi) < +\infty$, then the Gibbs measure of φ uniquely exists [1, Corollary 2.7.5]. In particular, P equals $P(\varphi)$.

For a continuous function $\varphi : E^\infty \to \mathbb{R}$, we call a σ-invariant probability measure μ an *equilibrium state* of the potential φ if $\mu(\varphi) > -\infty$ and

$$P(\varphi) = \int \varphi \, d\mu + h_\sigma(\mu),$$

where $h_\sigma(\mu)$ is the measure-theoretic entropy of μ with respect to σ. In this case, we may say that the measure-theoretic entropy of μ is given by $P(\varphi) - \mu(\varphi)$. It is known that if A is finitely irreducible and φ is in $F_\theta(E^\infty, \mathbb{R})$ with $P(\varphi) < +\infty$ and $\mu(\varphi) > -\infty$, then the Gibbs measure μ becomes the equilibrium state for φ [1, Theorem 2.2.9].

We end this section with the following result for Ruelle transfer operators. For function $\varphi : E^\infty \to \mathbb{R}$, the Ruelle operator $\mathcal{L}_\varphi$ associated to φ is defined by

$$\mathcal{L}_\varphi f(\omega) = \sum_{e \in E \,:\, A(e\omega_0)=1} e^{\varphi(e\cdot\omega)} f(e \cdot \omega)$$

if this series converges in $\mathbb{C}$ for a complex-valued function f on E^∞ and for $\omega \in E^\infty$. It is known that if the incidence matrix is finitely irreducible and φ is in $F_\theta(E^\infty, \mathbb{R})$ with finite topological pressure, then $\mathcal{L}_\varphi$ becomes a bounded linear operator both on the Banach spaces $C_b(E^\infty)$ and $F_{\theta,b}(E^\infty)$. We state a version of Ruelle-Perron-Frobenius Theorem:

Theorem 2.1 ([1]) *Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed graph with finite vertices V and countable edges E and assume that G has a finitely irreducible transition matrix. Let $\varphi \in F_\theta(E^\infty, \mathbb{R})$ with finite pressure. Then the Ruelle operator $\mathcal{L} : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$ of the potential φ has the spectral decomposition*

$$\mathcal{L} = \lambda \mathcal{P} + \mathcal{R}$$

such that

(1) λ is the spectral radius of $\mathcal{L}$ and is a simple eigenvalue of $\mathcal{L}$.

(2) $\mathcal{P}$ is the eigenprojection of the eigenvalue λ of $\mathcal{L}$ onto the one-dimensional eigenspace. In particular, $\mathcal{P}f = \int_{E^\infty} fh\,d\nu$, where $h \in F_\theta(E^\infty, \mathbb{R})$ is the corresponding positive eigenfunction of the eigenvalue λ and ν is the corresponding positive eigenvector of λ of the dual operator $\mathcal{L}^* : F_{\theta,b}(E^\infty)^* \to F_{\theta,b}(E^\infty)^*$ of $\mathcal{L}$ with $\nu(h) = 1$. In particular, there exists a constant $c \geq 1$ such that $c^{-1} \leq h \leq c$.

(3) $\mathcal{P}\mathcal{R} = \mathcal{R}\mathcal{P} = \emptyset$ and the spectrum of $\mathcal{R} : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$ is contained in $\{z \in \mathbb{C} : |z - \lambda| \geq \rho\}$ for some small $\rho > 0$.

(4) $P(\varphi)$ equals $\log \lambda$ and $h\nu$ becomes the Gibbs measure of the potential φ.

2.2 Conformal graph directed Markov systems

In this section, we recall the notion of conformal graph directed Markov systems given by [1, 10]. Let D be a positive integer, $\beta \in (0, 1]$ and $r \in (0, 1)$. We introduce a set $(G, (J_v), (O_v), (T_e))$ with $\sharp V < \infty$ satisfying the following conditions (i)-(iv):

(i) For each $v \in V$, J_v is a nonempty compact and connected subset of $\mathbb{R}^D$ satisfying $J_v = \overline{\mathrm{int} J_v}$, where $\mathrm{int} J_v$ is the interior of J_v.

(ii) For each $v \in V$, O_v is a bounded, open and connected subset of $\mathbb{R}^D$ containing J_v.

(iii) For each $e \in E$, a function $T_e : O_{t(e)} \to T_e(O_{t(e)}) \subset O_{i(e)}$ is a $C^{1+\beta}$-conformal diffeomorphism with $T_e(\mathrm{int} J_{t(e)}) \subset \mathrm{int} J_{i(e)}$ and $\sup_{x \in O_{t(e)}} \|T'_e(x)\| \leq r$, where $\|T'_e(x)\|$ means the operator norm of $T'_e(x)$. Moreover, for any $e, e' \in E$ with $e \neq e'$ and $i(e') = i(e)$, $T_e(\mathrm{int} J_{t(e)}) \cap T_{e'}(\mathrm{int} J_{t(e')}) = \emptyset$, namely the open set condition (OSC) is satisfied.

