ON EQUIVARIANT ASYMPTOTIC DIMENSION OF ACTIONS ON NON-COMPACT SPACES

防衛大学校 総合教育学群 数学教育室 知念 直紹
Naotsugu Chinen
Department of Mathematics,
National Defense Academy of Japan

愛媛大学 大学院理工学研究科 山内 貴光
Takamitsu Yamauchi
Graduate School of Science and Engineering,
Ehime University

1. Introduction

This note is an announcement of results in [8]. Let $\Gamma \curvearrowright X$ be a left action of a discrete group Γ on a topological space X by homeomorphisms, $\mathcal{F}$ a family of subgroups of Γ, and $N \in \mathbb{Z}_{\geq 0}$ (see Section 2 for undefined notations and terminology). Sawicki [21] defined equivariant asymptotic dimension $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X)$ of $\Gamma \curvearrowright X$ on a *compact* Hausdorff space X by means of N-$\mathcal{F}$-amenability. The notion of N-$\mathcal{F}$-amenability was introduced by Bartels, Lück and Reich [5, Theorem 1.2], [4, Assumption 1.4] to prove the Farrell-Jones conjecture for hyperbolic groups (see [2], [15]). In [14, Definition 4.6], being N-$\mathcal{F}$-amenable is said to be N-BLR for $\mathcal{F}$. For a free action $\Gamma \curvearrowright X$, eq-asdim$(\Gamma \curvearrowright X)$ is also called the amenability dimension of $\Gamma \curvearrowright X$ (see [22, Definition 9.2]).

The purposes of this note are the following.

(A) We extend $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X)$ to $\mathcal{F}$-ead$(\Gamma \curvearrowright X)$ of actions on (not necessarily compact) topological spaces (viewed in the theory of the topological dimension and geometric group theory), satisfying that
(A.1) $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X) = \mathcal{F}$-ead$(\Gamma \curvearrowright X)$ whenever X is compact (see Proposition 3.5), and
(A.2) $\operatorname{ead}(\Gamma \overset{\text{can.}}{\curvearrowright} \Gamma) = \operatorname{asdim}\Gamma$ (see Remark 5.11);

(B) We give a characterization theorem for $\mathcal{F}$-ead$(\Gamma \curvearrowright X)$ in terms of maps into $\ell_1(V)$, which is a generalization of [14, Proposition 4.5] due to Guentner, Willett and Yu;

In Section 2, we prepare notations and terminology. In Section 3, we recall the definition of $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X)$. With its definition in mind, we introduce

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the notion of $\mathcal{F}$-ead($\Gamma \curvearrowright X$) in (A), and give some properties of $\mathcal{F}$-ead($\Gamma \curvearrowright X$) including (A.1). In Section 4, we give an analogue of the well-known fact that $\dim X = \dim \beta X$ for $\mathcal{F}$-ead($\Gamma \curvearrowright X$). In Section 5, we recall the characterization of $\mathcal{F}$-eq-asdim($\Gamma \curvearrowright X$) in terms of conditions in [1, Theorem A, page 11] which is due to Guentner, Willett and Yu [14, Proposition 4.5]. Its characterization was applied to, for example, [22, Lemma 9.4] and [7, Lemma 4.2]. However, the authors think that a part of the proof of (ii) $\Rightarrow$ (i) of [14, Proposition 4.5] is unclear (see Remark 5.7). A characterization theorem for the purpose (B) is given in Theorem 5.3 with a sketch of a proof.

2. Preliminaries

Let $\mathbb{Z}$ denote the set of all integers. For $n \in \mathbb{Z}$, let $\mathbb{Z}_{\geq m} := \{i \in \mathbb{Z} \mid i \geq m\}$. The cardinal number of a set A is denoted by card$A$.

Convention 2.1. Throughout this paper, let Γ denote a nontrivial discrete group and X a nonempty topological space, and we assume that Γ acts on X, where by an *action* $\Gamma \curvearrowright X$ we mean a nontrivial homomorphism from Γ to the group of self-homeomorphisms of X. Each $\gamma \in \Gamma$ is regarded as a homeomorphism $\gamma : X \to X$ and the value of $x \in X$ under γ is denoted by γx.

Let 1_Γ denote the unit of Γ and $[\Gamma]^{<\omega}$ the collection of all finite subsets of Γ. The action $\Gamma \overset{\text{can.}}{\curvearrowright} \Gamma$ defined by $\gamma : \Gamma \to \Gamma; \eta \mapsto \gamma\eta$ for each $\gamma \in \Gamma$ is called the *canonical left action*.

For $A \subset X$ and a collection $\mathcal{U}$ of subsets of X, set

$$\mathcal{U}[A] := \{U \in \mathcal{U} \mid U \cap A \neq \emptyset\} \text{ and}$$
$$\mathrm{ord}(\mathcal{U}) := \sup_{x \in X} \mathrm{card}\{U \in \mathcal{U} \mid x \in U\}.$$

By a normal space, we mean a normal Hausdorff space. The covering dimension of a normal space X is denoted by $\dim X$ (see [11, Definition 1.6.7] or [10, p.385 and Theorem 7.1.7]).

