Optimal Lockdown Decisions of the Stochastic SIR Model with the Allocation Medical Resources*

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Abstract

We consider the optimal timing (start and end) of lockdowns during the COVID-19 pandemic to minimize the expected total cost, which includes the opportunity cost of economic activities and the disease burden in medical facilities, based on the stochastic SIR model. The model incorporates uncertainty around the contact rate and the choice of the medical resource level. The major differences from the standard SIR model are as follows: (i) the model takes into account two types of deaths, those due to infectious and non-infectious diseases; (ii) it allows decision-makers to make a sequence of decisions regarding lockdown during the pandemic; (iii) the model is formulated as a combination of optimal stopping and a stochastic control problem; (iv) it is shown, under the assumption of linear variety, that there exists a simple optimal lockdown policy and an optimal investment level in medical resources; and (v) these optimal policies depend not only on the parameters of the infectious process but also on the contact rate, the number of susceptible individuals remaining at the decision epoch, and the marginal cost with respect to the infection rate.

1 Introduction

More than three years have passed since COVID-19 was first recognized by the WHO in December 2019. During the early stages of the pandemic, we had no effective means of preventing the virus's spread, except through strong restrictions such as implementing lockdowns and promoting social distancing. The virus quickly spread worldwide, and most countries had failed in stopping the infection expansion among their nations. In the absence of a definitive vaccine or effective medical treatments, decision-makers in many countries faced three options: A) Implementing lockdowns to control societal interactions. B) Expanding medical services, including hospital capacity and essential medical workers. C) Doing nothing and relying on 'herd immunity,' which can only be achieved if recovered individuals do not become infectious again.

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Many countries opted for either option A or B, while Sweden and Brazil chose option C. In this paper, we present a model that combines options A and B, which may be more acceptable to decision-makers than option C. The model aims to address the challenge of determining the optimal timing for starting and ending lockdowns, which involves a severe trade-off between economic activities and public health care.

In this paper, we present a stochastic contagion control model based on the SIR model with several modifications. In our model, an infected patient can transition to either the recovered or death state, and a recovered patient becomes susceptible again immediately after a finite immunity period. When the length of the immunity period is relatively short, our model exhibits characteristics that are intermediate between the SIS and SIR models. Additionally, we account for the number of deaths caused by infectious and non-infectious diseases within the SIR framework. Our emphasis in this model is not solely on mathematical derivation but also on its practical relevance to epidemic control, particularly in the context of decision-making regarding lockdown measures. We explore how implementing lockdowns impacts in transmitting the virus to susceptible and consider how decision-makers should allocate resources to medical care in order to enhance infectious and fatality rates during the pandemic. In essence, our study delves into how decision-makers should utilize lockdown measures as a potent tool to compel individuals to significantly restrict their economic and social activities for effective pandemic management, while also evaluating the social cost associated with policies that curtail these activities.

While many stochastic epidemiological models exist in the literature, classical models consist of deterministic SIS and SIR models with homogeneous populations. It is widely acknowledged that introducing stochastic noise factors can contribute to a more realistic representation of the infectious disease process, adding a level of credibility (see references [4], [12] and [13]). Moreover, there is a substantial body of literature on heterogeneous populations and vaccination, as seen in the works of Fu et al. [8]. Acemoglu et al. [1] and Federico and Ferrari [7] explore lockdown policies that target multiple population groups, demonstrating that targeted strategies can minimize both economic losses and fatalities. Optimization for stochastic SIR models is also addressed in the literature, as evidenced by [2], [6], [9], [10] and [19]. Salman et al. [17] employ the deterministic version of Pontryagin's maximum principle. Jones et al. [11] and Pindyck [15] utilize the deterministic SIR model to analyze the response of social and economic activities to COVID-19, considering strategies for pandemic mitigation. Pindyck [15] and Alvarez et al. [3] consider the deterministic SIRD models, incorporating fatality rates. Huberts and Thijssen [10], in the setting of a continuous-time Markov chain model based on SIRD model, is closely related to our model where the system dynamics of the infection process is given by stochastic differential equations similar to the standard stochastic SIR model.They consider the model where the decision maker wants to find an optimal timing policy to minimise the expected discounted cost consisting of the present values of before, during and after the lockdown. However, it does not include a decision variable related to either the contact rate or the recovery rate. Many studies including Rezapour et al. [16] and Thul and Powell [19] have investigated optimal policies of the vaccine allocation problem and restriction policies such as traffic restrictions since any communicable virus is transmitted through human interactions where the infection process is a discrete-time Markov chain. Sato and Sawaki [18] have analyzed optimal lockdown policies based on the SIR model, although their objective functions differ from the model presented in this paper. Additionally, they consider uncertainty around the recovery rate in their models, while around the contact rate in the model.

We analyze the optimal lockdown policy and the allocation of medical resources, which empower decision-makers to manage medical services effectively and minimize the expected discounted cost during a pandemic. The primary distinctions between our work and existing papers are as follows: 1) Our optimal policy for determining the timing of lockdowns and allocating medical resources within the stochastic SIR model; 2) Our optimal policy depends not only on the infection level and the number of susceptibles remaining at the decision epoch but also on the marginal cost of the value function with respect to the number of infectious individuals; 3) The optimal policy can be represented by two straightforward threshold values.

The paper is structured as follows: In Section 2, we present a stochastic SIR model that accounts for two types of deaths, differentiating it from the standard SIR model. In Section 3, we conduct an analysis of a combined optimal stopping and stochastic control problem to derive optimal policies for lockdown measures and the allocation of medical resources. In Section 4, we provide numerical examples to illustrate the optimal threshold values for optimal lockdown and conduct sensitivity analyses with respect to various parameters. Finally, we present our conclusion in the last section.