(iv) (Bounded distortion) There exists a constant $c_1 > 0$ such that for any $e \in E$ and $x, y \in O_{t(e)}$, $|\|T'_e(x)\| - \|T'_e(y)\|| \leq c_1 \|T'_e(x)\| |x - y|^\beta$, where $| \cdot |$ means a norm of any Euclidean space.

Under these conditions (i)-(iv), we call the set $(G, (J_v), (O_v), (T_e))$ a conformal graph directed Markov system (CGDMS for short).

Remark 2.2 A cone condition was part of the definition of a CGDMS in [1]:

(v) (Cone condition) There exist $\gamma, l > 0$ with $\gamma < \pi/2$ such that for any $v \in V$, $x \in J_v$, there is $u \in \mathbb{R}^D$ with $|u| = 1$ so that the set $\{y \in \mathbb{R}^D : 0 < |y - x| < l \text{ and } (y - x, u) > |y - x| \cos \gamma\}$ is in $\mathrm{int} J_v$, where $(y - x, u)$ denotes the inner product of $y - x$ and u.

This condition is not necessary for generalized Bowen's formula (see [10, Remark 19.3.2(d)]).

The *coding map* $\pi : E^\infty \to \mathbb{R}^D$ is defined by $\pi\omega = \bigcap_{n=0}^{\infty} T_{\omega_0} \cdots T_{\omega_n}(J_{t(\omega_n)})$ for $\omega \in E^\infty$. Put $K = \pi(E^\infty)$. This set is called the *limit set* of the CGDMS. We define a function $\varphi : E^\infty \to \mathbb{R}$ by

$$\varphi(\omega) = \log \|T'_{\omega_0}(\pi\sigma\omega)\|.$$

Put

$$\underline{s} = \inf\{s \geq 0 : P(s\varphi) < +\infty\}.$$

We call the CGDMS *regular* if $P(s\varphi) = 0$ for some $s \geq \underline{s}$. The CGDMS is said to be *strongly regular* if $0 < P(s\varphi) < +\infty$ for some $s \geq \underline{s}$ (see [1, 3] for the terminology). It is known that the general Bowen's formula is satisfied:

Theorem 2.3 ([10, Theorem 19.6.4]) *Let $(G, (J_v), (O_v), (T_e))$ be a graph directed Markov system. Assume that E^∞ is finitely irreducible. Then $\dim_H K = \inf\{t \in \mathbb{R} : P(t\varphi) \leq 0\}$. In addition to the above condition, we also assume that the potential φ is regular. Then $s = \dim_H K$ if and only if $P(s\varphi) = 0$.*

3 Asymptotic perturbations of graph directed Markov systems

Now we formulate an asymptotic perturbation of graph directed Markov systems. Fix integers $n \geq 0$, $D \geq 1$ and a number $\beta \in (0, 1]$. Consider the following conditions $(G.1)_n$ and $(G.2)_n$:

$(G.1)_n$ A set $(G, (J_v), (O_v), (T_e))$ is a CGDMS on $\mathbb{R}^D$ with strongly regular and with finitely irreducible incidence matrix. The limit set K has positive Hausdorff dimension. Moreover, the function T_e is of class $C^{1+n+\beta}(O_{t(e)})$ for each $e \in E$.

$(G.2)_n$ The set $\{(G, (J_v), (O_v), (T_e(\epsilon, \cdot))) : \epsilon > 0\}$ is a CGDMS with a small parameter $\epsilon > 0$ satisfying the following (i)-(vi):

(i) For each $e \in E$, the function $T_e(\epsilon, \cdot)$ has the n-asymptotic expansion:

$$T_e(\epsilon, \cdot) = T_e + T_{e,1}\epsilon + \cdots + T_{e,n}\epsilon^n + \tilde{T}_{e,n}(\epsilon, \cdot)\epsilon^n \quad \text{on } J_{t(e)}$$

for some functions $T_{e,k} \in C^{1+n-k+\beta}(O_{t(e)}, \mathbb{R}^D)$ $(k = 1, 2, \ldots, n)$ and $\tilde{T}_{e,n}(\epsilon, \cdot) \in C^{1+\beta(\epsilon)}(O_{t(e)}, \mathbb{R}^D)$ $(\beta(\epsilon) > 0)$ satisfying $\sup_{e\in E} \sup_{x \in J_{t(e)}} |\tilde{T}_{e,n}(\epsilon, x)| \to 0$.

