RECONSTRUCTIONS OF DYNAMICS USING MULTIVARIATE OBSERVATIONS

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Abstract. In the paper [Kat23], the author proved Takens-type reconstruction theorems of one-sided topological dynamics from single time series data. This note explains some additional results obtained in [Kat24]. In principle, multivariate time series are not necessary for reconstructing 'one-sided' dynamical systems; only one time series should be needed. However, in many practical situations, multivariate time series data are available. In this note, we study the most effective multivariate observation functions of one-sided topological dynamics and investigate the relationships between the multivariate time series and the 'trajectory' embedding dimensions. We show that using multivariate time series can improve the trajectory embedding dimensions, and we give the minimal trajectory embedding dimensions.

1. Introduction

Deterministic chaotic systems have influenced thinking in many fields of science. Chaotic systems show rich and surprising mathematical structures. Time delay embedding- a method for reconstructing two-sided (reversible) chaotic dynamical systems by single time series data- is widely used to forecast nonlinear time series as a model-free approach (see [PCFS80], [Tak81] and [SYC91]). In principle, multivariate time series are not necessary for reconstructing (one-sided) dynamical systems; only one time series should be needed (see [Kat23] for reconstructions of one-sided topological dynamical systems). Single time series data and moreover, multivariate time series data are available in many practical situations, and so the dynamics from (multivariate) time series also have been examined and discussed experimentally by many authors (e.g, see [PCFS80], [CMJ98], [GL02] and [KS04]). In natural sciences and physical engineering etc., there has been an increase in importance of fractal sets and more complicated spaces, and also in mathematics, many topological and dynamical properties and stochastic analysis of such spaces have been studied. So we consider here a broader setting than smooth dynamical systems, namely topological dynamical systems of all continuous maps. Studies related to this paper are e.g. [AAM18, Coo15, Gut15, Gut16, GQS18, GT14, Jaw74, Lin99, MS93, Noa91].

In this note, we study reconstructions of one-sided topological dynamical systems from multivariate time series. We will study the questions of how to choose which time series to use, and how to select trajectory embedding dimensions. We show that for one-sided topological dynamical systems, there exist the most effective combinations of multivariate observation functions for determining the minimal trajectory embedding dimensions. Moreover, we study 'k-trajectory embedding' ($k \geq 2$) which means that the difference of any different k trajectories can be observed by single observation function. This notion is stronger than 'trajectory embedding'

2. Trajectory embeddings by Multivariate time series

Let $\mathbb{N} = \{0, 1, 2, ...\}$ and let $\mathbb{R}$ be the real line. All spaces in this note are separable metric spaces and maps are continuous.

A pair (X, T) is called a *one-sided dynamical system* (abbreviated as *dynamical system*) if X is a compact metric space and $T : X \to X$ is any map. Moreover, if $T : X \to X$ is a homeomorphism, i.e., invertible, then (X, T) is called a *two-sided dynamical system*. In general, one-sided dynamical systems are more diverse than two-sided dynamical systems. In fact we know that two-sided (invertible) dynamical systems of the unit interval $I = [0, 1]$ are very simple. However, in the case of the one-sided (irreversible) dynamical systems of I, the dynamics are so complex and diverse that many researchers are currently working on their elucidation.

Definition 2.1. (Dynamical systems with multivariate observations) *Let (X, T) be a one-sided dynamical systems and let $f_i : X \to \mathbb{R}$ ($i = 1, 2, ...m$) be any maps. We call the maps $f_i (i = 1, 2, .., m)$ observation functions of (X, T). Let $f = (f_1, f_2, .., f_m) := (f_i) : X \to \mathbb{R}^m$ and let $S \subset \mathbb{N}$ be a subset.*