2 Dynamics of in the SIR epidemic model

Standard SIR models assume that the population is calcified into three classes, the susceptible (S), the infectious (I), and the recovered (R). The standard SIR model does not distinguish the recovered and the deaths which are removed from the number of infected members in the population. Also, the standard model assumes that infectious individuals after recovered gain immunity. In the model we separate the removed patients into recovered and deaths where there are two kind of deaths, due to infectious or non -infectious (natural) diseases. Death due to non-infectious disease is called infectious free. The model incorporates the finite length of the immunity periods $\eta > 0$ which can be a random variable. This means that recovered patients can be infected multiple times after the immunity periods. This is realistic for COVID-19 since it was reported that recovered patients become infected again after a finite periods. Let $S(t)$, $I(t)$ and $R(t)$ represent the numbers of the susceptible, the infectious and the recovered at time t. Random noise plays a realistic role in the infectious transmission process compared with the deterministic model because it incorporates the volatility of the infectious rate. Let $D_1(t)$ and $D_2(t)$ be the number of deaths due to infectious free and infectious diseases with their fatality rates δ_1, $\delta_2(u)$, respectively, that is, $D_1(t) = \delta_1 S(t)$ and $D_2(t) = \delta_2(u)I(t)$ where u represents the medical care level and is a control variable and $0 < \delta_2(u) < 1$. Note that the (natural) fatality rate due to the infectious free does not depend on u but the death by infectious disease depends on the control variable. Let $c(t)$ be the contact rate and β be the infected rate. The product

$c(t)\beta$ presents the efficient infected rate (transmitting the virus to susceptible patients)[‡]. The system dynamics of the stochastic SIR model with deaths is given by

$$dS(t) = \{-\delta_1 S(t) - c(t)\beta S(t)I(t) + \gamma_r(u)I(t-\eta)\}dt - \sigma S(t)I(t)dB(t), \tag{2.1}$$

$$dI(t) = \{c(t)\beta S(t) - (\gamma_r(u) + \delta_2(u))\}I(t)dt + \sigma S(t)I(t)dB(t), \tag{2.2}$$

$$dR(t) = \{\gamma_r(u)I(t) - \gamma_r(u)I(t-\eta)\}dt, \tag{2.3}$$

$$D(t) \equiv D_1(t) + D_2(t) = \delta_1 S(t) + \delta_2(u)I(t), \tag{2.4}$$

where σ is a positive constant and $B(t)$ is a standard Wiener process defined on the σ-field $(\Omega, \mathcal{F}_t, t \geq 0, \mathcal{P})$. Since infectious becomes susceptible after the immunity periods, the number of $\gamma_r(u)I(t-\eta)$ is added in equation (1) and subtracted in equation (3). We assume that $I(t-\eta) = 0$ for $\eta > t$. Notice that $S(t)$ and $I(t)$ are mutually interdependent but independent from $R(t)$ and $D(t)$ which are both uniquely determined from $S(t)$ and $I(t)$, respectively. Defining $N(t)$ as the total population after subtracting the number of deaths at time $t > 0$, we have

$$N(t) = (1-\delta_1)S(t) + (1-\delta_2(u))I(t) + R(t), \tag{2.5}$$

where the numbers of natural and infectious patients and deaths are eliminated from the population. It is assumed that $N(0) = S(0) + I(0) = 1$ and $0 < I(0) < 1$. Hence, we can normalize for all variables $S(t)$, $I(t)$ and $R(t)$ to present the percentage of the population, that is, we assume without loss of generality that

$$(1-\delta_1)S(t) + (1-\delta_2(u))I(t) + R(t) = 1, \quad \text{for all } t > 0. \tag{2.6}$$

This is different from the standard SIR model where the cumulative number of deaths is included in the system dynamics. (see Pindyck [15]). From equations (2.1) to (2.4), we have $dN(t) = -D(t)dt$ which implies that the total population at time t is decrasing in t because of eliminating of the number of deaths.

Remark 1 *(i) In the standard SIR model $D(t)$ is defined as the cumulative number of deaths which causes a problem to explain reasonable interpretations of taking integration of cumulative number $D(t)$ over a time horizon. However, since $D(t)$ in the model is the number of deaths at time t, the integration of $D(t)$ over time enables us to give reasonable economic interpretations. (ii) In equation (2.2), the dynamics of the infectious disease process can be rewritten as*

$$dI(t) = S(t)I(t)[c(t)\beta dt + \sigma dB(t)] - (\gamma_r(u) + \delta_2(u))I(t)dt, \tag{2.7}$$

where the volatility parameter of the uncertainty is introduced to model around the efficient infectious rate $c(t)\beta$. This modeling is reasonable to differentiate the effect of control variable u

[‡]Let L be the restrictive level of the lockdown, $0 < L < 1$. If the efficient infected rate should not be separated into two parts $c(t)$ and β, $S(t)$ and $I(t)$ would depend on L. In our model $c(t)$ can be interpreted as $c(t) = c(1 - L(t))^2$, depending on the lockdown period (see Alvarez et al. [3]). Hence, the efficient infected rate should separately be presented because the contact rate $c(t)$ can be controlled by the decision maker and the infected rate β should be based on the epidemic evidence.

from the uncertainty of the infectious rate.
(iii) From equation (2.2), $I(t)$ is increasing in t whenever $c(t)\beta S(t) > \gamma_r(u)+\delta_2(u)$ for the events of $dB(t) > 0$ and is decreasing in t when $c(t)\beta S(t) < \gamma_r(u)+\delta_2(u)$ for the events of $dB(t) > 0$. Hence, the expectation of $I(t)$ attains the maximum at $S(t) = (\gamma_r(u)+\delta_2(u))/(c(t)\beta)$ which equals the inverse of the basic reproduction number.

Put $\gamma(u) = \gamma_r(u) + \delta_2(u)$ which is called the removed rate. Applying Ito's lemma for equation (2.2), we obtain

$$I(t) = I(0)e^{\beta\int_0^t c(s)S(s)ds-\gamma(u)t-\frac{\sigma^2}{2}\int_0^t S^2(s)ds+\sigma\int_0^t S(s)dB(s)} \tag{2.8}$$

and put

$$m(t,u) = \beta\int_0^t c(s)S(s)ds - \gamma(u)t. \tag{2.9}$$

Then, we have

$$m'(t,u) := \frac{\partial m(t,u)}{\partial t} = \beta c(t)S(t) - \gamma(u),$$

which implies that the growth rate of infections depends also on the susceptible, the residual number of non-infected individuals.

$$m'(0,u) = \beta c(0)S(0) - \gamma(u),$$
$$\mathbb{E}[I(t)|\mathcal{F}_t] = I(0)e^{m(t,u)}.$$

From equation (2.6), we have

$$(1-\delta_2(u))I(t) + D_2(t) = I(t)$$

and

$$(1-\delta_1)S(t) + R(t) = 1-(1-\delta_2(u))I(t),$$

which are both linear variety in $I(t)$ for all t. Two equations above present the numbers of infectious and deaths who need medical cares, and the number of individuals who carry economic activities at time t, respectively.

