Energy method for partial differential equations with time delay

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1 Introduction

In this article, we study a useful method to analyze partial differential equations with time delay. To treat partial differential equations with time delay, we have to focus on the multi-dimensional problem and the delay effect. These two properties make complicated problems, and few methods are known to handle them simultaneously. In this sense, we need a versatile approach to analyze partial differential equations with time delay. Under this situation, we employ the energy method and try to get a global-in-time solution to the Cauchy problem.

As a good example, we consider the following viscous Burgers equation with time delay:

$$\partial_t \rho - \nu \partial_x^2 \rho + \partial_x (\rho V(\rho_\tau)) = 0, \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $\rho = \rho(t,x)$ is an unknown function, the positive constant ν denotes the diffusion coefficient, and $\rho_\tau(t,x) = \rho(t-\tau, x)$ with a positive delay parameter τ. Also, $V(\rho)$ denotes a given function that depends only on ρ. The Burgers equation (1.1) is one of the simple models of traffic flow, in which ρ means the traffic density.

Here, we shall focus on the Cauchy problem and assign the initial history:

$$\rho(\theta, x) = \rho_0(\theta, x), \qquad -\tau \le \theta \le 0, \quad x \in \mathbb{R}. \tag{1.2}$$

Then, our main goal is to derive a global-in-time solution to the Cauchy problem (1.1), (1.2) and analyze the property of the solutions. To mention our main theorem, we introduce the notation:

$$I_0 := \left(\sup_{-\tau \le \theta \le 0} \|\rho_0(\theta)\|_{H^1}^2 + \int_{-\tau}^0 \|\partial_x \rho_0(\theta)\|_{L^2}^2 d\theta \right)^{1/2}.$$

The main theorem describes the existence of the global-in-time solution when the product of the size of the delay parameter and the one of the initial history is suitably small.

Theorem 1.1 ([1])**.** *Suppose that $\rho_0 \in C([-\tau, 0]; H^1)$ and V is C^1 class function of ρ under consideration. Then there exists a positive number δ such that if*

$$(1 + K_0\sqrt{\tau})^2 \sqrt{\tau}(1 + I_0^2) I_0 \le \delta,$$

then (1.1), (1.2) has a unique global-in-time solution $\rho \in C([-\tau, \infty); H^1)$ satisfying

$$\partial_t \rho \in L^2(0, \infty; L^2), \qquad \partial_x \rho \in L^2(0, \infty; H^1)$$

and the energy estimate:

$$\|\rho(t)\|_{H^1}^2 + \int_0^\infty \left(\|\partial_t \rho(s)\|_{L^2}^2 + \|\partial_x \rho(s)\|_{H^1}^2 \right) ds \le C_0 (1 + I_0^4) I_0^2$$

for $t \ge 0$, where $K_0 := |V(0)|$ and C_0 is a certain positive constant which does not depend on τ.

Furthermore, the solution ρ satisfies the asymptotic behavior: $\|\rho(t)\|_{L^\infty} \to 0$ as $t \to \infty$.

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2 A priori estimate

The key to the proof of Theorem 1.1 is to construct the following *a priori* estimate.

Proposition 2.1 (*A priori* estimate)**.** *Let $T > 0$ and suppose that $\rho \in C([-\tau, T]; H^1)$ is a solution to* (1.1)*,* (1.2)*, which satisfies $\partial_t \rho \in L^2(0,T;L^2)$ and $\partial_x \rho \in L^2(0,T;H^1)$. Then there exists a positive number δ_0 such that if*

$$(1 + K_0\sqrt{\tau})^2\sqrt{\tau} \sup_{-\tau \le s \le T} \|\rho(s)\|_{L^\infty} \le \delta_0, \tag{2.1}$$

then the solution ρ satisfies the following estimate:

$$\|\rho(t)\|_{H^1}^2 + \frac{1}{\nu}\int_0^t \|\partial_t \rho(s)\|_{L^2}^2 ds + \frac{\nu}{4}\int_0^t \|\partial_x \rho(s)\|_{H^1}^2 ds \le C_0(1 + I_0^4)I_0^2$$

for $t \in [0,T]$, where C_0 is a certain positive constant that does not depend on τ.