The action $\Gamma \curvearrowright X \times Y$ defined by $\gamma(x, y) = (\gamma x, \gamma y)$ for $\gamma \in \Gamma$ and $(x, y) \in X \times Y$ is called the *diagonal action* induced by $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$.

Convention 2.2. We assume that $\Gamma \times X$ is equipped with the diagonal action $\Gamma \curvearrowright \Gamma \times X$ induced by the canonical left action $\Gamma \overset{\text{can.}}{\curvearrowright} \Gamma$ and the action $\Gamma \curvearrowright X$.

For $\gamma \in \Gamma$, $\Lambda \subset \Gamma$, $A \subset X$ and a collection $\mathcal{U}$ of subsets of X, let

$$\gamma A := \{\gamma a \mid a \in A\}, \quad \Lambda A := \bigcup_{\lambda \in \Lambda} \lambda A, \quad \text{and} \quad \Lambda\mathcal{U} := \{\lambda U \mid \lambda \in \Lambda, U \in \mathcal{U}\}.$$

For $A \subset X$, let Γ_A denote the stabilizer, i.e.,

$$\Gamma_A := \{\gamma \in \Gamma \mid \gamma A = A\}.$$

For $x \in X$, we write Γ_x instead of $\Gamma_{\{x\}}$.

Suppose that Γ also acts on a space Y. A continuous map $f : X \to Y$ is said to be Γ-*equivariant* if $f(\gamma x) = \gamma f(x)$ for each $(\gamma, x) \in \Gamma \times X$, i.e., for every $\gamma \in \Gamma$ the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X \\ {\scriptstyle f}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow[\gamma]{} & Y \end{array}$$

For a subgroup Γ' of Γ, a subset A of X is said to be Γ'-*invariant* if $\Gamma' A = A$.

Definition 2.3 ([3, p.640], [21, Definition 1.1])**.** Let

$$\mathcal{F}_\Gamma := \{\Lambda \mid \Lambda \text{ is a subgroup of } \Gamma\} \text{ and}$$
$$\mathcal{F}_{\text{fin}} := \{\Lambda \mid \Lambda \text{ is a finite subgroup of } \Gamma\}.$$

A nonempty subcollection $\mathcal{F}$ of $\mathcal{F}_\Gamma$ is called a *family of subgroups of* Γ if $\mathcal{F}$ is

- closed under conjugation, i.e., $\gamma^{-1}\Lambda\gamma \in \mathcal{F}$ for any $\Lambda \in \mathcal{F}$ and $\gamma \in \Gamma$; and
- closed under taking subgroups, i.e., for any $\Lambda \in \mathcal{F}$, if Ω is a subgroup of Λ, then $\Omega \in \mathcal{F}$.

A family $\mathcal{F}$ of subgroups of Γ is said to be *virtually closed* if $\mathcal{F}$ is closed under taking finite index supergroups, i.e., for any $\Lambda \in \mathcal{F}$ and $\Omega \in \mathcal{F}_\Gamma$, if $\Omega \supset \Lambda$ and Ω/Λ is finite, then $\Omega \in \mathcal{F}$.

The collections $\mathcal{F}_\Gamma$ and $\mathcal{F}_{\text{fin}}$ are virtually closed families of subgroups of Γ, while $\{\{1_\Gamma\}\}$ is a family of subgroups of Γ such that it is virtually closed if and only if Γ is torsion free.

Convention 2.4. Throughout this paper, let $\mathcal{F}$ denote a family of subgroups of Γ.

Definition 2.5 ([5, Definition 2.2])**.** A collection $\mathcal{U}$ of subsets of X is said to be Γ-*equivariant* (or Γ-*invariant*) if $\gamma U \in \mathcal{U}$ for any $\gamma \in \Gamma$ and $U \in \mathcal{U}$, i.e., $\mathcal{U}$ is closed under $\Gamma \curvearrowright X$.

A subset A of X is called an $\mathcal{F}$-*subset* if $\Gamma_A \in \mathcal{F}$ and for every $\gamma \in \Gamma$, if $\gamma A \neq A$, then $\gamma A \cap A = \emptyset$. A collection $\mathcal{U}$ of subsets of X is called an $\mathcal{F}$-*cover* of X if $\mathcal{U}$ is Γ-equivariant, every $U \in \mathcal{U}$ is $\mathcal{F}$-subset, and $X = \bigcup \mathcal{U}$.

3. Equivariant asymptotic dimension

Recall the definition of equivariant asymptotic dimension.

Definition 3.1 ([21])**.** Let $\Gamma \curvearrowright X$ be an action of a discrete group Γ on a *compact* space X and $\mathcal{F}$ a family of subgroups of Γ. Then the *equivariant asymptotic dimension* of $\Gamma \curvearrowright X$ with respect to $\mathcal{F}$, denoted by $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X)$, is the smallest $N \in \mathbb{Z}_{\geq 0}$ such that for every $E \in [\Gamma]^{<\omega}$ there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $\Gamma \times X$ satisfying the following conditions:

(E1) $\operatorname{ord}(\mathcal{U}) \leq N + 1$;

(E2) $\mathcal{U}/\Gamma$ is finite, i.e., there exists a finite subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{U} = \Gamma\mathcal{U}'$, in other words, $\mathcal{U}'$ generates $\mathcal{U}$ by $\Gamma \curvearrowright \Gamma \times X$;

(E3) for every $(\gamma, x) \in \Gamma \times X$ there exists $U \in \mathcal{U}$ such that $\gamma E \times \{x\} \subset U$.