3 Optimal policies for lockdown periods and medical resources

In this section, we formulate an optimization problem as a combination of the optimal stopping model and the stochastic control problem for the decision maker. Suppose that after observing the realization of the variables $I(t)$, $R(t)$ and $D(t)$ at time t the decision maker wishes to make when to start and end the lockdown and to choose the level of medical cares. Let τ be a time to start the lockdown and $\tau+\theta$ be the time to end the lockdown where θ stands for the lockdown periods. The decision to start the lockdown is made at a stopping time τ. The decision to remove (end) it is made at a stopping time $\tau+\theta$. Assume that the economic activities are carried out by the susceptible and recovered individuals and a fraction L, $0 < L < 1$, of the sum of susceptibles and recovered individuals are forced to stay home during the lockdown periods, that is, $L\{(1-\delta_2(u))I(t)+R(t)\}$ must leave economic activities. The lockdown policy with the stronger restriction may reduce the infectious rate when the large enough number of susceptibles remain. This also implies that the fraction $(1-L)$ of the sum of the $(1-\delta_1)S(t)+R(t)$ and $R(t)$ can work during the lockdown. We assume that the decision maker simultaneously chooses the restriction level (intensity) of the lockdown at time t when she/he adapts the lockdown [§]. The larger she/he chooses the level L the more people must stay home. Let π be the continuous unite economic benefit generated by economic activities. Let p_1 be the unit medical cost caused by infectious and p_2 be the unit social cost of death due to the infectious disease. Let $w(L)$ denote the continuous unit incentive (financial aids) during the lockdown periods with the restriction level L, $0<L<1$. Then, the continuous immediate cost at time t during the lockdown is given by, from equation (2.6),

$$\begin{aligned}
&p_1(1-\delta_2(u))I(t)+p_2D_2(t)+w(L)-(1-L)\pi[(1-\delta_1)S(t)+R(t)]\\
&\quad=p_1(1-\delta_2(u))I(t)+p_2\delta_2(u)I(t)+w(L)-(1-L)\pi[1-(1-\delta_2(u))I(t)]\\
&\quad=[(p_1+(1-L)\pi)(1-\delta_2(u))+p_2\delta_2(u)]I(t)-(1-L)\pi+w(L) \qquad (3.1)
\end{aligned}$$

where the term $(1-L)\pi[(1-\delta_1)S(t)+R(t)]$ in equation (3.1) presents the economic benefit with workers who can work during the lockdown. The economic cost of fatality is often measured as the value of statistical life or as the opportunity cost consumed for the remained life time periods. Before the lockdown, we have the social cost as follows, from equation (2.6),

$$\begin{aligned}
&p_1(1-\delta_2(u))I(t)+p_2D_2(t)-\pi[(1-\delta_1)S(t)+R(t)]\\
&\quad=[(p_1+\pi)(1-\delta_2(u))+p_2\delta_2(u)]I(t)-\pi \qquad (3.2)
\end{aligned}$$

Equations (3.1) and (3.2) are both linear variety in $I(t)$. The sum of susceptible and recovered can produces the unite economic benefit π. Since the number of deaths is proportional to the number of infected patients, the benefit of reducing the number of infected patients is also the value of lives saved and can lighten the medical burden.

Since the choice of a medical care level affects the parameters $\gamma_r(u)$ and $\delta_2(u)$ in the dynamics of the state variable, we combine the optimal stopping problem with the stochastic control problem to choose the level of medical cares. From equations (3.1) and (3.2), the sum of infectious and death and the sum of susceptible and recovered are all linear variety in $I(t)$. The state variable of the control model for the epidemic system can be described by $I(t)$. Let $f_1(I,u)$ be the immediate cost at time t before the lockdown, $f_2(I,u)$ be the immediate cost at time t during the lockdown and $f_3(I,u)$ be the immediate cost at time t after the lockdown. We

[§]For instance, L can be chosen so as to depend on the tight medical capacity and the number of critical ill patients which are Tokyo Metropolitan's indicator to request "the State of Emergency Declaration".

rewrite equations (3.1) and (3.2) as follows;

$$\begin{aligned}
f_1(I,u) &= [(p_1+\pi)(1-\delta_2(u))+p_2\delta_2(u)]I(t)-\pi \\
&= -\pi+[p_1+\pi+A_1\delta_2(u)]I(t), \\
f_2(I,u) &= [(p_1+(1-L)\pi)(1-\delta_2(u))+p_2\delta_2(u)]I(t)-(1-L)\pi+w(L) \\
&= -(1-L)\pi+w(L)+[p_1+(1-L)\pi+A_2\delta_2(u)]I(t) \\
&= -\pi+[p_1+\pi+A_1\delta_2(u)]I(t)+L\pi[1-I(t)+\delta_2 I(t)]+w(L) \\
&= f_1(I,u)+L\pi[1-(1-\delta_2)I(t)]+w(L) \\
&= f_1(I,u)+L\pi[(1-\delta_1)S(t)+R(t)]+w(L)>f_1(I,u)
\end{aligned}$$

and

$$f_3(I,u)=f_1(I,u),$$

where $A_1=p_2-p_1-\pi$ and $A_2=p_2-p_1-(1-L)\pi=A_1+L\pi$, and the second term of $f_2(I,u)$ presents the opportunity cost due to the lockdown. Hence, the difference between $f_2(I,u)$ and $f_1(I,u)$ equals the additional cost caused by the lockdown decision, including the financial aid $w(L)$. By assuming $p_2>p_1+\pi$, we have $A_2>A_1>0$ and $A_2-A_1=L\pi$. The quantity A_2 can be interpreted as the incremental cost due to the lockdown. So is A_1 before the lockdown. The difference between A_1 and A_2 is the opportunity cost lost by the lockdown with restriction level L. It is worthwhile to remark that $f_1(I,u)=f_2(I,u)$ by putting $L=0$ and $w(0)=0$. All functions $f_i(I,u)$ are all linear varieties in I where the initial value of $I(t)$ in $f_3(I,u)$ after ending the lockdown is different from the initial value in $f_1(I,u)$ before starting the lockdown. Note from equations (3.1) and (3.2) that $f_i(I,u)$ is increasing in I, $i=1,2$.