1. *We say that points $x_1, x_2, .., x_k \in X$ are S-trajectory separated if*
$$T^j|\{x_1, x_2, .., x_k\} : \{x_1, x_2, .., x_k\} \to X$$
is an embedding for each $j \in S$.
2. *Let $I^S_{T,f}$ denote the map $(f_1T^j, f_2T^j, ..., f_mT^j)_{j\in S} : X \to (\mathbb{R}^m)^S$ defined by*
$$\begin{aligned}&(f_1T^j, f_2T^j, ..., f_mT^j)_{j\in S}(x)\\ &= (f_1T^j(x), f_2T^j(x), ..., f_mT^j(x))_{j\in S}\end{aligned}$$
for $x \in X$.
3. *We say that $I^S_{T,f}$ is a trajectory-embedding if $I^S_{T,f}(x) \neq I^S_{T,f}(y)$ whenever $x, y \in X$ are S-trajectory separated. Note that for an injective map $T : X \to X$, $I^S_{T,f}$ is an embedding if and only if it is a trajectory embedding.*
4. *Let $k \in \mathbb{N}$ with $k \geq 2$. Then we say that $I^S_{T,f} : X \to (\mathbb{R}^m)^S$ is a k-trajectory embedding if for any S-trajectory separated points $x_1, x_2, ..., x_k$ of X, there exsit a $j \in S$ and a single observation function f_i such that*
$$f_iT^j|\{x_1, x_2, ..., x_k\} : \{x_1, x_2, ..., x_k\} \to \mathbb{R}$$
is an embedding, i.e. the difference of S-trajectory separated points $x_1, x_2, ..., x_k$ is measured by a single observation function f_i at time

$j \in S$. Note that 'trajectory embedding' is equivatent to '2-trajectory embedding'.

(5) *Let $p \in \mathbb{N}$ with $p \geq 1$. Then p is called a k-trajectory embedding dimension of (T, f) if $I_{T,f}^{(0,1,..,p-1)} : X \to (\mathbb{R}^m)^p$ is a k-trajectory embedding (i.e. we consider $I_{T,f}^S$, where $S = \{0, 1, .., p-1\}$). In particular, p is called a trajectory embedding dimension of (T, f) if $I_{T,f}^{(0,1,..,p-1)} : X \to (\mathbb{R}^m)^p$ is a 2-trajectory embedding.*

If K is a subset of a space X, then $\mathrm{cl}(K)$, $\mathrm{bd}(K)$ and $\mathrm{int}(K)$ denote the closure, the boundary and the interior of K in X, respectively. A subset K of X is dense in X if $\mathrm{cl}(K) = X$. A subset A of X is a G_δ-set of X if A is an intersection of countably many open subsets of X.

Let X be a compact metric space and Y a space with a complete metric d_Y. Let $C(X, Y)$ denote the space consisting of all maps $f : X \to Y$. We equip $C(X, Y)$ with the metric d defined by

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Recall that $C(X, Y)$ is a complete metric space and hence Baire's category theorem holds in $C(X, Y)$, i.e. if $E_n (n \in \mathbb{N})$ is an open dense set of $C(X, Y)$, then $\bigcap_{n \in \mathbb{N}} E_n$ is a G_δ-dense set of $C(X, Y)$.

For a space X, $\dim X$ means the topological (covering) dimension of X (e.g. see [Eng95] and [Mil01]). The following is well-known as the classical embedding theorem of Menger-Nöbeling (see [Eng95]):

If X is a compact metric space with $\dim X = d$, then the set of embedding from X into $\mathbb{R}^{2d+1}$ is a G_δ-dense set of $C(X, \mathbb{R}^{2d+1})$.

A map $g : X \to Y$ of separable metric spaces is *n-dimensional* ($n = 0, 1, 2, ...$) if $\dim g^{-1}(y) \leq n$ for each $y \in Y$. Note that a closed map $g : X \to Y$ is 0-dimensional if and only if for any 0-dimensional subset D of Y, $\dim g^{-1}(D) \leq 0$ (see [Eng95, Hurewic's theorem (1.12.4)]). A map $T : X \to X$ is *doubly 0-dimensional*[1] if for each closed set $A \subset X$ of dimension 0, one has $\dim T^{-1}(A) \leq 0$ and $\dim T(A) = 0$.

An indexed family $(C_s)_{s \in S}$ of subsets of a set X will by abuse of notation also be denoted by $\{C_s\}_{s \in S}$ or $\{C_s : s \in S\}$. Hence if $\mathcal{C} = \{C_s\}_{s \in S}$ is such a family then its members C_s and C_t will be considered as different whenever $s \neq t$. We then put

$$\mathrm{ord}(\mathcal{C}) = \sup\{\mathrm{ord}_x(\mathcal{C}) : x \in X\}, \text{ where } \mathrm{ord}_x(\mathcal{C}) = |\{s \in S |\ x \in C_s\}|.$$

Note that $\mathrm{ord}(\mathcal{C})$ so defined is by 1 larger than it would be according to the usual definition, as e.g. in [Eng95, (1.6.6) Definition].