If such an $N \in \mathbb{Z}_{\geq 0}$ does not exist, we write $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X) = \infty$. We also write eq-asdim$(\Gamma \curvearrowright X)$ instead of $\{\{1_\Gamma\}\}$-eq-asdim$(\Gamma \curvearrowright X)$.

Note that condition (E2) in Definition 3.1 can be skipped by the compactness of X ([21, Remark 1.5]). In contrast to (E2), the compactness of X in Definition 3.1 is crucial in the following sense: If X is compact and $\mathcal{F} \subset \mathcal{F}_{\text{fin}}$, then $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X)$ is related to the asymptotic dimension of Γ by [14, Theorem 6.5] (see Remark 5.11 below). On the other hand, if we apply Definition 3.1 to $\Gamma \overset{\text{can.}}{\curvearrowright} \Gamma$, then $\mathcal{F}$-eq-asdim$(\Gamma \overset{\text{can.}}{\curvearrowright} \Gamma) = 0$. Indeed, for every $E \in [\Gamma]^{<\omega}$ the open $\mathcal{F}$-cover $\mathcal{U} := \Gamma\{\Gamma \times \{1_\Gamma\}\}$ of $\Gamma \times \Gamma$ satisfies (E1), (E2) and (E3) above for $N = 0$ ([21, Remark 1.7]). In general, we see that for any free action $\Gamma \curvearrowright X$ on a discrete space X, $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X) = 0$. Recall that the covering dimension $\dim X$ of a normal space X is defined by *finite* open covers of X. For an open $\mathcal{F}$-cover $\mathcal{U}$ of $\Gamma \times X$, we see that for every $\gamma \in \Gamma$, $\mathcal{U} = \Gamma(\mathcal{U}[\{\gamma\} \times X])$, i.e., $\mathcal{U}[\{\gamma\} \times X]$ generates $\mathcal{U}$ by $\Gamma \curvearrowright \Gamma \times X$. From the above point of view, by replacing $\mathcal{U}'$ in (E2) with $\mathcal{U}[\{\gamma\} \times X]$, we define the following:

Definition 3.2. Let $\Gamma \curvearrowright X$ be an action of a discrete group Γ on a topological space X and $\mathcal{F}$ a family of subgroups of Γ. By $\mathcal{F}$-ead$(\Gamma \curvearrowright X)$ we mean the smallest $N \in \mathbb{Z}_{\geq 0}$ such that for every $E \in [\Gamma]^{<\omega}$ there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $\Gamma \times X$ satisfying the following conditions:

(E1) $\operatorname{ord}(\mathcal{U}) \leq N + 1$;

(E2′) for every $\gamma \in \Gamma$, $\mathcal{U}[\{\gamma\} \times X]$ is finite, i.e., $\mathcal{U}$ is a *finite* open cover as seen from any γ-level $\{\gamma\} \times X$ of $\Gamma \times X$;

(E3) for every $(\gamma, x) \in \Gamma \times X$ there exists $U \in \mathcal{U}$ such that $\gamma E \times \{x\} \subset U$.

If such an $N \in \mathbb{Z}_{\geq 0}$ does not exist, we write $\mathcal{F}$-ead$(\Gamma \curvearrowright X) = \infty$. We also write ead$(\Gamma \curvearrowright X)$ instead of $\{\{1_\Gamma\}\}$-ead$(\Gamma \curvearrowright X)$.

Remark 3.3. Let $\mathcal{U}$ be a Γ-equivariant collection of subsets of $\Gamma \times X$ and $E \in [\Gamma]^{<\omega}$.

(1) Condition (E2′) in Definition 3.2 is equivalent to the following:

(E2″) $\mathcal{U}[\{1_\Gamma\} \times X]$ is finite.

(2) Condition (E3) in Definitions 3.1 and 3.2 is equivalent to the following:

(E3′) for every $x \in X$ there exists $U \in \mathcal{U}$ such that $E \times \{x\} \subset U$.

Moreover, if $E \neq \emptyset$, then $\mathcal{U}$ satisfying (E3′) is a cover of $\Gamma \times X$. Thus $\mathcal{F}$-ead$(\Gamma \curvearrowright X) \leq N$ if and only if for every $E \in [\Gamma]^{<\omega}$ there exists a Γ-equivariant collection $\mathcal{U}$ of open $\mathcal{F}$-subsets of $\Gamma \times X$ satisfying (E1), (E2″) and (E3′).