Let r be a discount rate which can be interpreted as the sum of the rate of the capital cost and the inverse value of the mean random time at which a determinative vaccine is discovered[¶]. Define the total cost $v^{\tau,\theta,u}(I)$ of the decision (τ,θ,u) for the initial state variable I by

$$\begin{aligned}
v^{\tau,\theta,u}(I)=\mathbb{E}^I\Bigg[\int_0^\tau e^{-rt}f_1(I(t),u)dt+e^{-r\tau}\Bigg\{K_1+\int_\tau^{\tau+\theta}e^{-r(t-\tau)}f_2(I(t),u)dt \\
+e^{-r\theta}\left(K_2+\int_{\tau+\theta}^{\infty}e^{-r(t-\tau-\theta)}f_3(I(t),u)dt\right)\Bigg\}\Bigg]. \qquad (3.3)
\end{aligned}$$

where the first term of the right-hand side of equation (3.3) is the discounted cost before the lockdown, the second term is the sum of the discounted cost during the lockdown and the fixed cost to start it, and the third one is the sum of the discounted cost after the lockdown and fixed cost to remove the lockdown. Since $D_1(t)$ is the number of deaths due to the infectious free diseases, the cost caused by $D_1(t)$ is not incorporated in the decision making but it reflects on the population of individuals who do not carry the economic activities. We assume in the stochastic equation of $I(t)$ given by equation (2.2) that $r>m'(t,u)$ as $t\to\infty$. This assumption is needed to ensure that the growth rate of the infectious process is positive before the lockdown

[¶]Since a discount factor in finance theory is small when the time horizon is a few years, the discount rate should reflect the time preference and the probability of discovering the vaccine during the small time interval dt.

and is non-positive after the lockdown. Define $v^*(I)$ for each $I \in [0, 1]$ by

$$v^*(I) = \inf_{\tau,\theta,u} v^{\tau,\theta,u}(I), \tag{3.4}$$

which can be rewritten from the Markov property as follows,

$$\begin{aligned} v^*(I) = \inf_{\tau,u_1} \mathbb{E}^I \Bigg[\int_0^\tau & e^{-rt} f_1(I(t), u_1)dt \\ & + e^{-r\tau} \Bigg\{ K_1 + \inf_{\theta,u_2} \mathbb{E}^{I(\tau)} \Bigg[\int_0^\theta e^{-r(t-\tau)} f_2(I(t), u_2)dt \\ & \qquad + e^{-r\theta} \left(K_2 + \int_\theta^\infty e^{-r(t-\tau-\theta)} f_1(I(t), u_3)dt \right) \Bigg\} \Bigg]. \end{aligned} \tag{3.5}$$

The optimization problem defined by equation (3.5) above involves an optimal stopping problem combined with choices of medical care level. It has been widely studied that such a problem can be formulated as a combination of optimal stopping problems and a stochastic control problem (Refer to Øksendal and Sulem [14]). It is easy to show from equations (3.1) and (3.2) that the following proposition holds, which guarantees that an optimal lockdown duration can be given by the connected interval (See Sato and Sawaki [18]).

Proposition 1 *(i) $v^{\tau,\theta,u}(I)$ is increasing in I for all τ, θ, u.*
(ii) $v^*(I)$ is increasing in I.

Let $v^1(I)$ be the optimal value of the option to start the lockdown and $v^2(I)$ be the value of the option to end the lockdown with an initial value I. Our optimization problem possesses the similar functional structure to the compound real option approach does. Duckworth and Zervos [5] assume that the decision variable does not affect the dynamics of the problem's state variable. This is different from our model where the decision variable u affects the system dynamics of the state variable which is the infection rate. This is why we adapt a combination of the stochastic control and optimal stopping problems. Therefore, we are not allowed to separate decision makings between the lockdown timing choice and the medical resource allocation. As Dockworth and Zervo's model, after formulating the entry (starting) and exit (ending) decisions as one-shot decision making, then every such problem can be decomposed into a sequence of decision-making problems consisting of optimal stopping and the choice of decision variable u. The problem can be formulated by "the Principle of Optimality" of dynamic programming and be solved sequentially backward in time. However, the decision to start the lockdown comes first and then the decision to remove the lockdown follows after it. The computation is in reverse order to the actual decisions. It is known from the theory of stochastic control that the value function $v^1(I)$ coincides with the value function of the original problem given by equation (3.5), that is, $v^1(I) = v^*(I)$ for all I.

From the Principle of Optimality and the Markovian property of the model, $v^1(I)$ and $v^2(I)$ satisfy

$$v^1(I) = \inf_{\tau,u_1} \mathbb{E}^I \left[\int_0^\tau e^{-rt} f_1(I(t), u_1)dt + e^{-r\tau}\{K_1 + v^2(I(\tau))\} \right] \tag{3.6}$$

and

$$v^2(I) = \inf_{\theta,u_2} \mathbb{E}^I \left[\int_0^\theta e^{-rt} f_2(I(t),u_2)dt + e^{-r\theta}\{K_2 + v^1(I(\theta))\} \right]. \tag{3.7}$$

Equation (3.6) presents the minimum payoff of the decision for an optimal starting time τ for given the optimal decision to end the lockdown. Equation (3.7) presents the minimum payoff of ending the lockdown for given the value function of $v^2(I)$ at the end of the lockdown periods. Using $f_2(I,u) = f_1(I,u) + \pi L(1 - I(t) + \delta_2(u)I(t)) + w(L)$, we obtain $\frac{\partial f_2}{\partial I} = \frac{\partial f_1}{\partial I} - \pi L(1 - \delta_2(u)) < \frac{\partial f_1}{\partial I}$. Hence, it can be shown from equations (3.5) and (3.7) that

$$\frac{\partial v^1}{\partial I} = \frac{\partial v^*}{\partial I} > \frac{\partial v^2}{\partial I}. \tag{3.8}$$