[1] The 'doubly 0-dimensional' is equal to 'two-sided 0-dimensional' in [KM20].

Modifying the definition of TSP in [Kat21], we define the notion of $(t,\eta)_m$ trajectory-separation property for $m \in \mathbb{N}$ with $m \geq 1$, $t \in \mathbb{N}$ and $\eta > 0$. The property is very important in this paper. For the case of homeomorphisms, the original idea comes from Kulesza [Kul95].

Definition 2.2. *Let $T : X \to X$ be a map of a compact metric space X with $\dim X = d < \infty$, and let $m \in \mathbb{N}$ with $m \geq 1$ and $t \in \mathbb{N}, \eta > 0$. Then T has the $(t,\eta)_m$ trajectory-separation property ($(t,\eta)_m$-TSP for short) provided that there are m closed sets H_i $(i = 1, 2, .., m)$ of X such that*
(1) for each $i = 1, 2, .., m$, $X \setminus H_i$ is a union of finitely many disjoint open sets of diameter at most η, and
(2) $\mathrm{ord}\{T^{-p}(H_i)|\ 1 \leq i \leq m, 0 \leq p \leq t\} \leq d$.

Let (X,T) be a one-sided dynamical systems. For $n \geq 1$, let $P_n(T)$ be the set of all periodic points of T with period $\leq n$ and $P(T)$ the set of all periodic points of T, i.e.

$$P_n(T) = \{x \in X|\ \text{there is an } i \text{ such that } 1 \leq i \leq n \text{ and } T^i(x) = x\}$$

$$\text{and } P(T) = \bigcup_{n\geq 1} P_n(T).$$

Let A be a (nonempty) closed subset of a compact metric space X. Here we consider the following condition: $D(A) < \eta$ if A can be decomposed into finitely many mutually disjoint closed sets A_i with $\mathrm{diam}(A_i) < \eta$ for each i, i.e., $A = \bigcup_i A_i$, $\mathrm{diam}(A_i) < \eta$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. Note that $\dim A = 0$ if and only if $D(A) < \eta$ for each $\eta > 0$.

We need the following lemmas.

Lemma 2.3. (c.f. [Kat23, Lemma 5.2]) *Let $\eta > 0$, $t \in \mathbb{N}$ and $m \in \mathbb{N}$ with $m \geq 1$. Suppose that $T : X \to X$ is a doubly 0-dimensional map of a compact metric space X such that $\dim X = d < \infty$ and $D(\mathrm{cl}[\bigcup_{p=0}^{4t} T^{-p}(P(T))]) < \eta$. Then T has $(t,\eta)_m$-TSP.*

Lemma 2.4. (A version of Borsuk's homotopy extension theorem, c.f. [Bor67, (8.1)Theorem] and [Mil01, Theorem 4.1.3]) *Let X be a compact metric space and M a closed subset of X, and let maps $f', g' : M \to \mathbb{R}$ satisfy $d(f', g') < \epsilon$. If $g : X \to \mathbb{R}$ is an extension of g', then f' has an extension $f : X \to \mathbb{R}$ such that $d(f, g) < \epsilon$.*

3. Main theorem

For a compact metric space (Y, d), 2^Y denotes the space whose elements are nonempty closed subsets of Y and the space 2^Y has the Hausdorf metric d_H, i.e. $d_H(A,B) = \inf\{\epsilon > 0|\ B \subset U(A,\epsilon), A \subset U(B,\epsilon)\}$, where $A, B \in 2^Y$ and $U(A,\epsilon)$ denotes the ϵ-neighborhood of A in Y. Note that $(2^Y, d_H)$ is a compact metric space. Let Z be a space and let $\varphi : Z \to 2^Y \cup \{\emptyset\}$ be a set-valued function, where we consider that the empty set $\emptyset$ is an isolated point of the space $2^Y \cup \{\emptyset\}$. Then $\varphi : Z \to 2^Y \cup \{\emptyset\}$ is *upper semi-continuous* if for any $z \in Z$ and any open neighborhood V of $\varphi(z)$ in Y, there is an open neighborhood U of z in Z such that $\varphi(z') \subset V$ for any $z' \in U$.

