Performance of benchmark execution algorithms*

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Abstract

For institutional investors who execute a large amount of securities, the difference in execution costs due to automated trading is a matter of great concern.In this study, we compare the performance of institutional investors executing securities based on benchmark targeting strategy. In particular, we compare optimal execution strategies with VWAP targeting strategy, and examine how strategies are affected by the market environment.

1 Introduction

Generally, large traders such as institutional investors who wish to execute a large number of securities often divide them into small pieces and execute them using appropriate computer algorithms. While various algorithms have been considered, execution algorithms targeting benchmarks are often taken in practice. One of the most popular benchmarks is the VWAP(Volume Weighted Average Price). This is because execution targeting VWAP benchmark allows for a straightforward assessment of the large trader's execution performance. However VWAP targeting execution is noto always optimal. The large trader needs to balance between targeting this easy-to-understand benchmark execution and optimal execution. The study of the optimal execution problem was started mainly by [1] and [2], and later, due to practical requirements, [8] studied the derivation of the optimal VWAP strategy, which has remained a central topic. As for the optimal execution strategy targeting VWAP, the POV (Percentage of Volume) strategy based on the deterministic intraday volume line has been the mainstream. On the other hand, as seen for example in [6], there has been a lot of research on the derivation of optimal execution strategies based on stochastic intraday volume lines. Specifically, [6] derives a strategy for liquidating large positions using gamma bridges on intraday volume lines.

In this study, by using the permanent price model defined in [9] and [10], we derive the optimal execution strategy in discrete time settings, where a large trader, similar to [11], tracks the VWAP of the market with market orders. Then we compare the optimal execution strategy derived in [9] and [10] with the optimal execution strategy targeting VWAP, and examine the appropriateness of VWAP, which is widely used in practice.

The remainder of this paper is organized as follows. Section 2 shows the permanent price model defined in [10] and targeting VWAP strategy. In Section 3, we derive the optimal execution

*This research was supported in part by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center, Kyoto University in 2023 and the Grant-in-Aid for Scientific Research (No. 19K13749) of the Japan Society for the Promotion of Science..

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strategy targeting VWAP. A comparison using numerical examples is presented in Section 4. Then in Section 5, we concludes the paper.

2 Model setup

In this section, we construct the price model used to derive the optimal execution strategy and explain the VWAP as the target of execution. For the pricing model, we mainly follow [9], [10], and [3].

2.1 Price model

Let p_t denote the price of the stock at time $t(= 1, 2, \cdots, T)$, and $\hat{p}_t$ denote the execution price at time t^+. Precisely because the difference between the execution price and the price at time t represents slippage, t^+ represents a time slightly later than t. This is denoted by $\lambda_t q_t$ as the price impact. where λ_t represents the deterministic instantaneous price impact and q_t represents the amount of order submitted at time t (in this case, the sale strategy). Thus, the execution price is as follows,

$$\hat{p}_t = p_t - \lambda_t q_t, \tag{2.1}$$

For large volume of a sell order, the price will decrease. However, it is also possible to purchase, in which case the execution volume will be negative and the price will rise. In addition, we set α_t as the level of permanent impact, then

$$p_{t+1} = p_t - (1 - \alpha_t)\lambda_t q_t + \epsilon_{t+1}, \tag{2.2}$$

where, $(1 - \alpha_t)\lambda_t q_t$ represents the permanent impact, which is the inclusion of information in the price due to large execution and ϵ_t is represented by

$$\epsilon_t := p_t^0 - p_{t-1}^0, \tag{2.3}$$

and i.i.d. with $\epsilon_t \sim N(0, \sigma_\epsilon^2)$, where p_t^0 represents the intrinsic value of the stock at time t.

