Limit Theorems for Renewal Hawkes Processes

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1 Introduction

Hawkes processes are point processes in which the ocurrence of new events is facilitated by the ocurrence of previous events, which is why they are often known as *self-exciting* processes. In Bacry–Delattre–Hoffmann–Muzy [2], a Law of Large Numbers and Central Limit Theorem for the classical (multivariate) Hawkes process have been established. We can understand the Hawkes process as a cluster process Hawkes–Oakes [7] in which the cluster centres are given by an *immigrant process* which consists of an homogeneous Poisson process of constant intensity, and satellite processes or *offspring processes* that consist of inhomogeneous Poisson processes originated at each one of the previous points.

The renewal Hawkes process (RHP) is a generalization of the classical one in which the immigrants arrive according to a renewal process. In Wheatley–Filimonov–Sornette [13], the (RHP) was proposed, and the evaluation of the likelihood through an EM algorithm was studied. In Chen–Stindl [3] and Chen–Stindl [4], the evaluation of the maximum likelihood was revisited and refined to be more quickly calculated. In this article, we summarize our results of construction of such a process and of two limit theorems, namely, a Law of Large Numbers and a Central Limit Theorem.

2 Classical and renewal Hawkes processes

Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of random variables on $[0,\infty)$ such that, for all $i \in \mathbb{N}$, $T_i \leq T_{i+1}$ and $T_i < T_{i+1}$ on the event $\{T_i < \infty\}$. We identify the *point process* $\{T_n\}_{n\in\mathbb{N}}$ with the associated counting process $N(t) = \sum_i 1_{\{T_i \leq t\}}$ for $t \geq 0$. Let $(\mathcal{F}_t)$ be a filtration to which N is adapted. An $(\mathcal{F}_t)$-*intensity* of N is a nonnegative, locally integrable process $\lambda(t)$ that is $(\mathcal{F}_t)$-progressive, and such that,

$$M(t) = N(t) - \int_0^t \lambda(s)\mathrm{d}s,$$

is an $(\mathcal{F}_t)$-martingale, which we call the *characteristic martingale*. It is always possible to find a predictable version of the $(\mathcal{F}_t)$-intensity, in which case it is unique up to modification. We can then define the classical Hawkes process by specifying its intensity.

Definition 2.1. A point process N is called a *classical Hawkes process* if N admits an $(\mathcal{F}_t)$-intensity,

$$\lambda(t) = \mu + \int_0^{t-} h(t-u)N(\mathrm{d}u), \tag{2.1}$$

where μ is a positive constant and h is a nonnegative function on $[0,\infty)$ such that $\int_0^\infty h(t)\mathrm{d}t < 1$.

In Bacry–Delattre–Hoffmann–Muzy [2], a uniform Law of Large Numbers (LLN) was established for classical multivariate Hawkes processes. In the case of one dimension, if $\alpha := \int_0^\infty h(t)\mathrm{d}t < 1$, this result takes the form,

Theorem 2.2. *For all $t \geq 0$, we have $N(t) \in L^2(\mathbb{P})$ and,*

$$\sup_{v\in[0,1]} \left| T^{-1}N(Tv) - v\frac{\mu}{1-\alpha} \right| \underset{T\to\infty}{\longrightarrow} 0, \tag{2.2}$$

a.s. and in $L^2(\mathbb{P})$.

A Central Limit Theorem was also proved, namely,

Theorem 2.3. *The processes*

$$\left(\frac{1}{\sqrt{T}}(N_{Tv} - \mathbb{E}[N_{Tv}])\right)_{v\in[0,1]} \overset{d}{\underset{T\to\infty}{\longrightarrow}} \left(\sqrt{\frac{\mu}{(1-\alpha)^3}}W_v\right)_{v\in[0,1]}, \tag{2.3}$$

where $(W_v)_{v\in[0,1]}$ is a standard Brownian motion.

We proceed then to define the renewal Hawkes process through its intensity function and we adopt a martingale approach similar to Bacry–Delattre–Hoffmann–Muzy [2] for the classical Hawkes process to establish the limit theorems. Let us consider a process describing the arrival of "immigrants" and their respective "offspring". We introduce the random variables D_i, $i = 1, 2, \ldots$, that take the values $D_i = 0$ if the i-th point is an immigrant, and $D_i = 1$ if it corresponds to offspring. The random variable $I(t) = \max\{i;\ T_i \leq t, D_i = 0\}$ represents the index of the last immigrant. We consider an enhanced filtration $(\widetilde{\mathcal{F}}_t)$ where $\widetilde{\mathcal{F}}_t = \sigma\{N(s), I(s);\ s \leq t\}$. Additionally, consider a function h satisfying the assumption:

(A0) h is a measurable function satisfying $h(t) \geq 0$ for $t \geq 0$, and $\alpha := \int_0^\infty h(t)\mathrm{d}t < 1$.