Let (X, T) be any one-sided dynamical system. A point $x \in X$ is a *chain recurrent point* of T if for any $\epsilon > 0$ there is a finite sequence $x = x_0, x_1, \cdots, x_p = x$ ($p \geq 1$) of points of X such that $d(T(x_i), x_{i+1}) < \epsilon$ for each $i = 0, 1, \cdots, p-1$. Let $CR(T)$ be the set of all chain recurrent points of T. Then $P(T) \subset CR(T)$ and $CR(T)$ is a nonempty closed subset of X. Note that the set-valued function

$$CR : C(X, X) \to 2^X, \ T \mapsto CR(T)$$

is upper semi-continuous (see [BF85, Theorem F]).

We will define a class 0-$\mathcal{DCR}$ of compact metric spaces.

Definition 3.1. *Let 0-$\mathcal{DCR}$ be the class of all compact metric spaces X such that X satisfies the following two conditions:*
(0-$\mathcal{D}$) *The set of doubly 0-dimensional maps $T : X \to X$ is dense in $C(X, X)$.*
(0-$\mathcal{CR}$) *The set of maps $T : X \to X$ with* $\dim CR(T) = 0$ *is dense in $C(X, X)$.*

Remark (3-1). Note that for a compact metric space X, both the set of 0-dimensional maps $T : X \to X$ and the set of maps $T : X \to X$ with $\dim CR(T) = 0$ are G_δ sets of $C(X, X)$ (e.g. see [KOU16]). So we see that if X belongs to 0-$\mathcal{DCR}$, then the set of all maps $T : X \to X$ such that T is a 0-dimensional map with $\dim CR(T) = 0$ is a G_δ-dense set of $C(X, X)$.

Theorem 3.2. ([Kat23, Theorem 6.9]) *If X is one of the following spaces: PL-manifold, manifold with branched structures, Menger manifold, Sierpiński carpet, Sierpiński gasket and dendrite, then X belongs to 0-$\mathcal{DCR}$.*

For any real number $t \geq 0$, let $\langle t \rangle = \min\{n \in \mathbb{N} | t < n\}$. Note that e.g. $\langle 0.1 \rangle = 1$ and $\langle n \rangle = n + 1$ for $n \in \mathbb{N}$.

The following result is the main theorem of this note whose proof needs more precise arguments than one of [Kat23, Theorem 5.4].

Theorem 3.3. (Takens-type reconstructions of dynamical systems from multivariate time series) *Let X be a d-dimensional compact metric space ($d < \infty$) which belongs the class 0-$\mathcal{DCR}$ and let $m \in \mathbb{N}$ with $m \geq 1$. Then:*

(I) Let $k \in \mathbb{N}$ with $k \geq 2$ and $S \subset \mathbb{N}$ with $|S| = \langle \frac{kd}{m} \rangle$. If $E_m^{k,S}$ is the set of all pair $(T, (f_i)) \in C(X, X) \times C(X, \mathbb{R})^m$ such that $I^S_{(T,(f_i))} : X \to (\mathbb{R}^m)^S$ is a k-trajectory embedding, then $E_m^{k,S}$ is a G_δ-dense set of $C(X, X) \times C(X, \mathbb{R})^m$.

(II) If E_m is the set of all pair $(T, (f_i)) \in C(X, X) \times C(X, \mathbb{R})^m$ such that for any $k \in \mathbb{N}$ with $k \geq 2$ and any $S \subset \mathbb{N}$ with $|S| = \langle \frac{kd}{m} \rangle$, $I^S_{(T,(f_i))}$:

$X \to (\mathbb{R}^m)^S$ *is a* k*-trajectory embedding, then* E_m *is a* G_δ*-dense set of* $C(X,X) \times C(X,\mathbb{R})^m$.

Remark (3-2). In a laboratory, an experimentalist considers a reconstruction of a one-sided dynamical system (X,T), where m observation functions $f_i (i = 1, 2, .., m)$ are available. Theorem 3.3 implies that for any k different trajectories, within a constant time $\langle \frac{k \times \dim X}{m} \rangle$, the experimentalist may be able to observe its difference by single observation function $f_i : X \to \mathbb{R}$.