Remark 3.4. For families $\mathcal{F}$ and $\mathcal{F}'$ of subgroups of Γ, if $\mathcal{F}' \subset \mathcal{F}$, then

$$\mathcal{F}\text{-ead}(\Gamma \curvearrowright X) \leq \mathcal{F}'\text{-ead}(\Gamma \curvearrowright X).$$

In particular,

$$0 = \mathcal{F}_\Gamma\text{-ead}(\Gamma \curvearrowright X) \leq \mathcal{F}_{\text{fin}}\text{-ead}(\Gamma \curvearrowright X) \leq \text{ead}(\Gamma \curvearrowright X).$$

Proposition 3.5. *If X is compact, then*

$$\mathcal{F}\text{-eq-asdim}(\Gamma \curvearrowright X) = \mathcal{F}\text{-ead}(\Gamma \curvearrowright X). \tag{A.1}$$

Sketch of proof. Assume that X is compact. We show that $N := \mathcal{F}\text{-eq-asdim}(\Gamma \curvearrowright X) \geq \mathcal{F}\text{-ead}(\Gamma \curvearrowright X)$. Let $E \in [\Gamma]^{<\omega}$. Since $\mathcal{F}\text{-eq-asdim}(\Gamma \curvearrowright X) = N$, there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $\Gamma \times X$ satisfying (E1) and (E3). For $U \in \mathcal{U}$, let

$$O_U := \{x \in X \mid E \times \{x\} \subset U\}.$$

Then there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $X = \bigcup_{i=1}^n O_{U_i}$ and each O_{U_i} is non-empty. Then $\Gamma\{\Gamma_{U_i}(E \times O_{U_i}) \mid i = 1, 2, \ldots, n\}$ is an open $\mathcal{F}$-cover of $\Gamma \times X$ satisfying (E1), (E2″) and (E3′). □

The following proposition is easy to verify from the definition:

Proposition 3.6. *Suppose that Γ acts on topological spaces Z and X. If there exists a continuous Γ-equivariant map $f : Z \to X$, then*

$$\mathcal{F}\text{-ead}(\Gamma \curvearrowright Z) \leq \mathcal{F}\text{-ead}(\Gamma \curvearrowright X).$$

In particular, if f is an inclusion map, i.e., Z is a Γ-invariant subspace of X, then the inequality holds.

Remark 3.7. For a subgroup Γ' of Γ, let $\mathcal{F}\restriction_{\Gamma'} = \{\Omega \cap \Gamma' \mid \Omega \in \mathcal{F}\}$. Then $\mathcal{F}\restriction_{\Gamma'}$ is a family of subgroups of Γ'. It is easy to see that

$$\mathcal{F}\restriction_{\Gamma'}\text{-ead}(\Gamma' \curvearrowright X) \leq \mathcal{F}\text{-ead}(\Gamma \curvearrowright X).$$

We also have the following analogue of [9, Theorem 2.1]:

Proposition 3.8.

$$\mathcal{F}\text{-ead}(\Gamma \curvearrowright X) = \sup_{E \in [\Gamma]^{<\omega}} \mathcal{F}\restriction_{\langle E\rangle}\text{-ead}(\langle E\rangle \curvearrowright X),$$

where $\langle E\rangle$ is the subgroup of Γ generated by E.

Sketch of proof. By Remark 3.7, it suffices to show that

$$\mathcal{F}\text{-ead}(\Gamma \curvearrowright X) \leq \sup_{E \in [\Gamma]^{<\omega}} \mathcal{F}\restriction_{\langle E\rangle}\text{-ead}(\langle E\rangle \curvearrowright X).$$

Suppose that $N := \sup_{E \in [\Gamma]^{<\omega}} \mathcal{F}\restriction_{\langle E\rangle}\text{-ead}(\langle E\rangle \curvearrowright X) < \infty$. To show that $\mathcal{F}\text{-ead}(\Gamma \curvearrowright X) \leq N$, let $E \in [\Gamma]^{<\omega}$. Since $\mathcal{F}\restriction_{\langle E\rangle}\text{-ead}(\langle E\rangle \curvearrowright X) \leq N$, there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $\langle E\rangle \times X$ such that each $U \in \mathcal{U}$ is non-empty and $\mathcal{U}$ satisfies (E1), (E2″) and (E3′). Let $\Lambda \subset \Gamma$ such that $\Gamma = \bigcup_{\lambda \in \Lambda} \lambda\langle E\rangle$ and $\{\lambda\langle E\rangle\}_{\lambda \in \Lambda}$ is pairwise disjoint. Then, $\Lambda\mathcal{U}$ is an open $\mathcal{F}$-cover of $\Gamma \times X$ satisfying (E1), (E2″) and (E3′). Thus $\mathcal{F}\text{-ead}(\Gamma \curvearrowright X) \leq N$. □

4. Actions on Stone-Čech compactifications

Suppose that X is a normal space, and let βX denote the Stone-Čech compactification of X. For each $\gamma \in \Gamma$, the homeomorphism $\gamma : X \to X; x \mapsto \gamma x$ can be extended to a unique homeomorphism $\widetilde{\gamma} : \beta X \to \beta X$, and this defines an action of Γ on βX. Let $\Gamma \curvearrowright \beta X$ denote the extended action of $\Gamma \curvearrowright X$, and the extension $\widetilde{\gamma} : \beta X \to \beta X$ is simply denoted by γ. Note that X and $\beta X \setminus X$ are Γ-invariant subspaces of βX.