In this context, we introduce the following definition:

Definition 2.4. A point process N is called a *renewal Hawkes process* (RHP) if N admits the $(\widetilde{\mathcal{F}}_t)$-intensity,

$$\lambda(t) = \mu\big(t - T_{I(t)}\big) + \int_0^{t-} h(t-u)N(\mathrm{d}u), \tag{2.4}$$

where h satisfies **(A0)** and μ is a function on $[0,\infty)$ given as

$$\mu(t) = \frac{f(t)}{1 - \int_0^t f(s)\mathrm{d}s}, \tag{2.5}$$

for some probability density function f on $[0,\infty)$.

3 Construction of the RHP

First we define the renewal process part. Let $\tau, \tau_1, \tau_2, \ldots,$ be positive i.i.d. random variables whose probability distribution function

$$F(t) := \mathbb{P}(\tau \leq t), \tag{3.1}$$

has zero mass at the origin (so that the process is orderly), and satisfies the assumption:

(B0) F has a density f, i.e. $F(t) = \int_0^t f(s)\mathrm{d}s$. Moreover, τ has finite mean, i.e. $m^{-1} := \mathbb{E}[\tau] = \int_0^\infty sF(\mathrm{d}s) < \infty$.

Take the hazard function $\mu(t)$ as in (2.5). Define the partial sums $S_0 = 0$, $S_n = \tau_1 + \cdots + \tau_n$; the associated counting process will be denoted by $N_R(t) = \sum_i 1_{\{S_i \leq t\}}$.

The RHP can be thought of as a cluster process in which the cluster centre process corresponds to a renewal process of immigrants, $N_c^{(0)}(\cdot) = N_R(\cdot)$. To each point in the centre process, corresponds a cluster, that consists of a branching process generated by the offspring of the immigrant and all the subsequent generations of their offspring. The construction is as follows:

Consider $\left\{N_s^{(n)}(\cdot \mid t);\ t \in \mathbb{R}_+, n \geq 1\right\}$ a (symbolic) measurable family of i.i.d. point processes and independent of $N_c^{(0)}$, such that for each $t \in \mathbb{R}_+$ and $n \geq 1$, $N_s^{(n)}(\cdot \mid t)$ is an inhomogeneous Poisson processes of characteristic intensity $h(\cdot + t)$. Given that there is a centre at $t_0 \geq 0$, we construct higher-level centre processes $N_c^{(n)}(\cdot \mid t_0)$ for $n \geq 1$ from a superposition of the processes $N_s^{(n)}$ with the following recursive structure:

$$N_c^{(0)}(\cdot \mid t_0) := \delta_{t_0}, \quad N_c^{(n+1)}(\cdot \mid t_0) = \sum_{t \in N_c^{(n)}(\cdot \mid t_0)} N_s^{(n+1)}(\cdot \mid t), \tag{3.2}$$

where $N_c^{(0)}(\cdot \mid t_0)$ is the original immigrant at t_0 and $N_c^{(n)}(\cdot \mid t_0)$ represents its n-th generation offspring. We define as well some processes of interest, namely, the total number of n-th generation descendants,

$$N_c^{(n)}(\cdot) = \sum_{t_0 \in N_R(\cdot)} N_c^{(n)}(\cdot \mid t_0), \tag{3.3}$$

and the complete offspring of the imimigrant at t_0 (including the immigrant),

$$N_c(\cdot \mid t_0) = \sum_{n \geq 0} N_c^{(n)}(\cdot \mid t_0). \tag{3.4}$$

We take the processes defined as in (3.4) as the satellite processes of our construction for a centre located at t_0. Finally, the RHP is given by the superposition:

$$N(\cdot) = \int_0^\infty N_c(\cdot \mid t) N_R(\mathrm{d}t) = \sum_{t_0 \in N_R(\cdot)} \sum_{n \geq 0} N_c^{(n)}(\cdot \mid t_0). \tag{3.5}$$