Remark (3-3). In Theorem 3.3, the case $m = 1, k = 2$ (i.e. $\langle \frac{2d}{1} \rangle = 2d + 1$) means the main theorem of [Kat23, Theorem 5.4], and the case $m = 2d+1, k = 2$ (i.e. the case of $\langle \frac{2d}{2d+1} \rangle = 1$ and $S = \{0\} \subset \mathbb{N}$) means the classical embedding theorem of Menger-Nöbeling.

A map $f = (f_i) \in C(X,\mathbb{R})^m$ is called *a* k*-embedding* if for any different k points $x_1, x_2, .., x_k$ of X, there is an i such that $f_i|\{x_1, x_2, .., x_k\} : \{x_1, x_2, .., x_k\} \to \mathbb{R}$ is an embedding. Note that the "embedding" is the same as "2-embedding". The following corollary can be proved according the special case $m = kd + 1, \langle \frac{kd}{kd+1} \rangle = 1$ and $S = \{0\} \subset \mathbb{N}$ of Theorem 3.3. In this case, we do not need the condition $X \in 0 - \mathcal{DCR}$. The corollary is an extension of the embedding theorem of Menger-Nöbeling.

Corollary 3.4. *Suppose that* X *is a compact metric space with* $\dim X = d < \infty$ *and* $k \in \mathbb{N}$ *with* $k \geq 2$*. If* F_{kd+1} *is the set of all* k*-embeddings in* $C(X,\mathbb{R})^{kd+1}$*, then* F_{kd+1} *is a* G_δ*-dense set of* $C(X,\mathbb{R})^{kd+1}$.

Theorem 3.5. (Jaworski-type embeddings of dynamical systems from multivariate time series) *Let* X *be a compact metric space with* $\dim X = d < \infty$ *and* $T : X \to X$ *a doubly 0-dimensional map with* $\dim P(T) \leq 0$*. Let* $m \in \mathbb{N}$ *with* $m \geq 1$*. If* F_m *is the set of all* $f = (f_i) \in C(X,\mathbb{R})^m$ *such that for any* $k \in \mathbb{N}$ *with* $k \geq 2$ *and any* $S \subset \mathbb{N}$ *with* $|S| = \langle \frac{kd}{m} \rangle$, $I^S_{(T,(f_i))} : X \to (\mathbb{R}^m)^S$ *is a* k*-trajectory embedding, then* F_m *is a* G_δ*-dense set of* $C(X,\mathbb{R})^m$*. Hence* $\langle \frac{kd}{m} \rangle$ *is a* k*-trajectory embedding dimension of* $(T,(f_i))$ $((f_i) \in F_m)$*. In particular, for the case* $k = 2$, $\langle \frac{2d}{m} \rangle$ *is a trajectory embedding dimension of* $(T,(f_i))$.

Corollary 3.6. *Suppose* $X \in 0 - \mathcal{DCR}$*, and* $T : X \to X$ *is a doubly 0-dimensional map with* $\dim P(T) \leq 0$ *and* $\dim X = d < \infty$*. Let* E_m *be the subset of* $C(X,X) \times C(X,\mathbb{R})^m$ *as in Theorem 3.3, and* F_m *be the subset of* $C(X,\mathbb{R})^m$ *as in Theorem 3.5. Then* $\{T\} \times F_m \subset E_m$.

4. Trajectory isomorphisms of dynamical systems and inverse limits

For a space K, we consider the (one-sided) *shift map* $\sigma : K^{\mathbb{N}} \to K^{\mathbb{N}}$ which is defined by

$$\sigma(x_0, x_1, x_2, x_3....) = (x_1, x_2, x_3....), \ x_i \in K.$$

Let (X,T) and (X',T') be dynamical systems. If $h : X \to X'$ is a map such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \downarrow T & & \downarrow T' \\ X & \xrightarrow{h} & X' \end{array}$$

is commutative, then we say that $h : (X,T) \to (X',T')$ is a *morphism* of dynamical systems. For the case of $K = \mathbb{R}^m$ and $f = (f_i) \in C(X,\mathbb{R})^m$, $I_{T,f} := I^{\mathbb{N}}_{T,f} : (X,T) \to ((\mathbb{R}^m)^{\mathbb{N}}, \sigma)$ is a morphism of dynamical systems. We call $I_{T,f}$ the *infinite delay observation map* for (X,T).