First, we solve the combined problem of the optimal stopping and stochastic control problems defined by equation (3.7), and then solve the combined problem defined by equation (3.6). It is well known from a standard stochastic control theory that the value function $v^1(I)$ given by equation (3.6) equals the value function $v^*(I)$ defined by equation (3.4) under the assumption that "verification theorem" holds[‖]. Since there is a possibility of many waves of the pandemics, the lockdown decisions shall be made sequentially by the decision maker following equations (3.6) and (3.7). To derive an optimal policy the value function $v^2(I)$ shall be computed first from equation (3.7) for a given equation (3.6) and then compute $v^1(I)$ from equation (3.6). The HJB equation corresponding to equation (3.7) is given by

$$\begin{aligned} \min \Bigg\{ -rv^2(I) + \min_{u_2} \Bigg\{ &(c(t)\beta S(t) - \gamma(u_2))I \frac{\partial v^2}{\partial I} + \frac{1}{2}\sigma^2 S^2(t) I^2 \frac{\partial^2 v^2}{\partial I^2} \\ &- (1-L)\pi + w(L) + [p_1 + (1-L)\pi + A_2\delta_2(u_2)]I \Bigg\}, K_2 + v^1(I) - v^2(I) \Bigg\} = 0. \end{aligned} \tag{3.9}$$

Since $v^*(I)$ is increasing in I from Proposition 1(ii) and $v^*(I) = v^1(I)$, we have $\frac{\partial v^1}{\partial I} = \frac{\partial v^*}{\partial I} > 0$ and then $\frac{\partial v^2}{\partial I} > 0$ from equation (3.7). From minimization with respect to u_2, we have, for $I > 0$

$$u_2^* = \arg\min_{u_2} \left\{ -\gamma(u_2)\frac{\partial v^2}{\partial I} + A_2\delta_2(u_2) \right\} \equiv u_2^*\left(\frac{\partial v^2}{\partial I}\right). \tag{3.10}$$

Since $\gamma(u_2)$ is increasing and $\delta_2(u_2)$ is decreasing in u_2, respectively, there exists a value u_2^* satisfying $(\frac{\partial v^2}{\partial I})\frac{d\gamma(u_2)}{du_2} = A_2\frac{d\delta_2(u_2)}{du_2}$, provide that $\gamma(0)\frac{\partial v^2}{\partial I} < A_2\delta_2(0)$ and $\gamma(\bar{u})\frac{\partial v^2}{\partial I} > A_2\delta_2(\bar{u})$ for $u \in [0, \bar{u}]$.

The HJB equation corresponding to equation (3.6) is given by

$$\begin{aligned} \min \Bigg\{ -rv^1(I) + \min_{u_1} \Bigg\{ &(c(t)\beta S(t) - \gamma(u_1))I \frac{\partial v^1}{\partial I} + \frac{1}{2}\sigma^2 S^2(t) I^2 \frac{\partial^2 v^1}{\partial^2 I^2} \\ &- \pi + [p_1 + \pi + A_1\delta_2(u_1)]I \Bigg\}, K_1 + v^2(I) - v^1(I) \Bigg\} = 0. \end{aligned} \tag{3.11}$$

Similarly, we have an optimal level of u_1 is given by

$$u_1^* = \arg\min_{u_1} \left\{ -\gamma(u_1)\frac{\partial v^1}{\partial I} + A_1\delta_2(u_1) \right\}. \tag{3.12}$$

[‖]The verification theorem holds in the model (see Duckworth and Zervos [5])

Since $A_2 > A_1 > 0$ and $\delta_2(u)$ is decreasing, $A_2\delta_2(u) > A_1\delta_2(u)$ and $\gamma(u_i)\frac{\partial v^1}{\partial I}$ is increasing in u_i for $\frac{\partial v^i}{\partial I} > 0$ for each $i = 1, 2$. Then, we obtain the relation $u_1^* > u_2^*$, that is, the decision maker should choose the higher level of the medical resources before the lockdown. Equations (3.10) and (3.12) tell us that we should invest the medical resources to the level at which the marginal cost with the coefficient $\gamma(u)$ is equal to the fatality rate with A_i, $i = 1, 2$. However, the optimal level of the medical cares depends on the marginal value of $v^i(I)$, $i = 1, 2$. Let $\mathcal{L}^u v$ be the differencial operator of v for each u. To carry out to derive optimal policies given by equations (3.6), (3.7), we need to compute $\frac{\partial v^i}{\partial I}$, $i = 1, 2$, solving by the following the HJB equations from equation (3.9):

$$\begin{aligned}\inf_{u\geq 0}\{\mathcal{L}^u v^2(I) + f_2(I,u)\} &= \mathcal{L}^0 v^2(I) + \inf_{u>0}\left\{-\gamma(u)I\frac{\partial v^2}{\partial I} + f_2(I,u)\right\}\\ &= \mathcal{L}^0 v^2(I) - \gamma(u_2^*)I\frac{\partial v^2}{\partial I} + f_2(I,u_2^*) = 0,\ \ u_2^* > 0,\end{aligned} \tag{3.13}$$

and

$$\inf_{u\geq 0}\{\mathcal{L}^u v^1(I) + f_1(I,u)\} = \mathcal{L}^0 v^1(I) - \gamma(u_1^*)I\frac{\partial v^1}{\partial I} + f_1(I,u_1^*) = 0,\ \ u_1^* > 0, \tag{3.14}$$

with the boundary conditions $v^1(0) = -\pi/r$ and $v^2(0) = 0$ since there is no option for the lockdown and no chance to remove it as I goes to 0. Note that the operators $\mathcal{L}^u$ and $\mathcal{L}^0$ are defined by

$$\begin{aligned}(\mathcal{L}^u v)(I) &= (c(t)\beta S(t) - \gamma(u))I\frac{\partial v}{\partial I} + \frac{1}{2}\sigma^2 S^2(t)I^2\frac{\partial^2 v}{\partial I^2},\ \ u > 0,\\ (\mathcal{L}^0 v)(I) &= (c(t)\beta S(t) - \gamma(0))I\frac{\partial v}{\partial I} + \frac{1}{2}\sigma^2 S^2(t)I^2\frac{\partial^2 v}{\partial I^2}.\end{aligned}$$

Put $\phi(u) = -\gamma(u)\frac{\partial v^2}{\partial I} + A_2\delta_2(u)$. After taking the derivative of $\phi(u)$ an optimal value u_2^* must satisfy $\frac{\partial v^2}{\partial I} = \frac{A_2\delta_2'(u_2^*)}{\gamma'(u_2^*)}$ from equation (3.10). Then we have $\phi(u_2^*) = A_2\{\delta_2(u_2^*) - \frac{\gamma(u_2^*)}{\gamma'(u_2^*)}\delta_2'(u_2^*)\}$. If $\phi'(u_2^*)$ is constant (independent of u_2^*), HJB equation can be computed.