(ii) There exist constants $t(l, k) \in (0, 1]$ $(l = 0, 1, \ldots, n,\ k = 1, \ldots, n - l + 1)$ such that the function $x \mapsto T^{(k)}_{e,l}(x)/\|T'_e(x)\|^{t(l,k)}$ is bounded, β-Hölder continuous and its Hölder constant is bounded uniformly in $e \in E$.

(iii) $c_2(\epsilon) := \sup_{e\in E}\sup_{x\in J_{t(e)}}(\|\frac{\partial}{\partial x}\tilde{T}_{e,n}(\epsilon,x)\|/\|T_e'(x)\|^{\tilde{t}_0}) \to 0$ as $\epsilon \to 0$ for some $\tilde{t}_0 \in (0,1]$.

(iv) $\dim_H K/D > p(n)$, where $p(n)$ is defined by

$$p(n) := \begin{cases} \underline{p}/\tilde{t}, & (n=0) \\ \max\big(\underline{p}+n(1-t_1), \underline{p}+n(1-t_2)/2, \cdots, \underline{p}+n(1-t_n)/n, \\ \qquad \underline{p}/t_1,\ \underline{p}/t_2,\ \cdots, \underline{p}/t_n,\ \underline{p}+1-\tilde{t},\ \underline{p}/\tilde{t}\big), & (n\geq 1) \end{cases}$$

$$\underline{p} := \underline{s}/D$$

$$t_k := \min\{\frac{1}{D}\sum_{p=1}^{D} t(i_p, j_p+1)\ :\ i := i_1+\cdots+i_D \text{ and } j := j_1+\cdots+j_D \text{ satisfy}$$
$$i = k \text{ and } j = 0 \text{ or } 0\leq i<k \text{ and } 1\leq j\leq k-i\} \tag{3.1}$$

$$\tilde{t} := \min\left\{t_n,\ \tilde{t}_0,\ \frac{\tilde{t}_0}{D}+\frac{D-1}{D}t(1,1), \ldots,\ \frac{\tilde{t}_0}{D}+\frac{D-1}{D}t(n,1)\right\}. \tag{3.2}$$

Remark 3.1 If the edge set E is finite, then the conditions (ii) and (iv) are always satisfied because $\|T_e'(x)\|$ is uniformly bounded away from zero, and $p(n)$ becomes $\underline{s}/D$ by taking $t(l,k) = \tilde{t} = 1$. Moreover, $c_2(\epsilon)$ in (iii) can be taken as $\sup_{e\in E}\sup_{x\in J_{t(e)}}\|\frac{\partial}{\partial x}\tilde{T}_{e,n}(\epsilon,x)\|$ when E is finite.

Let $K(\epsilon)$ be the limit set of the perturbed CGDMS $(G,(J_v),(O_v),(T_e(\epsilon,\cdot)))$. We put $s(\epsilon) = \dim_H K(\epsilon)$. Under those conditions, we obtained the following:

Theorem 3.2 ([8, Theorem 2.2]) *Assume that the conditions $(G.1)_n$ and $(G.2)_n$ are satisfied with fixed integer $n \geq 0$. Then the perturbed CGDMS $(G,(J_v),(O_v),(T_e(\epsilon,\cdot)))$ is strongly regular for any small $\epsilon > 0$, and there exist $s_1,\ldots,s_n \in \mathbb{R}$ such that the Hausdorff dimension $s(\epsilon)$ of the limit set $K(\epsilon)$ of the perturbed system has the form $s(\epsilon) = s(0)+s_1\epsilon+\cdots+s_n\epsilon^n+o(\epsilon^n)$ as $\epsilon \to 0$ with $s(0) = \dim_H K$.*

Remark 3.3 Roy and Urbański [3] considered continuous perturbation of infinitely conformal iterated function systems given as a special CGDMS. They also studied analytic perturbation of CGDMS with $D \geq 3$ in [4]. We investigated an asymptotic perturbation of CGDMS with finite graph in [5, Theorem 1.1]. Theorem 3.2 is an infinite graph version of this previous result.

4 Main results

In order to investigate an asymptotic perturbation of the Gibbs measure associated with the dimension $\dim_H K(\epsilon)$ and of the measure-theoretic entropy of this measure, we further introduce the following condition:

$$(G.3)_n \ \sup_{\epsilon>0}\sup_{e\in E}\sup_{x,y\in O_{t(e)}:x\neq y}\frac{|\frac{\partial}{\partial x}\tilde{T}_{e,n}(\epsilon,x)-\frac{\partial}{\partial x}\tilde{T}_{e,n}(\epsilon,y)|}{|x-y|^\beta}<\infty.$$

Note that such a condition is firstly given in [5] (see the condition $(G)'_n$ in this paper). We denote the physical potential for the perturbed CGDMS by

$$\varphi(\epsilon,\cdot)=\log\|\tfrac{\partial}{\partial x}T_{\omega_0}(\epsilon,\pi(\epsilon,\sigma\omega))\| \tag{4.1}$$

for $\omega\in E^\infty$, where $\pi(\epsilon,\cdot)$ means the coding map of $K(\epsilon)$ which is defined by $\pi(\epsilon,\omega)=\bigcap_{n=0}^\infty T_{\omega_0}(\epsilon,\cdot)\cdots T_{\omega_n}(\epsilon,J_{t(\omega_n)})$ for $\omega\in E^\infty$.