$$\begin{array}{ccc} X & \xrightarrow{I_{T,f}} & I_{T,f}(X) \subset (\mathbb{R}^m)^{\mathbb{N}} \\ \downarrow T & & \downarrow \sigma_{T,f} \\ X & \xrightarrow{I_{T,f}} & I_{T,f}(X) \subset (\mathbb{R}^m)^{\mathbb{N}} \end{array}$$

where $\sigma_{T,f} = \sigma | I_{T,f}(X)$. A morphism $h : (X,T) \to (X',T')$ is a *trajectory-monomorphism* if $h(x), h(y)$ are $\mathbb{N}$-trajectory separated for T', whenever $x, y \in X$ are $\mathbb{N}$-trajectory separated for T. Let $x, y \in X$ and let $o_T(x) = (T^i(x))_{i\in\mathbb{N}}$ and $o_T(y) = (T^i(y))_{i\in\mathbb{N}}$ be orbits of x and y for T respectively. We say that the orbit $o_T(x)$ is *eventually equivalent* to the orbit $o_T(y)$ if the orbits will be equal in the future, i.e., there exists an $n \in \mathbb{N}$ such that $T^i(x) = T^i(y)$ for each $i \geq n$. In this case, we wright $o_T(x) \sim_e o_T(y)$. We see that this relation is an equivalence relation. So we have the equivalence class

$$[o_T(x)] = \{o_T(y) |\ o_T(x) \sim_e o_T(y)\}$$

containing $o_T(x)$, and we put

$$[O(T)] = \{[o_T(x)] |\ x \in X\}.$$

Note that if $T : X \to X$ is injective, the function $o : X \to [O(T)]$ defined by $x \mapsto [o_T(x)]$ is bijective, i.e., $o : X \cong [O(T)]$. Also, note that if $h : (X,T) \to (X',T')$ is a morphism of dynamical systems, then h induces the function $h : [O(T)] \to [O(T')]$ defined by $h([o_T(x)]) = [o_{T'}(h(x))]$ for $x \in X$. A morphism $h : (X,T) \to (X',T')$ of dynamical systems is a *trajectory-isomorphism* if h induces the bijection $h : [O(T)] \cong [O(T')]$.

Let (X,T) and (X',T') be one-sided dynamical systems of compact metric spaces. The *inverse limit* of T is the space

$$\varprojlim(X,T) = \{(x_i)_{i=0}^{\infty} \in X^{\mathbb{N}} \mid T(x_{i+1}) = x_i \text{ for each } i \in \mathbb{N}\}$$

which has the topology inherited as a subspace of the product space $X^{\mathbb{N}}$. If $h : (X,T) \to (X',T')$ is a morphism of dynamical systems, then the map

$$\varprojlim h : \varprojlim(X,T) \to \varprojlim(X',T')$$

is defined by $\varprojlim h((x_i)_i) = (h(x_i))_i$ for $(x_i)_i \in \varprojlim(X,T)$. Note that if T is a homeomorphism, then $X \cong \varprojlim(X,T)$.

In the statement of [Kat23, Proposition 4.1], if $f : X \to \mathbb{R}$ is replaced by $f : X \to K$, and $I_{T,f}^{(0,1,2,..,k)} : X \to \mathbb{R}^{k+1}$ by $I_{T,f}^{(0,1,2,..,k)} : X \to K^{k+1}$, we can prove the following proposition (see the proof of [Kat23, Proposition 4.1]). In fact, in this note, we consider the case $K = \mathbb{R}^m$.