It is well-known that $\dim X = \dim \beta X$ (see [11, Theorem 3.1.25] or [10, Theorem 7.1.17]). By [10, p.388] we can show the following analogue of this fact for $\mathcal{F}$-ead$(\Gamma \curvearrowright X)$:

Proposition 4.1. *Suppose that X is normal. Then*

$$\mathcal{F}\text{-eq-asdim}(\Gamma \curvearrowright \beta X) = \mathcal{F}\text{-ead}(\Gamma \curvearrowright \beta X) = \mathcal{F}\text{-ead}(\Gamma \curvearrowright X).$$

Sketch of proof. Suppose that $N := \mathcal{F}\text{-ead}(\Gamma \curvearrowright X) < \infty$. It suffices to show that $\mathcal{F}\text{-ead}(\Gamma \curvearrowright \beta X) \leq N$. Let $E \in [\Gamma]^{<\omega}$ with $1_\Gamma \in E = E^{-1}$. Since $\mathcal{F}\text{-ead}(\Gamma \curvearrowright X) = N$, there exists a Γ-equivariant collection $\mathcal{U}$ of open $\mathcal{F}$-subsets of $\Gamma \times X$ satisfying (E1), (E2$''$) and (E3$'$).

For $U \in \mathcal{U}$, let $\widetilde{\operatorname{Ex}} U := (\Gamma \times \beta X) \setminus \overline{(\Gamma \times X) \setminus U}$, where $\overline{(\Gamma \times X) \setminus U}$ is the closure of $(\Gamma \times X) \setminus U$ in $\Gamma \times \beta X$. Set $\widetilde{\mathcal{U}} := \{\widetilde{\operatorname{Ex}} U \mid U \in \mathcal{U}\}$. Then $\widetilde{\mathcal{U}}$ is a Γ-equivariant collection of open $\mathcal{F}$-subsets of $\Gamma \times \beta X$ satisfying (E1), (E2$''$) and (E3$'$). Thus $\mathcal{F}\text{-ead}(\Gamma \curvearrowright \beta X) \leq N$. $\square$

According to [23, Theorem 1.1], if X is a paracompact Hausdorff space with $\dim X < \infty$ and $\Gamma \curvearrowright X$ is free, then $\Gamma \curvearrowright \beta X$ is also free. We also have the following analogue:

Corollary 4.2. *Let $\Gamma \curvearrowright X$ be a free action on a non-compact normal space X such that* $\operatorname{ead}(\Gamma \curvearrowright X) < \infty$. *Then $\Gamma \curvearrowright \beta X$ is also free.*

For an example of a free action $\Gamma \curvearrowright X$ such that $\Gamma \curvearrowright \beta X$ is not free, see [23, Section 3].

5. A characterization theorem

We first prepare terminology on ℓ_1-metric polyhedron. For undefined terminology on simplicial complexes, we refer to [20].

Let K be a simplicial complex, $|K| := \bigcup K = \bigcup_{\sigma \in K} \sigma$, and $\dim K := \sup\{\dim \sigma \mid \sigma \in K\}$. For $n \in \mathbb{Z}_{\geq 0}$, let $K^{(n)}$ denote the n-skeleton of K. For each $\sigma \in K$, let $\sigma^{(0)}$, $\operatorname{rint} \sigma$ and $\widehat{\sigma}$ denote the set of all vertices of σ, the interior of σ and the barycenter of σ, respectively.

Let $\ell_1(K^{(0)})$ denote the ℓ_1-space $\{x : K^{(0)} \to \mathbb{R} \mid \sum_{v \in K^{(0)}} |x(v)| < \infty\}$ with the norm $\|\cdot\|_{\ell_1}$ defined by $\|x\|_{\ell_1} = \sum_{v \in K^{(0)}} |x(v)|$ for $x \in \ell_1(K^{(0)})$. Then $|K|$ can be regarded as a subset of $\ell_1(K^{(0)})$ by identifying $v \in K^{(0)}$ with the unit vector $\mathbf{e}_v \in \ell_1(K^{(0)})$ defined by $\mathbf{e}_v(w) = 1$ if $v = w$; and $\mathbf{e}_v(w) = 0$ if $v \neq w$ for $w \in K^{(0)}$. Let d_{ℓ_1} be the metric on $|K|$ defined by $d_{\ell_1}(x, y) := \|x - y\|_{\ell_1} = \sum_{v \in K^{(0)}} |x(v) - y(v)|$

for $x, y \in |K|$, and let $|K|_{\ell_1}$ denote the metric space $(|K|, d_{\ell_1})$. We call $|K|_{\ell_1}$ the ℓ_1-*metric polyhedron* (or metric polyhedron [20, §4.5]) of K.

An action $\Gamma \curvearrowright |K|$ is said to be *simplicial* if for each $\gamma \in \Gamma$ the map $\gamma : |K| \to |K|$ is simplicial, i.e., $\gamma\sigma \in K$ and $\gamma\restriction_\sigma$ is affine for every $\sigma \in K$. A simplicial complex K equipped with a simplicial action $\Gamma \curvearrowright |K|$ is called a *simplicial* Γ-*complex*. Note that every simplicial action $\Gamma \curvearrowright |K|_{\ell_1}$ on the ℓ_1-metric polyhedron is isometric.