Now we are in a position to state our main results.

Theorem 4.1 *Assume that the conditions $(G.1)_n$-$(G.3)_n$ are satisfied with fixed integer $n\geq 0$. Then there exist bounded functionals $\mu_1,\dots,\mu_n$ in the dual $(F_{\theta,b}(E^\infty))^*$ of $F_{\theta,b}(E^\infty)$ such that the Gibbs measure $\mu(\epsilon,\cdot)$ of the potential $(\dim_H K(\epsilon))\varphi(\epsilon,\cdot)$ satisfies the n-order asymptotic expansions $\mu(\epsilon,f)=\mu(f)+\mu_1(f)\epsilon+\cdots+\mu_n(f)\epsilon^n+o(\epsilon^n)$ as $\epsilon\to 0$ for $f\in F_{\theta,b}(E^\infty)$, where μ is the Gibbs measure of $(\dim_H K(0))\varphi$.*

Theorem 4.2 *Assume that the conditions $(G.1)_n$-$(G.3)_n$ are satisfied with fixed integer $n\geq 1$. Then there exist numbers $H_1,\dots,H_{n-1}\in\mathbb{R}$ such that the measure-theoretic entropy $h_\sigma(\mu(\epsilon,\cdot))$ of the Gibbs measure $\mu(\epsilon,\cdot)$ of the potential $(\dim_H K(\epsilon))\varphi(\epsilon,\cdot)$ satisfies the asymptotic expansions $h_\sigma(\mu(\epsilon,\cdot))=h_\sigma(\mu)+H_1\epsilon+\cdots+H_{n-1}\epsilon^{n-1}+o(\epsilon^{n-1})$ as $\epsilon\to 0$, where μ is the Gibbs measure of $(\dim_H K(0))\varphi$.*

Remark 4.3 (1) Those theorems are infinite graph versions of [5, Theorem 1.3] which considered asymptotic expansions of the Gibbs measure and of this entropy in the case $\sharp E<+\infty$.

(2) We expect the expansion of the entropy in Theorem 4.2 to expand to the length of n, which is still an open problem.

To show Theorem 4.1 and Theorem 4.2, we need some lemmas as follows. We start with the asymptotic expansion of the perturbed coding map $\pi(\epsilon,\cdot)$.

Lemma 4.4 *Assume that the conditions $(G.1)_n$ and $(G.2)_n$ are satisfied. Choose any $r_1\in(r,1)$. Then there exist functions $\pi_1,\pi_2,\dots,\pi_n\in F_{r_1,b}(E^\infty,\mathbb{R}^D)$ and $\tilde{\pi}(\epsilon,\cdot)\in C_b(E^\infty,\mathbb{R}^D)$ such that $\pi(\epsilon,\cdot)=\pi+\pi_1\epsilon+\cdots+\pi_n\epsilon^n+\tilde{\pi}_n(\epsilon,\cdot)\epsilon^n$ and $\sup_{\omega\in E^\infty}|\tilde{\pi}_n(\epsilon,\omega)|\to 0$ as $\epsilon\to 0$.*

Proof. See [8, Lemma 3.13] and [5, Lemma 3.1]. □

Lemma 4.5 *Assume that the conditions $(G.1)_n$-$(G.3)_n$ are satisfied. Then for any number $r_2\in(r_1,1)$, $\limsup_{\epsilon\to 0}[\tilde{\pi}_n(\epsilon,\cdot)]_{r_2}<\infty$.*

Proof. This follows from [5, Lemma 3.3] replacing $\max_{e\in E}$ by $\sup_{e\in E}$. □