To find a closed form of the optimal level of the medical resources, we assume that $\gamma_r(u) \equiv \gamma_r^0(1+u)$ and $\delta_2(u) \equiv \delta_2^0(1-u)$ for $\underline{u} \leq u \leq \overline{u}$. Then we have $\gamma(u) = \gamma_r^0 + \delta_2^0 + (\gamma_r^0 - \delta_2^0)u$. If $\gamma_r^0 > \delta_2^0$, we obtain $\gamma'(u) = \gamma_r^0 - \delta_2^0 > 0$ and $\delta_2'(u) = -\delta_2^0 < 0$. The function in the brackets of equation (3.10) is given by

$$\begin{aligned}\psi_2(u_2) &\equiv -\gamma(u_2)\frac{\partial v^2}{\partial I} + A_2\delta_2(u_2)\\ &= -\left\{\gamma_r^0\frac{\partial v^2}{\partial I} - \left(\frac{\partial v^2}{\partial I} - A_2\right)\delta_2^0\right\}u_2 - (\gamma_r^0 + \delta_2^0)\frac{\partial v^2}{\partial I} + A_2\delta_2^0.\end{aligned} \tag{3.15}$$

If $\gamma_r^0 > \delta_2^0$ and $A_2 > 0$, $\psi_2(u_2)$ is decreasing in u_2. Thus, the optimal level of u_2 is

$$u_2^* = \begin{cases}\overline{u}_2, & \text{if } \gamma_r^0\frac{\partial v^2}{\partial I} > \left(\frac{\partial v^2}{\partial I} - A_2\right)\delta_2^0,\\ \underline{u}_2, & \text{if } \gamma_r^0\frac{\partial v^2}{\partial I} \leq \left(\frac{\partial v^2}{\partial I} - A_2\right)\delta_2^0.\end{cases} \tag{3.16}$$

Similarly, the function in the brackets of equation (3.12) is given by

$$\begin{aligned}\psi_1(u_1) &\equiv -\gamma(u_1)\frac{\partial v^1}{\partial I} + A_1\delta_2(u_1) \\ &= -\left\{\gamma_r^0\frac{\partial v^1}{\partial I} - \left(\frac{\partial v^1}{\partial I} - A_1\right)\delta_2^0\delta_2^0\right\}u_1 - (\gamma_r^0 + \delta_2^0)\frac{\partial v^1}{\partial I} + A_1\delta_2^0.\end{aligned} \tag{3.17}$$

Thus, the optimal level of u_1 is

$$u_1^* = \begin{cases} \overline{u}_1, & \text{if } \gamma_r^0\dfrac{\partial v^1}{\partial I} > \left(\dfrac{\partial v^1}{\partial I} - A_1\right)\delta_2^0, \\ \underline{u}_1, & \text{if } \gamma_r^0\dfrac{\partial v^1}{\partial I} \leq \left(\dfrac{\partial v^1}{\partial I} - A_1\right)\delta_2^0. \end{cases} \tag{3.18}$$

Sufficient conditions for equations (3.16) and (3.18) are as follows**:

$$u_i^* = \begin{cases} \overline{u}_i, & \text{if } A_i > 0 \text{ and } \gamma_i^0 > \delta_2^0, \\ \underline{u}_i, & \text{if } A_i < 0 \text{ and } \gamma_i^0 < \delta_2^0, \end{cases} \quad \text{for } i = 1, 2. \tag{3.19}$$

Note that in equations (3.16) and (3.18) the optimal levels of u_i turn out to be the lower value $\underline{u}_i$ if the fatality rate δ_2^0 is bigger than the recovery rate and the penalty cost of the fatality is smaller than the costs of the infection and the opportunity cost $(1-L)\pi$ due to the lockdown.

Furthermore, we approximate $S(t)$ and $c(t)$ to be both piecewise constant as follows,

$$S(t) = \begin{cases} s_0, & \text{if } 0 < t < \tau, \\ s_1, & \text{if } \tau < t < \tau+\theta, \\ s_2, & \text{if } \tau+\theta < t < \infty \end{cases} \tag{3.20}$$

and

$$c(t) = \begin{cases} c_0, & \text{if } 0 < t < \tau, \\ c_1, & \text{if } \tau < t < \tau+\theta, \\ c_2, & \text{if } \tau+\theta < t < \infty. \end{cases} \tag{3.21}$$

Solving equation (18), we obtain the value function of the option to end the lockdown

$$\begin{aligned} &v^2(I) \\ &= \begin{cases} v^1(I) + K_2, & \text{if } I \in [0, I_*], \\ BI^{\lambda_2} + \dfrac{p_1 + (1-L)\pi + A_2\delta_2(u_2^*)}{r - (c_1\beta s_1 - \gamma(u_2^*))}I - \dfrac{(1-L)\pi - w(L)}{r}, & \text{if } I \in (I_*, 1]. \end{cases}\end{aligned} \tag{3.22}$$

From equation (20) similarly, we have the value function of the option to start the lockdown

$$v^1(I) = \begin{cases} AI^{\lambda_{1,0}} + \dfrac{p_1 + \pi + A_1\delta_2(u_1^*)}{r - (c_0\beta s_0 - \gamma(u_1^*))}I - \dfrac{\pi}{r}, & \text{if } I \in [0, I^*), \\ v^2(I) + K_1, & \text{if } I \in [I^*, 1). \end{cases} \tag{3.23}$$

**In South Africa where the Ebola epidemic exploded, the fatality rate exceeded 50% ($\delta_2^0 > 0.5$). In under-developed countries, medical costs p_1 are high, and conditions where $p_2 < p_1 + \pi$ are likely to occur for those who must work during epidemics to support their families.

where A and B are the solutions of quadratic equation and

$$\lambda_{1,i} = -\frac{1}{\sigma^2}\left(\tilde{\mu}_{1,i} - \frac{1}{2}\sigma^2\right) + \frac{1}{\sigma^2}\sqrt{\left(\tilde{\mu}_{1,i} - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}, \ i = 0, 2 \tag{3.24}$$

$$\lambda_2 = -\frac{1}{\sigma^2}\left(\tilde{\mu}_2 - \frac{1}{2}\sigma^2\right) - \frac{1}{\sigma^2}\sqrt{\left(\tilde{\mu}_2 - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}, \tag{3.25}$$

in which $\tilde{\mu}_{1,i} = c_i\beta s_i - \gamma(u_1^*)$, $i = 0, 2$, and $\tilde{\mu}_2 = c_1\beta s_1 - \gamma(u_2^*)$. The threshold values I_* and I^* to end and start the lockdown can be computed from the value-matching and smooth-pasting conditions, respectively. The optimal level of the medical cares satisfying the HJB equation can numerically solved.