Definition 5.1 ([14, Definition 4.3]). Let (Y, d) be a metric space equipped with $\Gamma \curvearrowright Y$. For $E \subset \Gamma$ and $\epsilon > 0$, a map $f : X \to Y$ is said to be (E, ϵ)-*equivariant* if $\sup_{(\gamma,x)\in E\times X} d(f(\gamma x), \gamma f(x)) < \epsilon$, i.e., for every $\gamma \in E$ the following diagram ϵ-commutes:

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X \\ {\scriptstyle f}\downarrow & \circlearrowright\epsilon & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\gamma]{} & Y \end{array}$$

In [14] the following result was shown:

Theorem 5.2 ([14, Proposition 4.5]). *Let $\Gamma \curvearrowright X$ be an action of a discrete group Γ on a* compact *Hausdorff space X, $\mathcal{F}$ a* virtually closed *family of subgroups of Γ and $N \in \mathbb{Z}_{\geq 0}$. Then the following conditions are equivalent:*

(1) $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X) \leq N$.

(2) *For every $E \in [\Gamma]^{<\omega}$ and every $\epsilon > 0$ there exist a simplicial Γ-complex K and a continuous (E, ϵ)-equivariant map $f : X \to |K|_{\ell_1}$ such that* $\dim K \leq N$ *and $\{\Gamma_v \mid v \in K^{(0)}\} \subset \mathcal{F}$.*

A characterization theorem for (B) in Section 1 is as follows:

Theorem 5.3. *Let $\Gamma \curvearrowright X$ be an action of a discrete group Γ on a normal space, $\mathcal{F}$ a family of subgroups of Γ and $N \in \mathbb{Z}_{\geq 0}$. Then the following conditions are equivalent:*

(I) $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright \beta X) \leq N$.

(II) $\mathcal{F}$-ead$(\Gamma \curvearrowright X) \leq N$.

(III) *For every $E \in [\Gamma]^{<\omega}$ there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $\Gamma \times X$ satisfying* (E2′) *and* (E3) *in Definition 3.2 and the following condition:*

(E1′) *there exist $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_N \subset \mathcal{U}$ such that $\bigcup_{i=0}^N \mathcal{U}_i = \mathcal{U}$ and each $\mathcal{U}_i$ is Γ-equivariant and pairwise disjoint.*

(IV) *For every $E \in [\Gamma]^{<\omega}$ and every $\epsilon > 0$ there exist a simplicial Γ-complex K and a continuous (E, ϵ)-equivariant map $f : X \to |K|_{\ell_1}$ such that* $\dim K \leq N$, *$\{\Gamma_x \mid x \in |K|\} \subset \mathcal{F}$ and the closure $\overline{f(X)}$ of $f(X)$ in $|K|_{\ell_1}$ is compact.*

(V) *For every $E \in [\Gamma]^{<\omega}$ and every $\epsilon > 0$ there exist a simplicial Γ-complex K and a continuous (E, ϵ)-equivariant map $f : X \to |K|_{\ell_1}$ such that* $\dim K \leq N$, *$\{\Gamma_x \mid x \in |K|\} \subset \mathcal{F}$ and $f(X) \subset |L|_{\ell_1}$ for some finite subcomplex L of K.*

Remark 5.4. In (IV) and (V) of Theorem 5.3, K need not be a full simplicial complex, see [14, Remark 4.8].

Remark 5.5. The following fact was used in the proof of [14, Proposition 4.5] implicitly and mentioned in the proof of [7, Lemma 4.2]:

Fact 5.6. *Let K be a simplicial Γ-complex such that $\{\Gamma_v \mid v \in K^{(0)}\} \subset \mathcal{F}$. If $\mathcal{F}$ is virtually closed, then $\{\Gamma_x \mid x \in |K|\} \subset \mathcal{F}$.*

It follows from Fact 5.6 that Theorem 5.3 implies Theorem 5.2.

Remark 5.7. As for a proof of Theorem 5.3, the equivalence (I) $\Leftrightarrow$ (II) follows from Proposition 4.1, and the implication (V) $\Rightarrow$ (IV) and (III) $\Rightarrow$ (II) are clear. The implication (IV) $\Rightarrow$ (III) can be shown by the same argument as in the proof of [14, (i)$\Rightarrow$(ii) of Proposition 4.5] ((2)$\Rightarrow$(1) of Theorem 5.2).

Our proof of (II) $\Rightarrow$ (V) in [8] is based on the proof of [14, (ii)$\Rightarrow$(i) of Proposition 4.5] ((1)$\Rightarrow$(2) of Theorem 5.2). However, in the proof of [14, (ii)$\Rightarrow$(i) of Proposition 4.5], it is unclear to the authors whether the equation "$d(f(gx), gf(x)) = \sum_{U\in\mathcal{U}} |\phi_U(gx, e) - \phi_U(gx, g)|$" in [14, page 799, line 4] holds, because the authors do not see why the equation $\psi_{g^{-1}U}(x, e) = \psi_U(gx, g)$ holds for every $U \in \mathcal{U}$ from the construction of ψ_U on [14, p.798]. In order to obtain the equation, we prove the following lemma in [8].