We define an operator $\mathcal{L}_k : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$ by $\mathcal{L}_k f = \mathcal{L}_{s(0)\varphi}(G_k f)$ with

$$G_k = \sum_{q=0}^{k} \frac{s_{q,k}}{q!}\varphi^q + \sum_{v=1}^{k}\sum_{q=0}^{k-v}\sum_{l=0}^{v}\sum_{j=0}^{\min(l,q)} \frac{s_{q,k-v}\cdot a_{l,j}}{(q-j)!}\varphi^{q-j} l! \sum_{\substack{j_1,\dots,j_v\ge 0:\\ j_1+\cdots+j_v=l\\ j_1+2j_2+\cdots+vj_v=v}} \prod_{u=1}^{v}\Big(\frac{(g_u)^{j_u}}{j_u! g^{j_u}}\Big),$$

where $a_{l,j}$ and $s_{q,k-v}$ are given by expanding $\binom{p}{l} = a_{l,0} + \sum_{j=1}^{l} a_{l,j}(p-s)^j$ with $a_{l,0} = \binom{s}{l}$ of binomial coefficient and $(s(\epsilon)-s(0))^k = \sum_{i=0}^{n} s_{k,i}\epsilon^i + o(\epsilon^n)$, respectively. Here $g, g_1, \dots, g_n, g(\epsilon,\cdot)$ satisfy the asymptotic expansion $g(\epsilon,\cdot) = g + \sum_{k=1}^{n} g_k\epsilon^k + \tilde{g}_n(\epsilon,\cdot)\epsilon^n$ with $g(\epsilon,\omega) := \det(\frac{\partial}{\partial x}T_{\omega_0}(\epsilon,\pi(\epsilon,\sigma\omega)))$ and $g(\omega) := \det(T'_{\omega_0}(\pi\sigma\omega))$. Observe the equations $\varphi(\epsilon,\cdot) = (1/D)\log|g(\epsilon,\cdot)|$ and $\varphi = (1/D)\log|g|$. We have the following:

Lemma 4.6 *Assume that the conditions $(G.1)_n$-$(G.3)_n$ are satisfied. Then each $\mathcal{L}_k$ is bounded acting on $F_{\theta,b}(E^\infty)$ with $\theta = r_2^\beta$ and the Ruelle operator $\mathcal{L}(\epsilon,\cdot)$ of the potential $s(\epsilon)\varphi(\epsilon,\cdot)$ has the expansion $\mathcal{L}(\epsilon,\cdot) = \mathcal{L} + \sum_{k=1}^{n}\mathcal{L}_k\epsilon^k + \tilde{\mathcal{L}}_n(\epsilon,\cdot)\epsilon^n$ with $\|\tilde{\mathcal{L}}_n(\epsilon,\cdot)\|_\infty \to 0$, where $\mathcal{L}$ is the Ruelle operator of $s\varphi$.*

Proof. This expansion is yielded by [8, (3.31)], and convergence of $\tilde{\mathcal{L}}_n(\epsilon,\cdot)$ is guaranteed by the proof of [8, Theorem 1.1]. □

Lemma 4.7 *Assume that the conditions $(G.1)_n$-$(G.3)_n$ are satisfied. Let $\theta = r_2^\beta$, where r_2 is given in Lemma 4.5. Then the potential $\varphi(\epsilon,\cdot)$ which appears in Lemma 4.6 satisfies* $\limsup_{\epsilon\to 0}[\tilde{\varphi}(\epsilon,\cdot)]_\theta < +\infty$. *In particular,* $\limsup_{\epsilon\to 0}\|\tilde{\mathcal{L}}_n(\epsilon,\cdot)\|_\theta < +\infty$.

Proof. This is due to [9, Lemma 3.15]. □

(*Proof of Theorem 4.1*). The convergence of the reminder $\tilde{\mu}_n(\epsilon,f)$ of the perturbed Gibbs measure $\mu(\epsilon,f)$ of $f\in F_{\theta,b}(E^\infty,\mathbb{R})$ is obtained by [9, Theorem 3.5] together with Lemma 4.6 and Lemma 4.7. □

Lemma 4.8 *Assume that the conditions $(G.1)_n$-$(G.3)_n$ with $n \ge 1$ are satisfied. Then there exists $\xi_1,\dots,\xi_{n-1},\tilde{\xi}_{n-1}(\epsilon,\cdot)\in F_{\theta,b}(E^\infty,\mathbb{R})$ such that*

$$\mathcal{L}(\epsilon, h(\epsilon,\cdot)\varphi(\epsilon,\cdot)) = \mathcal{L}\varphi + \xi_1\epsilon + \cdots + \xi_{n-1}\epsilon^{n-1} + \tilde{\xi}_{n-1}(\epsilon,\cdot)\epsilon^{n-1}$$

and $\|\tilde{\xi}_{n-1}(\epsilon,\cdot)\|_\infty \to 0$, where $h(\epsilon,\cdot)$ is the eigenfunction given in Theorem 2.1 for the Ruelle operator $\mathcal{L}(\epsilon,\cdot)$ of the potential $s(\epsilon)\varphi(\epsilon,\cdot)$.