Note that (3.5) can also be written as

$$N(\cdot) = \sum_{n \geq 0} \sum_{t_0 \in N_R(\cdot)} N_c^{(n)}(\cdot \mid t_0) = N_R(\cdot) + \sum_{n \geq 1} N_c^{(n)}(\cdot). \tag{3.6}$$

3.1 Notation

If f and g are both functions, we will denote their convolution as

$$f * g(t) = \int_0^t f(t-s)g(s)\mathrm{d}s, \tag{3.7}$$

whereas, if F is a measure and g is a function, the convention that $F * g$ is a function is used, and we write,

$$F * g(t) = \int_0^t g(t-s)F(\mathrm{d}s). \tag{3.8}$$

We sometimes identify the measure $F(\mathrm{d}s)$ with its cumulative distribution function $F(t) = \int_0^t F(\mathrm{d}s)$. If F and G are both measures, we will denote their convolution as

$$F * G(t) = \int_0^t F(t-s)G(\mathrm{d}s) = \int_0^t G(t-s)F(\mathrm{d}s). \tag{3.9}$$

Associated with the renewal process $N_R(\cdot)$, we have a function $\Phi : [0,\infty) \to [0,\infty)$, such that $\Phi(t) := \mathbb{E}[N_R(t)]$ for all $t \geq 0$. This function can be expressed as $\Phi(t) = \sum_{n\geq 1} F^{*n}(t)$, where F^{*n} denotes the n-fold convolution of F with itself (see for example [5, Chapter 4.1, p. 67]). If the distribution $F(t) = \int_0^t f(s)\mathrm{d}s$, then the *renewal measure* $\Phi(\mathrm{d}t) = \sum_{n\geq 1} F^{*n}(\mathrm{d}t)$ is absolutely continuous and has a density φ called the *renewal density*, given by $\varphi(t) = \sum_{n\geq 1} f^{*n}(t)$ for all $t \geq 0$ (see for example [1, Sec V, Proposition 2.7, p. 148]). In addition, we denote $\psi(t) = \sum_{n\geq 1} h^{*n}(t)$, and $\Psi(t) = \int_0^t \psi(s)\mathrm{d}s$.

3.2 Existence result

Lemma 3.1 (Hernández–Yano [10]). *If $N_R(\cdot)$ is a renewal process satisfying* **(B0)** *and h satisfies* **(A0)***, then the cluster process defined as in* (3.6) *exists and has a.s. finite clusters.*

For the proof of this result we use a result by Westcott [12, Corollary 3.3].

We can construct the cluster process (3.6) by using Lemma 3.1 if we check the following conditions,

(i) $\sup_t \mathbb{E}[N_c(I-t)] < \infty$ for all bounded interval I.

(ii) $N_s(\cdot \mid t) \overset{\mathrm{d}}{=} N_s(\cdot - t \mid 0)$ for all $t \in \mathbb{R}$.

(iii) $\mathbb{E}[N_s(\mathbb{R} \mid 0)] < \infty$.

Claim (i) can be obtained by using the subadditive properties of the renewal function [1, Sec. V, Theorem 2.4, p.146]. Claim (ii) follows from the construction of the clusters as inhomogeneous Poisson processes that originate at previous points of the process. Finally, to prove claim (iii), we notice that the total size of a cluster can be written as the total size of a Galton–Watson process. From the Galton–Watson theory [6, Theorem 6.1, p.7] we know that in this case, the mean cluster size is $\frac{1}{1-\alpha} < \infty$, concluding the proof.

4 Results for the mean number of events

The following expressions are key to the proofs of our limit theorems and they can be obtained by writing appropiate renewal equations.

Lemma 4.1 (**Hernández** [**9**]). *For any $t \geq 0$, the mean number of events $\mathbb{E}[N(t)]$ is given as,*

$$\mathbb{E}[N(t)] = \Phi(t) + \int_0^t \psi(t-s)\Phi(s)\mathrm{d}s. \tag{4.1}$$

This formula can be derived from the renewal type equation,

$$\mathbb{E}[N(t)] = \Phi(t) + \int_0^t h(t-s)\mathbb{E}[N(s)]\,\mathrm{d}s, \tag{4.2}$$

which is obtained from the defition of the intensity of N. In a similar way, we can also derive,

Lemma 4.2 (**Hernández** [**9**]). *Recall that $M(t) = N(t) - \int_0^t \lambda(s)\mathrm{d}s$ and set $A(t) := M(t) + \int_0^t \mu\big(s - T_{I(s)}\big)\,\mathrm{d}s - \Phi(t)$. Then, the process $X(t) = N(t) - \mathbb{E}[N(t)]$ satisfies,*

$$X(t) = A(t) + \int_0^t \psi(t-s)A(s)\mathrm{d}s, \tag{4.3}$$

for all $t \geq 0$.