Lemma 5.8. *Suppose that X is normal. Let $E \in [\Gamma]^{<\omega}$ with $1_\Gamma \in E$ and $n \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{U}$ be an open $\mathcal{F}$-cover of $\Gamma \times X$ satisfying the following conditions:*
(E2$''$) *$\mathcal{U}[\{1_\Gamma\} \times X]$ is finite;*
(E3$'$)^n *for every $x \in X$ there exists $U \in \mathcal{U}$ such that $E^n \times \{x\} \subset U$.*

Then there exist a subcollection $\mathcal{U}'$ of $\mathcal{U}$, families $\{V_U^{(l)}\}_{U\in\mathcal{U}'}$, $l = 0, 1, \dots, n$, of subsets of $\Gamma\times X$ and families $\{\psi_U^{(m)}\}_{U\in\mathcal{U}'}$, $m = 1, 2, \dots, n$, of continuous functions $\psi_U^{(m)} : \Gamma \times X \to [0, 1]$ such that

- *$\mathcal{U}'$ is an open $\mathcal{F}$-cover of $\Gamma \times X$ satisfying* (E2$''$),
- *each $\{V_U^{(l)} \mid U \in \mathcal{U}'\}$ is an open $\mathcal{F}$-cover of $\Gamma \times X$ and*
- *for any $m = 1, 2, \dots, n$ and $U \in \mathcal{U}'$,*
  - (i) *$(V_U^{(m)})_E \subset V_U^{(m-1)} \subset U$ and $\Gamma_{V_U^{(m)}} = \Gamma_U$;*
  - (ii) *$\gamma V_U^{(m)} = V_{\gamma U}^{(m)}$ for any $\gamma \in \Gamma$;*
  - (iii) *$\psi_U^{(m)}$ is Γ_U-invariant;*
  - (iv) *$\psi_{\gamma' U}^{(m)}(\gamma, x) = \psi_U^{(m)}(\gamma'^{-1}(\gamma, x))$ for any $\gamma' \in \Gamma$ and $(\gamma, x) \in \Gamma \times X$;*
  - (v) *$\overline{V_U^{(m)}} \subset (\psi_U^{(m)})^{-1}(\{1\}) \subset (\psi_U^{(m)})^{-1}((0, 1]) \subset V_U^{(m-1)}$.*

Sketch of proof of (II) $\Rightarrow$ (V) *in Theorem 5.3.* Let $E \in [\Gamma]^{<\omega}$ with $1_\Gamma \in E$ and $\epsilon > 0$. Choose $n \in \mathbb{Z}_{>0}$ such that $2(2N + 2)(4N + 6) < n\epsilon$. By (II) for the finite subset E^n of Γ, there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $\Gamma\times X$ satisfying $\mathrm{ord}(\mathcal{U}) \leq N+1$ and (E2$''$) and (E3$'$)^n in Lemma 5.8. Then there exist a subcollection $\mathcal{U}'$ of $\mathcal{U}$, families $\{V_U^{(l)}\}_{U\in\mathcal{U}'}$, $l = 0, 1, \dots, n$, of subsets of $\Gamma \times X$ and families $\{\psi_U^{(m)}\}_{U\in\mathcal{U}'}$,

$m = 1, 2, \ldots, n$, of continuous functions $\psi_U^{(m)} : \Gamma \times X \to [0,1]$ as in Lemma 5.8. Let $N(\mathcal{U}')$ be the nerve of $\mathcal{U}'$. Then $N(\mathcal{U}')$ have the natural simplicial action $\Gamma \curvearrowright N(\mathcal{U}')$ and is a simplicial Γ-complex such that $\{\Gamma_x \mid x \in |N(\mathcal{U}')|\} \subset \mathcal{F}$ and $\dim N(\mathcal{U}') \leq N$. Since $\mathcal{U}'$ satisfies (E2″), $\mathcal{U}'[\{1_\Gamma\} \times X]$ is finite. Let $N(\mathcal{U}'[\{1_\Gamma\} \times X])$ be the nerve of $\mathcal{U}'[\{1_\Gamma\} \times X]$, which is regarded as a finite subcomplex of $N(\mathcal{U}')$.

For $U \in \mathcal{U}'$, define $\psi_U : \Gamma \times X \to [0, n]$ by

$$\psi_U(\gamma, x) := \sum_{m=1}^{n} \psi_U^{(m)}(\gamma, x)$$

for each $(\gamma, x) \in \Gamma \times X$. Then $|\psi_U(\gamma, x) - \psi_U(\gamma s, x)| \leq 2$ for each $U \in \mathcal{U}'$, $(\gamma, x) \in \Gamma \times X$ and $s \in E$. For $U \in \mathcal{U}'$, define $\phi_U : \Gamma \times X \to [0, 1]$ by

$$\phi_U(\gamma, x) := \frac{\psi_U(\gamma, x)}{\sum_{U' \in \mathcal{U}'} \psi_{U'}(\gamma, x)}$$

for each $(\gamma, x) \in \Gamma \times X$. Then ϕ_U is continuous and $\{\phi_U\}_{U \in \mathcal{U}'}$ is a partition of unity on $\Gamma \times X$. Finally, define $f : X \to |N(\mathcal{U}')|_{\ell_1}$ by

$$f(x) := \sum_{U \in \mathcal{U}'} \phi_U(1_\Gamma, x)\, U$$

for each $x \in X$. Then $K := N(\mathcal{U}')$, $L := N(\mathcal{U}'[\{1_\Gamma\} \times X])$ and f are the desired simplicial complexes and map in (V). $\square$