Proof. Consider the form

$$\mathcal{L}(\epsilon, h(\epsilon,\cdot)\varphi(\epsilon,\cdot))(\omega) = \sum_{e\in E\,:\,A(e\omega_0)=1} |g(\epsilon, e\cdot\omega)|^{s(\epsilon)/D} \log|g(\epsilon, e\cdot\omega)| h(\epsilon, e\cdot\omega)\frac{s(\epsilon)}{D}.$$

We can give an $(n-1)$-order asymptotic expansion of $|g(\epsilon, a\cdot\omega)|^{s(\epsilon)/D}\log|g(\epsilon, a\cdot\omega)|$. In addition to asymptotic expansions of $h(\epsilon,\cdot)$ and of $s(\epsilon)$, the assertion follows from this expansion. □

(*Proof of Theorem 4.2*). Recall $h_\sigma(\mu(\epsilon,\cdot)) = P(s(\epsilon)\varphi(\epsilon,\cdot)) - \mu(\epsilon, s(\epsilon)\varphi(\epsilon,\cdot))$ and $\mu(\epsilon,\cdot) = h(\epsilon,\cdot)\nu(\epsilon,\cdot)$, where $\nu(\epsilon,\cdot)$ is the eigenvector given in Theorem 2.1 for the Ruelle operator $\mathcal{L}(\epsilon,\cdot)$. The pressure $P(s(\epsilon)\varphi(\epsilon,\cdot))$ equals 0 by Bowen's formula. We have

$$\begin{aligned}\mu(\epsilon, s(\epsilon)\varphi(\epsilon,\cdot)) =& s(\epsilon)\nu(\epsilon, \mathcal{L}(\epsilon, h(\epsilon,\cdot)\varphi(\epsilon,\cdot)))\\ \mathcal{L}(\epsilon, h(\epsilon,\cdot)\varphi(\epsilon,\cdot)) =& \mathcal{L}(h\varphi) + \xi_1\epsilon + \cdots + \xi_{n-1}\epsilon^{n-1} + \tilde{\xi}_{n-1}(\epsilon,\cdot)\epsilon^{n-1}\end{aligned}$$

with $\|\tilde{\xi}_{n-1}(\epsilon,\cdot)\|_\infty \to 0$ by Lemma 4.8. Moreover, we see

$$\begin{aligned}\nu(\epsilon, \mathcal{L}(\epsilon, h(\epsilon,\cdot)\varphi(\epsilon,\cdot))) =& \nu(\epsilon, \mathcal{L}(h\varphi) + \xi_1\epsilon + \cdots + \xi_{n-1}\epsilon^{n-1} + \tilde{\xi}_{n-1}(\epsilon,\cdot)\epsilon^{n-1})\\ =& \sum_{k=0}^{n-1}\sum_{i,j\geq 0\,:\,i+j=k} \nu_i(\xi_j)\epsilon^k + (\sum_{i=0}^{n-1}\tilde{\nu}_i(\epsilon, \xi_{n-i-1}) + \nu(\epsilon, \tilde{\xi}_{n-1}(\epsilon,\cdot)))\epsilon^{n-1},\end{aligned}$$

$\tilde{\nu}_i(\epsilon, \xi_{n-i-1}) \to 0$ and $|\nu(\epsilon, \tilde{\xi}_{n-1}(\epsilon,\cdot))| \leq \|\tilde{\xi}_{n-1}(\epsilon,\cdot)\|_\infty \to 0$. Thus $\mu(\epsilon, \varphi(\epsilon,\cdot))$ has an $(n-1)$-order asymptotic expansion in addition to the expansion of $s(\epsilon)$. Hence the entropy $h_\sigma(\mu(\epsilon,\cdot))$ behaves asymptotically with order $n-1$. □

5 Concrete example: countable linear IFS

Let $a > 1$ be a number, $V := \{v\}$ and $E := \{1, 2, \dots\}$. We take an infinite graph $G = (V, E, i(\cdot), t(\cdot))$ with $i(e) = t(e) = v$ for $e \in E$, and two intervals $J_v = [0,1]$ and $O_v = (-\eta, 1+\eta)$ for a small $\eta > 0$. For $e \in E$ and $\epsilon \geq 0$, we define a function $T_e(\epsilon,\cdot)$ by

$$T_e(\epsilon, x) = \Big(\frac{1}{5^e} + \frac{1}{a^e}\epsilon\Big)x + b(e).$$

Here we choose $b(e)$ so that the set $(G, (J_v), (O_v), T_e(\epsilon,\cdot))$ satisfies the conditions of the CGDMS for any small $\epsilon > 0$. It is not hard to check that the condition $(G.1)_n$ is valid with $T_e(x) = x/5^e + b(e)$, $T_{e,1}(x) = x/a^e$ and $T_{e,n} \equiv 0$ for $n \geq 2$. Therefore, $T_e(\epsilon,\cdot)$ has the n-order asymptotic expansion with $\tilde{T}_{e,n}(\epsilon,\cdot) \equiv 0$ for any $n \geq 1$. Moreover, the topological pressure of $s\varphi(\epsilon,\cdot)$ has the equation