Proposition 4.1. *Let (X, T) be a dynamical system and $k \in \mathbb{N}$, and let K be a space. If $f : X \to K$ is a map such that $I_{T,f}^{(0,1,..,k)} : X \to K^{k+1}$ is a trajectory-embedding, then the following assertions* (1)-(3) *hold.*

(1) *There is the unique map $\sigma_{T,f}^{(0,1,..,k)} : I_{T,f}^{(0,1,..,k)}(X) \to I_{T,f}^{(0,1,..,k)}(X)$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{I_{T,f}^{(0,1,..,k)}} & I_{T,f}^{(0,1,..,k)}(X) \subset K^{k+1} \\ \downarrow T & & \downarrow \sigma_{T,f}^{(0,1,..,k)} \\ X & \xrightarrow{I_{T,f}^{(0,1,..,k)}} & I_{T,f}^{(0,1,..,k)}(X) \subset K^{k+1}. \end{array}$$

is commutative. In other words, the map $\sigma_{T,f}^{(0,1,..,k)}$ defined by

$$(fT^i(x))_{i=0}^k) \mapsto (fT^i(x))_{i=1}^{k+1}) \ (x \in X)$$

is well-defined. And the morphism

$$I_{T,f}^{(0,1,..,k)} : (X, T) \to (I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$$

is a trajectory-isomorphism. In particular, $I_{T,f} : (X, T) \to (K^{\mathbb{N}}, \sigma)$ is a trajectory-monomorphism.

(2) *There is a map $\pi : I_{T,f}^{(0,1,..,k)}(X) \to X$ such that*

$$\pi \circ I_{T,f}^{(0,1,..,k)} = T^k \text{ and } I_{T,f}^{(0,1,..,k)} \circ \pi = (\sigma_{T,f}^{(0,1,..,k)})^k$$

and so the map $\varprojlim I_{T,f}^{(0,1,..,k)} : \varprojlim(X, T) \to \varprojlim(I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$ is a homeomorphism.

(3) *Let $p_{(0,1,..,k)} : K^{\mathbb{N}} \to K^{k+1}$ be the projection defined by $(x_i)_{i\in\mathbb{N}} \mapsto (x_i)_{i=0}^k$. Then $p_{(0,1,..,k)} : (I_{T,f}(X), \sigma_{T,f}) \to (I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$ is an isomorphism of dynamical systems, i.e., $p_{(0,1,..,k)} : I_{T,f}(X) \to I_{T,f}^{(0,1,..,k)}(X)$ is a homeomorphism.*

Theorem 4.2. *In the setup of Theorem 3.3, the following assertions (i) and (ii) hold.*

(i) $\langle \frac{kd}{m} \rangle$ is a k-trajectory embedding dimension with respect to a G_δ-dense set of $C(X, T) \times C(X, \mathbb{R})^m$. In particular, for the case $k = 2$, $\langle \frac{2d}{m} \rangle$ is the minimal trajectory embedding dimension with respect to E_m^{2,S_2}, where $S_2 = \{0, 1, 2, .., \langle \frac{2d}{m} \rangle - 1\}$.

(ii) If $(T, (f_i)) \in E_m^{2,S_2}$, then there is a map $\sigma_{T,f}^{S_2} : I_{T,f}^{S_2}(X) \to I_{T,f}^{S_2}(X)$ such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{I_{T,f}^{S_2}} & I_{T,f}^{S_2}(X) \subset (\mathbb{R}^m)^{\langle \frac{2d}{m} \rangle} \\ \downarrow T & & \downarrow \sigma_{T,f}^{S_2} \\ X & \xrightarrow{I_{T,f}^{S_2}} & I_{T,f}^{S_2}(X) \subset (\mathbb{R}^m)^{\langle \frac{2d}{m} \rangle}, \end{array}$$

is commutative, and satisfies the following conditions:
(1) $I_{T,f}^{S_2} : (X,T) \to (I_{T,f}^{S_2}(X), \sigma_{T,f}^{S_2})$ *is a trajectory isomorphism (i.e.* $I_{T,f}^{S_2} : [O(T)] \cong [O(\sigma_{T,f}^{S_2})]$*), and* $p_{S_2} : (I_{T,f}(X), \sigma_{T,f}) \to (I_{T,f}^{S_2}(X), \sigma_{T,f}^{S_2})$ *is an isomorphism of dynamical systems (i.e. topologically conjugate).*
(2) $\varprojlim I_{T,f}^{S_2} : \varprojlim(X,T) \cong \varprojlim(I_{T,f}^{S_2}(X), \sigma_{T,f}^{S_2})$ *is a homeomorphism.*

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