2.2 VWAP

The VWAP targeting strategy aims for a volume-weighted average, so that the execution price is matched to the VWAP. This strategy is preferred by investors such as pension and mutual fund that aim for passive investment. The VWAP calculation, in discrete time, is as follows,

$$VWAP = \frac{\sum_{t=1}^{T} \mu_t p_t}{\sum_{t=1}^{T} \mu_t}. \tag{2.4}$$

When considering equal time intervals, the VWAP targeting strategy is based on tracking with a POV (percent of volume) strategy using intraday volume lines. The VWAP strategy by POV uses ρ as the ratio between the large trader's intraday execution volume $\bar{Q}$ and the market's trading volume, $q_t = \rho\mu_t$, where

$$\rho = \frac{\bar{Q}}{\sum_{t=1}^{T} \mu_t}. \tag{2.5}$$

3 Optimal execution strategies

This section presents optimal execution strategies with performance criteria in a discrete-time framework using the price model defined in the previous section. Details of the proof follow in [9] and [3]. As mentioned in the previous section, we consider a large trader holding a large amount of a single security $\bar{Q}$ and selling the entire quantity in small pieces during the day (intraday), considering the trade-off between impact risk and price fluctuation risk. The optimal execution strategy at time t using permanent price model without execution benchmark is given by the following procedure then given proposition 1.

Under the price model (2.1) and (2.2), We consider the expected utility maximization problem from maturity wealth in a predetermined execution volume sell strategy. Let the expected utility,

$$V_t^\pi = E_t^\pi \left[-\exp\{-Rw_{t+1}\} \cdot \mathbf{1}_{\{Q_{T+1}=0\}} + (-\infty) \cdot \mathbf{1}_{\{Q_{T+1}\neq 0\}} \right] \tag{3.1}$$

then optimal value function defined as

$$V_t = \operatorname*{ess\,sup}_t V_t^\pi, t = 1, \cdots, T. \tag{3.2}$$

We therefore obtain the following proposition 1 by solving the following equations backward from maturity,

$$V_t(w_t, p_t, Q_t) = \sup_{q_t \in \mathbb{R}} E_t^\pi \left[V_{t+1}(w_{t+1}, p_{t+1}, Q_{t+1}) \mid w_t, p_t, Q_t, q_t \right]. \tag{3.3}$$

Proposition 1 (Optimal execution strategy using permanent price model)

$$q_t^* = \beta_t Q_t \tag{3.4}$$

where

$$\beta_t = \frac{2A_{t+1} - (1-\alpha)\lambda + R\sigma^2}{2A_{t+1} + 2\alpha\lambda + R\sigma^2} \tag{3.5}$$

and,

$$A_t = A_{t+1} + R\sigma^2 - \frac{(2A_{t+1} - (1-\alpha)\lambda + R\sigma^2)^2}{4(2A_{t+1} + 2\alpha\lambda + R\sigma^2)} \tag{3.6}$$

From this proposition, we see that the optimal execution volume in each period is a function only of the remaining amount Q of execution at that time, and that the strategy of executing more at earlier points in time is optimal, as shown in the numerical example in the next section.

On the other hand, in practice, it is often taken for to target some benchmark and to minimize the deviation from that benchmark as much as possible. Here, as in the derivation of the optimal execution strategy using the permanent pricing model, we also consider the problem of maximizing the utility from maturity wealth. However, the penalty for leaving the volume of intraday execution is expressed by C, and the penalty for deviating from the VWAP target is expressed by ψ. Let U_t^π be the expected utility of the large trader at time t, we apply the stochastic differential utility as in [5] and solve by backward induction. Then,

$$U_t^\pi = E_t^\pi \left[-\exp\{-Rw_{T+1}\} + \psi R \sum_{k=t+1}^{T} U_k^\pi (q_t - \rho\mu_k)^2 \mid w_t, p_t, Q_t, \mu_t, q_t \right] \tag{3.7}$$

Since U is negative and ψ and R are defined by positive values, the $\sum$ term makes sense as an expression of the penalty. Therefore,

$$U_t = \sup_{q_t} E\left[U_{t+1} + \psi R U_{t+1}(q_k - \rho\mu_k)^2\right], \tag{3.8}$$

we get the value function as following,

$$U_t(w_t, p_t, Q_t, \mu_t) = \sup_{q_t} E_t^{\pi}\left[U_{t+1}(w_{t+1}, p_{t+1}, Q_{t+1}, \mu_{t+1}) \mid w_t, p_t, Q_t, \mu_t, q_t\right] \tag{3.9}$$