$$P(s\varphi(\epsilon,\cdot)) = \log\sum_{e\in E}\left(\frac{1}{5^e} + \frac{1}{a^e}\epsilon\right)^s$$

for $s > 0$. Therefore, $P(s\varphi) = 0$ if and only if $\sum_{e\in E}(1/5^e)^s = 1$ if and only if $s = \log 2/\log 5 =: s(0)$. Moreover, $\underline{s} = \inf\{s : P(s\varphi) < +\infty\}$ is equal to 0.

Let $s(\epsilon) = \dim_H K(\epsilon)$ be the Hausdorff dimension of the limit set $K(\epsilon)$ of the perturbed CGDMS $(G, (J_v), (O_v), (T_e(\epsilon, \cdot)))$ for $\epsilon \geq 0$. The Ruelle operator $\mathcal{L}$ of $s(0)\varphi$ is given by

$$\mathcal{L}f(\omega) = \sum_{e\in E} \frac{1}{2^e} f(e\cdot\omega).$$

The Ruelle operator $\mathcal{L}(\epsilon, \cdot)$ of $s(\epsilon)\varphi(\epsilon, \cdot)$ has the form

$$\mathcal{L}(\epsilon, f)(\omega) = \sum_{e\in E}\left(\frac{1}{5^e} + \frac{\epsilon}{a^e}\right)^{s(\epsilon)} f(e\cdot\omega).$$

Then we summarize results for $(T_e(\epsilon, \cdot))$ as follows.

Proposition 5.1 *Assume the above conditions for $T_e(\epsilon, \cdot)$. Then we have the following:*

(1) If (λ, h, ν) is the Perron spectral triplet of $\mathcal{L}$, i.e. $\mathcal{L}h = \lambda h$, $\mathcal{L}^*\nu = \lambda\nu$, $\nu(h) = \nu(E^\infty) = 1$, then $\lambda = 1$, $h \equiv 1$ and ν satisfies $\nu([e]) = 1/2^e$ for $e \in E$.

(2) If $(\lambda(\epsilon), h(\epsilon, \cdot), \nu(\epsilon, \cdot))$ is the Perron spectral triplet of $\mathcal{L}(\epsilon, \cdot)$, i.e. $\mathcal{L}(\epsilon, h(\epsilon, \cdot)) = \lambda(\epsilon)h(\epsilon, \cdot)$, $\mathcal{L}(\epsilon, \cdot)^*\nu(\epsilon, \cdot) = \lambda(\epsilon)\nu(\epsilon, \cdot)$, $\nu(\epsilon, h(\epsilon, \cdot)) = \nu(\epsilon, E^\infty) = 1$, then $\lambda(\epsilon) \equiv 1$, $h(\epsilon, \cdot) \equiv 1$ and ν satisfies $\nu(\epsilon, [e]) = (1/5^e + \epsilon/a^e)^{s(\epsilon)}$ for $e \in E$.

(3) The Ruelle operator $\mathcal{L}(\epsilon, \cdot)$ of the potential $s(\epsilon)\varphi(\epsilon, \cdot)$ has n-order asymptotic expansion $\mathcal{L}(\epsilon, \cdot) = \mathcal{L} + \mathcal{L}_1\epsilon + \cdots + \mathcal{L}_n\epsilon^n + \tilde{\mathcal{L}}_n(\epsilon, \cdot)e^n$ with

$$\mathcal{L}_k f(\omega) = \sum_{e=1}^{\infty}\sum_{v=0}^{k}\sum_{q=0}^{k-v}\sum_{j=0}^{\min(v,q)} \frac{s_{q,k-v}\cdot a_{v,j}}{(q-j)!}(-e\log 5)^{q-j}\left(\frac{5^v}{2a^v}\right)^e f(e\cdot\omega),$$

where $s_{q,k-v}$ and $a_{v,j}$ are decided by

$$(s_1\epsilon + \cdots + s_{k-1}\epsilon^{k-1})^q = s_{q,0} + s_{q,1}\epsilon + \cdots + s_{q,n}\epsilon^n + o(\epsilon^n)$$
$$\binom{t}{v} = a_{v,0} + a_{v,1}(t - s(0)) + \cdots + a_{v,v}(t - s(0))^v \quad (a_{0,0} := 1).$$