The unknown variables A, B, I_*, and I^* are determined by the value matching and smooth pasting conditions. At the time of starting the lockdown, the value matching condition is given by

$$v^1(I^*) = v^2(I^*) + K_1 \tag{3.26}$$

and the smooth pasting condition is given by

$$\frac{\partial}{\partial I}v^1(I^*) = \frac{\partial}{\partial I}v^2(I^*). \tag{3.27}$$

At the time of ending the lockdown period, the value matching condition is given by

$$v^2(I_*) = v^1(I_*) + K_2 \tag{3.28}$$

and the smooth pasting condition is given by

$$\frac{\partial}{\partial I}v^2(I_*) = \frac{\partial}{\partial I}v^1(I_*). \tag{3.29}$$

The unknown variable A can be determined by solving equations (3.28) and (3.29), as given by

$$\begin{aligned} A &= \frac{\lambda_2}{\lambda_2 - \lambda_{1,0}}(I^*)^{-\lambda_{1,0}}\left\{\left(1 - \frac{1}{\lambda_2}\right)(g_1 - g_0)I^* + \frac{L\pi}{r} + \frac{w(L)}{r} + K_1\right\} \\ &= \frac{\lambda_2}{\lambda_2 - \lambda_{1,2}}(I_*)^{-\lambda_{1,2}}\left\{\left(1 - \frac{1}{\lambda_2}\right)(g_1 - g_2)I_* + \frac{L\pi}{r} + \frac{w(L)}{r} - K_2\right\}. \end{aligned} \tag{3.30}$$

In addition, from equations (3.26) and (3.27), the unknown variable B is given by

$$\begin{aligned} B &= \frac{\lambda_{1,0}}{\lambda_2 - \lambda_{1,0}}(I^*)^{-\lambda_2}\left\{\left(1 - \frac{1}{\lambda_{1,0}}\right)(g_1 - g_0)I^* + \frac{L\pi}{r} + \frac{w(L)}{r} + K_1\right\} \\ &= \frac{\lambda_{1,2}}{\lambda_2 - \lambda_{1,2}}(I_*)^{-\lambda_2}\left\{\left(1 - \frac{1}{\lambda_{1,2}}\right)(g_1 - g_2)I_* + \frac{L\pi}{r} + \frac{w(L)}{r} - K_2\right\}, \end{aligned} \tag{3.31}$$

where

$$g_i = \frac{p_1 + \pi + A_1\delta_2(u_1^*)}{r - (c_i\beta s_i - \gamma(u_1^*))}, \ i = 0, 2, \tag{3.32}$$

$$g_1 = \frac{p_1 + (1-L)\pi + A_2\delta_2(u_2^*)}{r - (c_1\beta s_1 - \gamma(u_2^*))}. \tag{3.33}$$

The unknown variables I^* and I_* can be determined by equations (3.30) and (3.31). However, we cannot obtain analytical solutions satisfying the four equations above. To produce numerical solutions, we assume that there exist s_i, $i = 0, 1, 2$, satisfying $\int_0^t S(z)dz = s_i t$ for all $0 < t \leq \tau$, $\tau < t \leq \tau + \theta$, and $t > \tau + \theta$, respectively.

Table 1: Summary of the model parameters

Infected rate before LD	$\beta_0 = 0.165$
Infected rate during LD	$\beta_1 = 0.134$
Infected rate after LD	$\beta_2 = 0.15$
Contact rate before LD	$c_0 = 0.4$
Contact rate during LD	$c_1 = 0.1$
Contact rate after LD	$c_2 = 0.2$
Recovered rate before LD	$\gamma_{r,0}^0 = 0.015$
Recovered rate during LD	$\gamma_{r,1}^0 = 0.028$
Recovered rate after LD	$\gamma_{r,2}^0 = 0.015$
Fatality rate	$\gamma_d = 0.01$
UB of revised service level	$u_1^* = 0.01$
LB of revised service level	$u_2^* = 0.012$
Mean value of # of suspicious before LD	$s_0 = 0.97$
Mean value of # of suspicious during LD	$s_1 = 0.95$
Mean value of # of suspicious after LD	$s_2 = 0.97$
Economic activity level	$L = 0.7$
Fatality rates	$\delta_2^0 = 0.01$
Volatility of # of infectious	$\sigma = 0.1$
Fixed cost to start LD	$K_1 = 1.5$
Fixed cost to end LD	$K_2 = 0.2$
Discount rate	$r = 0.03$
Unite benefit of economic activities	$\pi = 0.1$
Social cost of infectious	$p_1 = 0.1$
Social cost of fatality	$p_2 = 0.25$

4 Numerical Examples and Comparative Static

In this section, we conducted several numerical experiments to see the effect of the parameters on the optimal thresholds. Table 1 shows the values of the parameters based by data of Tokyo Metropolitan Office which are revised to take account of the existence of asymptotic patients and un-reported light and mild ill patients. Under the parameters, the drifts of the infectious disease process $I(t)$ for before, during and after the lockdown are given by

$$c(t)\beta S(t) - \gamma(u) \approx \begin{cases} c_0\beta_0 s_0 - \gamma(u_1^*) = 0.039, & \text{for } 0 < t < \tau, \\ c_1\beta_1 s_1 - \gamma(u_2^*) = -0.0255, & \text{for } 0 < t < \tau + \theta, \\ c_2\beta_2 s_2 - \gamma(u_1^*) = 0.004, & \text{for } \tau + \theta < t < \infty. \end{cases} \tag{4.1}$$

In Figure 1, the larger the volatility σ in the contact rate should be, the earlier the lockdown period starts which insists that the lockdown duration becomes longer. Figure 2 presents that the more restrictive of the lockdown level should be, the earlier the lockdown starts, but less sensitive to end the lockdown as well as in Figure 3. When the upper threshold I^* increases with respect to each parameter, it implies a later start to the lockdown. Conversely, a decreasing function I^* means the opposite. On the other hand, when the lower threshould I_* increases, it implies an earlier end to the lockdown, and a decrease in I_* means a later end to the lockdown. As a result, the lockdown period becomes shorter when both I^* and I_* are increasing, while it becomes longer when both are decreasing. In Figure 4, the fixed cost is much more sensitive to start the lockdown but has no effect to end it. These results are consistent with our economic intuition. In figure 5, since the discount rate reflects the probability of discovering the vaccine, the threshold to start the lockdown sharply decreases in r. Hence, the lockdown should start earlier as the mean time of the vaccine discovered is smaller.