5 Law of Large Numbers

In order to prove the LLN for the RHP, we make use of the renewal Theorems. For this, it is necessary to introduce the following assumptions.

(A1) The function h in (2.4) is bounded and $h(t) \underset{t\to\infty}{\longrightarrow} 0$.

(A2) The integral $\int_0^\infty x^{r'}\, h(x)\mathrm{d}x$ is finite for some $r' > 1$.

(B1) The integral $\int_0^\infty x^{r''}\, F(\mathrm{d}x)$ is finite for some $r'' > 1$.

Remark. If we take $r = \min\{r', r''\}$, then both **(A2)** and **(B1)** hold for r. Also **(B1)** implies **(B0)**.

We can now state the LLN,

Theorem 5.1 (**Hernández** [**9**]). *Assume* **(A0, A1, A2)** *and* **(B1)**. *Then,*

$$\sup_{v\in[0,1]} \left| T^{-1}N(Tv) - v\frac{m}{1-\alpha} \right| \underset{T\to\infty}{\longrightarrow} 0, \tag{5.1}$$

almost surely.

The proof of this Theorem is obtained by observing that,

$$\sup_{v\in[0,1]} |X(Tv)| \leq \sup_{t\leq T} |A(t)| \left(1 + \int_0^\infty |\psi(s)| \, \mathrm{d}s\right), \tag{5.2}$$

and

$$\sup_{t\leq T} |A(t)| \leq \sup_{t\leq T} |M(t)| + \sup_{t\leq T} \left| \int_0^t \mu(s - T_{I(s)}) \mathrm{d}s - \Phi(t) \right|. \tag{5.3}$$

Then, the result follows from the folowing Proposition and subsequent Lemmas:

Proposition 5.2 (**Hernández** [**8**]). *Assume* $\int_0^\infty x^s F(\mathrm{d}x) < \infty$ *for some* $p > 1$. *Then, for* q *such that* $1 < q < s$,

(i) $\|\mathbb{P}_0(B_t \in \cdot) - F_0\|_{\text{t.v.}} = O(t^{-q})$ *as* $t \to \infty$.

(ii) $U_2[x, \infty) = O(x^{-q})$, $\quad u_1(x) = m + O(x^{-q})$ *as* $x \to \infty$

(iii) If z *is measurable and bounded with* $z(x) = O(x^{-r})$ *as* $x \to \infty$ *for some* $r > 0$, *then*

$$\Phi * z(x) = m \int_0^\infty z(y) \mathrm{d}y + O\big(x^{1-\min\{r,q\}}\big) \quad \textit{as } x \to \infty. \tag{5.4}$$

Lemma 5.3 (**Hernández** [**9**]). *Assume* **(A0, A1, A2)** *and* **(B1)**. *We have, for* $0 \leq p < \min\{r - 1, 1\}$,

$$T^p \sup_{v\in[0,1]} \left| \left(T^{-1}\mathbb{E}[N(Tv)] - v\frac{m}{1-\alpha} \right) \right| \underset{T\to\infty}{\longrightarrow} 0. \tag{5.5}$$

This result is proved by using the Key renewal Theorem and assumption **(A2)** to obtain the speed of convergence. Furthermore, we also prove,

Lemma 5.4 (**Hernández** [**9**]). *Under* **(B0)**, *we have almost surely that,*

$$\frac{1}{T} \sup_{t\leq T} \left| \int_0^t \mu(s - T_{I(s)}) \mathrm{d}s - \Phi(t) \right| \underset{T\to\infty}{\longrightarrow} 0, \tag{5.6}$$

which follows by noting that $\int_0^t \mu(s - T_{I(s)}) \mathrm{d}s$ can be written as a sum of i.i.d. random variables ξ_j and an overshoot,

$$\int_0^t \mu(s - T_{I(s)}) \mathrm{d}s = \sum_{j=1}^{N_R(t)} \int_{S_{j-1}}^{S_j} \mu(s - S_{j-1}) \mathrm{d}s + \int_{S_{N_R(t)}}^t \mu(s - S_{N_R(t)}) \mathrm{d}s \tag{5.7}$$

$$= \sum_{j=1}^{N_R(t)} \xi_j + \int_{S_{N_R(t)}}^t \mu(s - S_{N_R(t)}) \mathrm{d}s, \tag{5.8}$$

and then the result follows by appealing to the LLN for sums of i.i.d. random variables. Notice that we recover the result in [2] if we take the renewal process of immigrants as an homogeneous Poisson process of intensity μ.