Proposition 2 (Optimal VWAP targeting strategy using permanent price model)

$$q_t^* = \gamma_t^1 Q_t + \gamma_t^2 \mu_t + \gamma_t^3 \tag{3.10}$$

$$\left(\begin{array}{l} \gamma_t^1 = -\frac{1+\zeta e^{2\omega_1(T-t)}}{1-\zeta e^{2\omega_1(T-t)}}\omega_1, \\ \gamma_t^2 = -\frac{\psi\rho - B_t^1}{\lambda+\psi}, \\ \gamma_t^3 = -\frac{B_t^2}{\lambda+\psi}, \end{array}\right. \tag{3.11}$$

and,

$$\left(\begin{array}{l} B_t^1 = \frac{\psi\rho(\zeta e^{\omega_1(T-t)} - e^{-\omega_1(T-t)} + 1 - \zeta)}{\zeta e^{\omega_1(T-t)} - e^{-\omega_1(T-t)}}, \\ B_t^2 = \frac{\psi\rho}{e^{-\omega_1(T-t)} - \zeta e^{-\omega_1(T-t)}} \sum_{k=t}^{T} g(k)\left(\zeta e^{\omega_1(T-k)} - e^{\omega_1(T-k)} + 1 - \zeta\right), \\ \quad\; = -B_t^1 \sum_{k=t+1}^{T} g(k)\left(\zeta e^{\omega_1(T-k)} - e^{\omega_1(T-k)} + 1 - \zeta\right), \end{array}\right. \tag{3.12}$$

where,

$$\zeta = \frac{C - \frac{1-\alpha}{2}\lambda + \omega_2}{C - \frac{1-\alpha}{2}\lambda - \omega_2}, \quad \omega_1 = \sqrt{\frac{(\lambda+\psi)^{-1}\sigma^2 R}{2}}, \quad \omega_2 = \sqrt{\frac{(\lambda+\psi)\sigma^2 R}{2}}. \tag{3.13}$$

The proof is a straightforward calculation, see [3] and [9]. In particular, since $\zeta \to 1$ when $C \to \infty$, the optimal execution strategy is q, where the constant term disappears and it becomes a function of the residual execution amount only. The first term is related to optimal execution, the second term is mainly to match the market VWAP, and finally the third term can be given an interpretation as mainly related to price impact.

4 Numerical examples

In this section, we numerically compare the optimal execution strategy with the optimal target VWAP strategy when using the permanent price model. We assume that the single large trader will sell 100,000 shares in 10 equal intraday. We set the instantaneous impact $\lambda = 0.01$ and the level of the permanent impact $1 - \alpha = 0.1$, i.e., $\alpha = 0.9$. We also assume $\sigma^2 = 0.5$ for market volatility and $R = 0.001$ for the risk aversion coefficient of the risk averse large trader. The historical volume lines are assumed to be shown in Table 1 and Figure 1 below.

From this, we get

$$\rho = \frac{\bar{Q}}{\sum_{t=1}^{10}\mu_t} = \frac{100,000}{3,350,000} = 0.02985. \tag{4.1}$$

Table 1: average daily volume

period	1	2	3	4	5	6	7	8	9	10
volume	500,000	450,000	350,000	300,000	250,000	225,000	225,000	300,000	350,000	400,000

Figure 1: daily volume line

We set $\psi = 10$ which means a large penalty. This shows that the optimal VWAP strategy tracks the market VWAP as much as possible. For simplicity, we will not leave any volume after the maturity, but will consider executing all volume intraday. In other words, from equation (3.13), $\zeta \to 1$ when $C \to \infty$.

Figure 2 shows the optimal execution strategy (dot line) and optimal VWAP tracking strategy (solid line). The optimal execution strategy is in principle a monotonically decreasing function with respect to the trade time when using the permanent price model. (However, it may rise in the last adjustment.) The optimal VWAP targeting strategy follows the historical volume curve to some extent, but has an overall trend to execute more volume initially.

Figure 3 illustrates the expected execution price and the expected value of accumulated wealth when the initial price is set at 1,000. Since the optimal VWAP we assumes a U-shape for the market volume curve, the expected price using optimal VWAP is found to fluctuate more largely during the trading period. The expected cumulative wealth for the optimal execution strategy is 84,628,958, while the expected cumulative wealth for the optimal VWAP targeting strategy is 84,836,985, the VWAP execution needs to consider penalties, but here we consider only simple wealth to match optimal execution. Even when market impact and market volume are independent, the optimal VWAP targeting strategy has higher expected wealth than the optimal execution strategy. From this, it follows that in the setting on which these strategies depend, the expected wealth of the particularly penalized VWAP targeting strategy with a particularly severe penalty is expected to have more wealth. On the other hand, too strong constraints in the VWAP strategy will sacrifice optimality.