(4) The Perron eigenvector $\nu(\epsilon, \cdot)$ of $\mathcal{L}(\epsilon, \cdot)^*$ has the form $\nu(\epsilon, f) = \nu(f) + \nu_1(f)\epsilon + \cdots + \nu_n(f)\epsilon^n + o(\epsilon^n)$ for each $f \in F_{\theta,b}(E^\infty)$, where

$$\nu_k(f) = \sum_{j=1}^{k}\sum_{\substack{i_1,\ldots,i_j\geq 1:\\ i_1+\cdots+i_j=k}} (-1)^j\nu(\mathcal{L}_{i_1}\mathcal{S}\cdots\mathcal{L}_{i_j}\mathcal{S}f)$$

and $\mathcal{S}f = (\mathcal{L} - 1\otimes\nu - \mathcal{I})^{-1}(\mathcal{I} - 1\otimes\nu)$.

Theorem 5.2 *Assume the above conditions for $T_e(\epsilon,\cdot)$. Then we have the following:*

(1) If $a \geq 5$ then the Hausdorff dimension $s(\epsilon) = \dim_H K(\epsilon)$ of the limit set of this CGDMS has n-asymptotic expansion $s(\epsilon) = s(0) + s_1\epsilon + \cdots + s_n\epsilon^n + \tilde{s}_n(\epsilon)\epsilon^n$ with $\tilde{s}_n(\epsilon) \to 0$ for any $n \geq 0$. Each coefficient s_k $(k = 1, 2, \ldots, n)$ is decided as

$$s_k = \sum_{e=1}^{\infty} \sum_{\substack{0\le v\le u, 0\le q\le u-v:\\ (v,q)\neq(0,1)}} \sum_{j=0}^{\min(v,q)} \frac{s_{q,u-v}\cdot a_{v,j}}{(q-j)!}(-e\log 5)^{q-j}\left(\frac{5^v}{2a^v}\right)^e. \tag{5.1}$$

In particular,

$$s_1 = \frac{\log 2}{(\log 5)^2}\frac{5}{4a-10}$$
$$s_2 = \frac{25\log 2}{(\log 5)^3}\left(\frac{1}{2(2a-5)^2} - \frac{a\log 2}{(2a-5)(4a^2-5)^2} + \frac{\log(2/5)}{8a^2-100}\right).$$

(2) If $1 < a < 5$ then take the largest integer $k \geq 0$ satisfying $a \leq 5/2^{1/(k+1)}$. In this case, $s(\epsilon)$ has the form

$$s(\epsilon) = \begin{cases} s(0) + s_1\epsilon + \cdots + s_k\epsilon^k + \hat{s}(\epsilon)\epsilon^{k+1}\log\epsilon, & a = 5/2^{1/(k+1)} \text{ for some } k \geq 0 \\ s(0) + s_1\epsilon + \cdots + s_k\epsilon^k + \hat{s}(\epsilon)\epsilon^{\frac{\log 2}{\log(5/a)}}, & \text{otherwise} \end{cases}$$

with $|\hat{s}(\epsilon)| \asymp 1$ as $\epsilon \to 0$, where each s_i is equal to (5.1) and $|\hat{s}(\epsilon)| \asymp 1$ as $\epsilon \to 0$ means $c^{-1} \leq |\hat{s}(\epsilon)| \leq c$ for any small $\epsilon > 0$ for some constant $c \geq 1$.

(3) For $a > 1$, the Gibbs measure $\mu(\epsilon,\cdot)$ of the potential $s(\epsilon)\varphi(\epsilon,\cdot)$ has the n-order asymptotic expansion $\mu(\epsilon, f) = \mu(f) + \mu_1(f)\epsilon + \cdots + \mu_n(f)\epsilon^n + o(\epsilon^n)$ for each $f \in F_{\theta,b}(E^\infty)$ with the coefficient

$$\mu_k(f) = \sum_{j=1}^{k} \sum_{\substack{i_1,\ldots,i_j\geq 1:\\ i_1+\cdots+i_j=k}} (-1)^j \mu(\mathcal{L}_{i_1}\mathcal{S}\cdots\mathcal{L}_{i_j}\mathcal{S}f).$$

(4) For $a > 1$, the measure-theoretic entropy $h_\sigma(\mu(\epsilon,\cdot))$ of $\mu(\epsilon,\cdot)$ has the n-order asymptotic expansion $h_\sigma(\mu(\epsilon,\cdot)) = h_\sigma(\mu) + H_1\epsilon + \cdots + H_n\epsilon^n + o(\epsilon^n)$ with

$$H_k = -\sum_{i_1,\ldots,i_4\geq 0\,:\, i_1+\cdots+i_4=k} s_{i_1}\mu_{i_2}(\mathcal{L}_{i_3}\varphi_{i_4})$$

and $\varphi_k(\omega) = ((-1)^{k-1}/k)\,(5/a)^{\omega_0}$.

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