6 Central Limit Theorem

For the proof of this theorem, we introduce an additional assumption:

(B2) τ has finite variance σ_f^2, i.e. $\int_0^\infty x^2 F(\mathrm{d}x) < \infty$.

The main objective of this last section is to prove,

Theorem 6.1 (Hernández [9]). *Under assumptions* **(A0,1,2)** *and* **(B2)**, *the processes*

$$\left(\frac{1}{\sqrt{T}}(N(Tv) - \mathbb{E}[N(Tv)])\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} (\sigma W(v))_{v\in[0,1]}\,, \tag{6.1}$$

where $(W(v))_{v\in[0,1]}$ *is a standard Brownian motion and*

$$\sigma = \frac{\sqrt{\sigma_M^2 + \sigma_R^2}}{(1-\alpha)}, \quad \sigma_M^2 = \frac{m}{1-\alpha}, \quad \frac{\sigma_R^2}{m} = 3 + m^2\sigma_f^2 - 2m\mathbb{E}\left[\tau\int_0^\tau \mu(s)\mathrm{d}s\right]. \tag{6.2}$$

For this purpose, we first obtain the results,

Lemma 6.2 (Hernández [8]). *Assume* **(B0)**. *Then we have,*

$$\left(\frac{1}{\sqrt{T}}\int_{S_{N_R(Tv)}}^{Tv} \mu(s - S_{N_R(Tv)})\mathrm{d}s\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} 0. \tag{6.3}$$

Moreover, under assumption **(B2)**, *we have,*

$$\left(\frac{1}{\sqrt{T}}(Tv - S_{N_R(Tv)})\right)_{v\in[0,1]} \xrightarrow[T\to\infty]{d} 0. \tag{6.4}$$

Lemma 6.3 (Hernández [9]). *Assume* **(A0, A1, A2)** *and* **(B2)**, *and set*

$$Q(t) := \int_0^t \mu\big(s - T_{I(s)}\big)\,\mathrm{d}s - \Phi(t). \tag{6.5}$$

Define for each $T > 0$,

$$Q^{(T)}(v) := \big(T^{-1/2}Q(Tv)\big)_{v\in[0,1]} \quad \text{and} \quad M^{(T)} := \big(T^{-1/2}M(Tv)\big)_{v\in[0,1]}\,. \tag{6.6}$$

Then,

$$\big(M^{(T)}, Q^{(T)}\big) \xrightarrow[T\to\infty]{d} \left(\sigma_M W, \sigma_R \widetilde{W}\right), \tag{6.7}$$

where $W = (W(v))_{v\in[0,1]}$ *and* $\widetilde{W} = (\widetilde{W}(v))_{v\in[0,1]}$ *are independent standard Brownian motions.*

The former is a direct application of [1, Proposition VI.4.7, p.183] to approximate the distribution of the maximum of a process. For the latter, we first decompose the process Q as a sum of a martingale $M_Q^{(T)}$ and both of the processes in (6.2) and we then make use of the result [11, Theorem 14.17, p.280] to find the limits of the martingales $\left(M^{(T)}, M_Q^{(T)}\right)$. Finally, we show that if $X^{(T)}(v) := T^{-1/2}(N(Tv) - \mathbb{E}[N(Tv)])$, then

$$\sup_{v\in[0,1]} \left| X^{(T)}(v) - \frac{1}{1-\alpha}M^{(T)}(v) - \frac{1}{1-\alpha}Q^{(T)}(v) \right| \xrightarrow[T\to\infty]{p} 0. \tag{6.8}$$

Then the limit process is given by the sum of two independent Brownian motions of known variances. This shows the convergence proposed by Theorem 6.1. Notice that once again we recover the result in [2] if we take the renewal process of immigrants as an homogeneous Poisson process of intensity μ.

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