Figure 4 illustrates the penalties for errors with the market VWAP for each penalty factor. The larger the error penalty, the more the execution tracks the market volume curve. When the

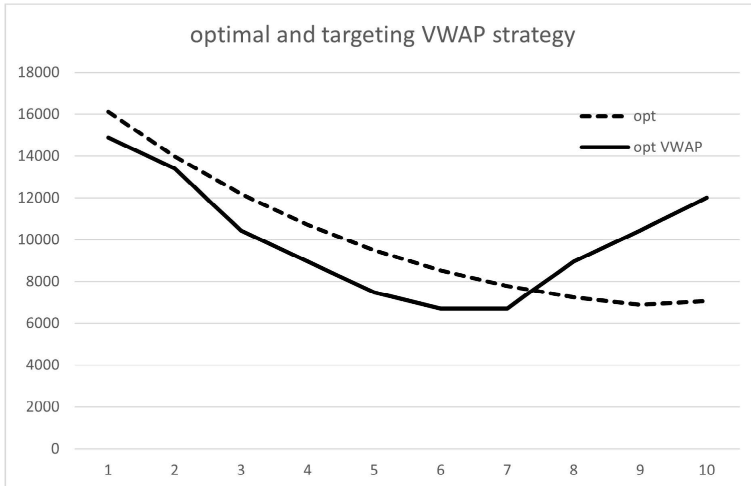

Figure 2: optimal strategy

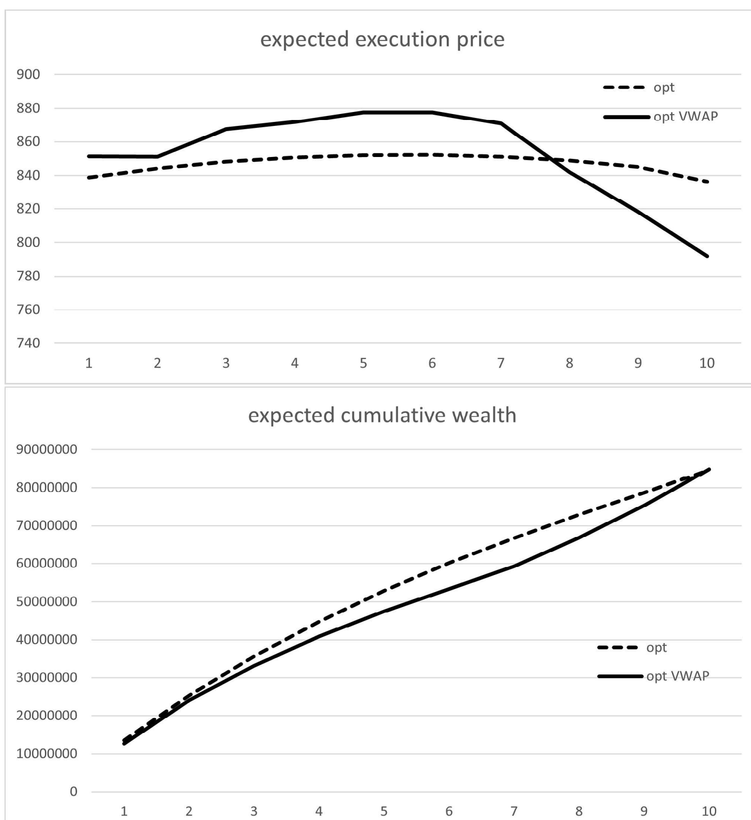

Figure 3: Expected execution price and cumulative wealth

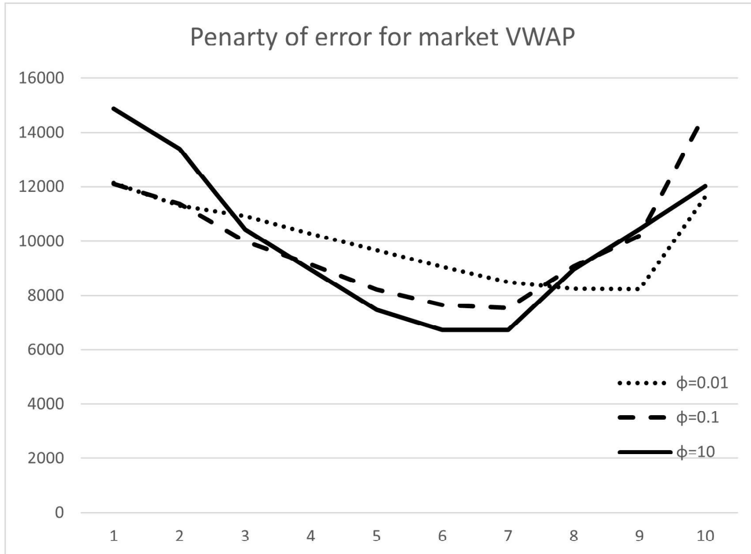

Figure 4: optimal VWAP tracking strategy

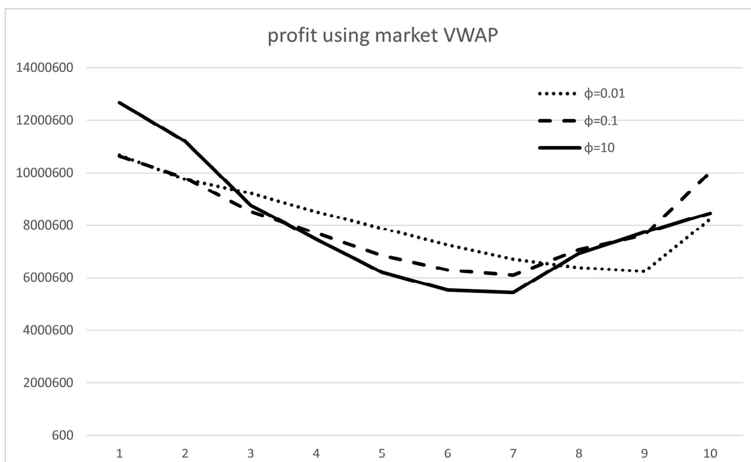

Figure 5: Profit for liquidation using targeting VWAP

penalty is small, it equals to the optimal execution strategy (after adjusting for the last part). The level of the error penalty is a subject for our future research.

Figure 5 shows the profit on the sell due to error penalty with Market VWAP, which is derived from the execution volume × expected execution price for each period. The simple gain on sale and the gain on sale taking into account the penalty at the respective penalty coefficients are shown in Table 2.

The lower the penalty coefficient, the easier it is for the optimal execution strategy to generate a profit even if the deviation from VWAP is large. In addition, the more severe the penalty, the smaller the simple profit and the penalty adjusted profit will be slightly. This is because the large trader tries to get closer to VWAP than to optimality. Finally, we see that when the VWAP tracking and optimization are considered at the same time, conversely, the penalty adjusted profit is lower and the large trader's assessment is the lowest.

Table 2: simple profit and profit taking into account the penalty

penalty coefficient	$\psi = 0.01$	$\psi = 0.01$	$\psi = 0.01$
simple profit	80,826,388	80,596,879	80,371,880
penalty adjusted profit	80,493,438	78,362,159	80,267,954

5 Conclusion

In this study, we compares the execution performance of optimal execution strategies and VWAP targeting strategies. We find that the stronger the VWAP execution penalty, the more the trader sacrifices optimality, resulting in lower profit. On the other hand, the trader's reputation is higher because it closely tracks market VWAP. That is, while overtracking to the VWAP benchmark reduces the profit for large trader's liquidation, the trader's assessment increases. Also, by making the penalty coefficient extremely small, the optimal execution strategy will result in a larger expected sell profit and a larger error-adjusted sell profit, but it will not track VWAP, and the large trader will receive a lower valuation when VWAP is used as the benchmark. In other words, a small penalty mitigation will result in a drop in both execution performance and trading assessment.

When market impact depends on market volume, an optimal execution strategy that takes volume into account is expected to lower market impact costs and thus increase profits from the liquidation. That is, we need to consider the trade-off between maximizing expected profit for the liquidation and evaluating execution based on VWAP. It can be concluded that a perfect track to VWAP would lower the market impact cost, on the other hand, it would be less than optimal.

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