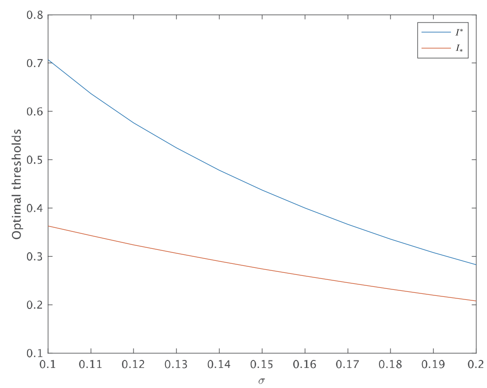

Figure 1: Optimal thresholds with respect to volatility

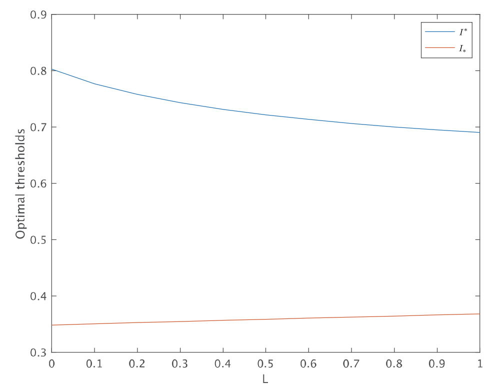

Figure 2: Optimal thresholds with respect to the lockdown restriction level

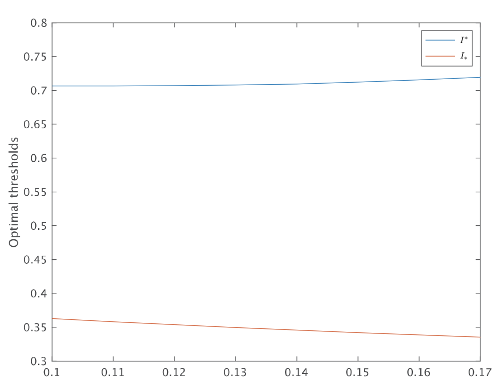

Figure 3: Optimal thresholds with respect to unit benefit of economic activities

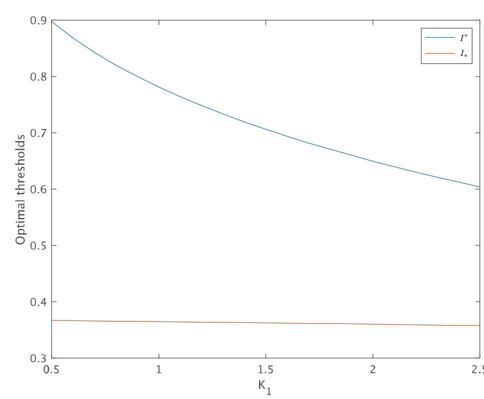

Figure 4: Optimal thresholds with respect to the fixed lockdown cost

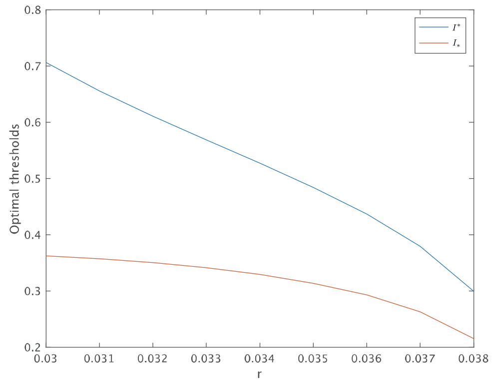

Figure 5: Optimal thresholds with respect to discount rates reflecting the probability of the vaccine availability

5 Concluding Summary

In this paper, we employ a combination of the compound real option approach and stochastic control theory to analyze the evaluation of lockdown decision-making during the COVID-19 pandemic. The model insists that it is optimal to invest the medical resources to the level that equates the product of the marginal cost and recovery rate to the fatality rate which reflects the social value of a life. The devastating impact of COVID-19 on our society and economy, as well as our way of life and work, cannot be overstated. Consequently, the mathematical epidemic model and its applications have garnered significant attention from researchers across various fields, including not only engineering but also the social sciences. The analysis presented in this model provides recommendations for policymakers seeking to make evidence-based decisions to enhance risk management. When a decision-maker delays implementing a lockdown to prioritize economic activities, the society may experience short-term economic benefits; however, it also faces a serious risk to human life due to the spread of infectious diseases. This underscores the need to advise decision-makers on when to start and end lockdowns, as well as how much to invest in improving recovery and fatality rates, in addition to facilitating the prompt recovery of social and economic activities following an epidemic disaster. The model offers valuable guidance to decision-makers regarding the duration of lockdowns and the optimal allocation of financial assistance to healthcare and essential workers. Reports have indicated that significant financial support is required to increase hospital capacity for infectious patients. It's important to note that this model, unfortunately, does not address the important aspect of COVID-19 related to population heterogeneity, mainly due to our limited knowledge of epidemiological findings. Nevertheless, the model proves useful for addressing regional outbreaks and aids decision-makers in understanding why lockdown measures are crucial for safeguarding society from the spread of the virus.

The model endeavors to address the challenging issue of predicting and preventing the COVID-19 pandemic. Overall, the paper supports the importance of government intervention and the allocation of medical resources to mitigate the severity of this pandemic. Additionally, the model elucidates that optimal lockdown decisions and allocation of medical resources depend not only on the parameters of the infection process but also on the marginal cost associated with the infectious level at the decision epoch. What we have learned from this pandemic is that it represented a unique opportunity for government authorities at both central and local levels, as well as researchers from diverse fields, to collaborate in order to guide society toward a more sustainable life and work balance. Finally, we would like to conclude by emphasizing the importance of the fatality rate proportionally to the number of infected patients.

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