If $\mathcal{F} \subset \mathcal{F}_{\mathrm{fin}}$, then a simplicial complex K in Theorem 5.3 can be taken to be locally finite by the following proposition:

Proposition 5.9. *Let K be a simplicial Γ-complex having a finite subcomplex K_0 such that $\{\Gamma_v \mid v \in K^{(0)}\} \subset \mathcal{F}_{\mathrm{fin}}$ and $|K| = \Gamma |K_0|$. Then K is locally finite.*

By Proposition 4.1 and Theorem 5.3 with $\mathcal{F} = \{\{1_\Gamma\}\}$, we have the following corollary which extends [22, (1) $\Leftrightarrow$ (2) of Lemma 9.4] to free actions of discrete groups on normal spaces:

Corollary 5.10. *Let $\Gamma \curvearrowright X$ be a free action of a discrete group Γ on a normal space X and $N \in \mathbb{Z}_{\geq 0}$. Then the following conditions are equivalent:*

(1) eq-asdim$(\Gamma \curvearrowright \beta X) \leq N$.
(2) ead$(\Gamma \curvearrowright X) \leq N$.
(3) *For every $E \in [\Gamma]^{<\omega}$ and every $\epsilon > 0$ there exist a locally finite simplicial complex K equipped with a free simplicial action $\Gamma \curvearrowright |K|$ and a continuous (E, ϵ)-equivariant map $f : X \to |K|_{\ell_1}$ such that $\dim K \leq N$ and $f(X) \subset |L|_{\ell_1}$ for some finite subcomplex L of K.*

The notion of asymptotic dimension was introduced by Gromov [13, 1.E] for metric spaces (see also [18, Definition 2.2.1]), and extended to coarse spaces by Roe [19, Definition 9.4] (see also [12, Definition]). Following [14, Definition 6.3

and Theorem 6.5 (iv)], let $\operatorname{asdim}\Gamma$ denote the asymptotic dimension with respect to the coarse structure

$$\mathcal{E}_{\mathrm{fin}(\Gamma)} = \{D \subset \Gamma \times \Gamma \mid \{s^{-1}t \in \Gamma \mid (s,t) \in D\} \in [\Gamma]^{<\omega}\}$$

(see also [16, Example 2.13] for the coarse structure $\mathcal{E}_{\mathrm{fin}(\Gamma)}$). Note that, if Γ is countable, then $\operatorname{asdim}\Gamma$ coincides with the asymptotic dimension with respect to a uniformly discrete left-invariant proper metric on Γ (see [18, Definitions 1.2.5 and 2.2.1] and [6, Theorem 2.1.2]).

Remark 5.11. Let $\Gamma \curvearrowright X$ be an action on a normal space X and $\Gamma \curvearrowright \beta X$ the extended action of $\Gamma \curvearrowright X$. If X is normal, then

$$\operatorname{asdim}\Gamma = \text{eq-asdim}(\Gamma \overset{\text{can.}}{\curvearrowright} \beta\Gamma) \leq \mathcal{F}_{\mathrm{fin}}\text{-eq-asdim}(\Gamma \curvearrowright \beta X) \text{ [14, Theorem 6.5]}.$$

By this fact and Proposition 4.1, it is easy to see that if X is normal and $\mathcal{F} \subset \mathcal{F}_{\mathrm{fin}}$, then

$$\operatorname{asdim}\Gamma = \mathcal{F}\text{-ead}(\Gamma \overset{\text{can.}}{\curvearrowright} \Gamma) \leq \mathcal{F}\text{-ead}(\Gamma \curvearrowright X). \tag{A.2}$$

Since $\operatorname{asdim}\mathbb{Z} = 1 > 0 = \mathcal{F}_{\mathbb{Z}}\text{-ead}(\mathbb{Z} \curvearrowright X)$, the assumption that $\mathcal{F} \subset \mathcal{F}_{\mathrm{fin}}$ cannot be skipped.

See [17, Theorem 4.4] for another inequality regarding $\operatorname{asdim}\Gamma$ and $\mathcal{F}$-eq-asdim$(\Gamma \curvearrowright X)$.

By Theorem 5.3 and [14, Theorem 6.5] (see also Remark 5.11), we obtain a characterization of asymptotic dimension of groups (see [6, Theorem 2.1.2]).

Corollary 5.12. *Let Γ be a discrete group and $N \in \mathbb{Z}_{\geq 0}$. Then the following conditions are equivalent:*

(1) $\operatorname{asdim}\Gamma \leq N$.
(2) *For every $E \in [\Gamma]^{<\omega}$ and every $\epsilon > 0$, there exist a locally finite simplicial complex K equipped with a free simplicial action $\Gamma \curvearrowright |K|$, an (E,ϵ)-equivariant map $f : \Gamma \to |K|_{\ell_1}$ and a finite subcomplex L of K such that* $\dim K \leq N$ *and* $f(K) \subset |L|_{\ell_1}$.

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Department of Mathematics, National Defense Academy of Japan, Yokosuka 239-8686, Japan
Email address: `naochin@nda.ac.jp`

Graduate School of Science and Engineering, Ehime University, Matsuyama, 790-8577, Japan
Email address: `yamauchi.takamitsu.ts@ehime-u.ac.jp`