The *a priori* estimate is derived by the energy method. To construct the energy estimate, we reformulate our Cauchy problem. More precisely, we introduce a new function:

$$z(t,\theta,x) := \rho(t+\theta, x), \qquad t > 0, \quad \theta \in [-\tau, 0], \quad x \in \mathbb{R}.$$

Then, we find $z(t,0,x) = \rho(t,x)$, $z(t,-\tau,x) = \rho_\tau(t,x)$, $z(0,\theta,x) = \rho_0(\theta,x)$, and

$$\partial_t z - \partial_\theta z = 0, \qquad t > 0, \quad \theta \in [-\tau, 0], \quad x \in \mathbb{R}. \tag{2.2}$$

By this reformulation, (1.1) and (2.2) are regarded as the dynamical boundary problem, and we can apply the energy method to (1.1) and (2.2). Then we also construct the desired energy estimate. Proposition 2.1 consists of the lower-order estimate and higher-order estimate of solutions. The lower-order estimate is given as follows.

Lemma 2.2. *Suppose the same assumption as in Proposition 2.1. Then the solution ρ satisfies the following estimate:*

$$\begin{aligned} &\|\rho(t)\|_{L^2}^2 + \nu \int_{-\tau}^0 e^{\theta/\tau}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta + \frac{\nu}{4}\int_0^t \|\partial_x \rho(s)\|_{L^2}^2 ds + \frac{\nu}{4e}\int_0^t \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds \\ &+ \frac{\nu}{\tau}\int_0^t\int_{-\tau}^0 e^{\theta/\tau}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds + \frac{\nu}{4(e\nu + 2K_0^2\tau)}\omega(t)\int_\tau^t\int_{s-\tau}^s \|\partial_t \rho(\sigma)\|_{L^2}^2 d\sigma ds \le \tilde{C}_0 \tilde{I}_0^2 \end{aligned} \tag{2.3}$$

for $t \in [0,T]$, where $\tilde{C}_0$ is a certain positive constant which does not depend on τ.

In Lemma 2.2, we used notations

$$\omega(t) := \begin{cases} 1 & \text{for} \quad t > \tau, \\ 0 & \text{for} \quad 0 \le t \le \tau, \end{cases}$$

and

$$\tilde{I}_0 := \left(\sup_{-\tau \le \theta \le 0} \|\rho_0(\theta)\|_{L^2}^2 + \int_{-\tau}^0 \|\partial_x \rho_0(\theta)\|_{L^2}^2 d\theta \right)^{1/2}.$$

At the end of this section, we give a proof of Lemma 2.1.

Proof of Lemma 2.1. To obtain (2.3), we derive *a priori* estimates for $t \in [0, \tau]$ and $t \in [\tau, T]$, respectively. Since (2.2), we have $\partial_x \partial_t z - \partial_x \partial_\theta z = 0$. Then, multiplying this equation by $e^{\varepsilon\theta}\partial_x z$ and integrating the resultant equation with respect to θ over $[-\tau, 0]$, we obtain

$$\partial_t \left(\int_{-\tau}^{0} e^{\varepsilon\theta} (\partial_x z)^2 d\theta \right) - (\partial_x \rho)^2 + e^{-\varepsilon\tau} (\partial_x \rho_\tau)^2 + \varepsilon \int_{-\tau}^{0} e^{\varepsilon\theta} (\partial_x z)^2 d\theta = 0, \tag{2.4}$$

where ε is a positive parameter determined later. On the other hand, multiplying (1.1) by ρ and taking into account the relation

$$V(\rho)\rho\partial_x\rho = \partial_x \left(\int_0^\rho V(\eta)\eta d\eta \right),$$

we get

$$\frac{1}{2}\partial_t(\rho^2) + \partial_x \left(\rho^2 V(\rho_\tau) - \nu\rho\partial_x\rho - \int_0^\rho V(\eta)\eta d\eta \right) + \nu(\partial_x\rho)^2 + (V(\rho) - V(\rho_\tau))\rho\partial_x\rho = 0. \tag{2.5}$$

Furthermore, multiplying (1.1) by $\partial_t\rho$, we have

$$\frac{\nu}{2}\partial_t(\partial_x\rho)^2 - \nu\partial_x(\partial_t\rho\partial_x\rho) + (\partial_t\rho)^2 + \partial_t\rho\partial_x(V(\rho_\tau)\rho) = 0. \tag{2.6}$$

Then, integrating (2.6) with respect to t over $[t-\tau, t]$, this yields

$$\nu(\partial_x\rho)^2 - \nu(\partial_x\rho_\tau)^2 - 2\nu\partial_x \left(\int_{t-\tau}^{t} \partial_t\rho\partial_x\rho ds \right) + 2\int_{t-\tau}^{t} (\partial_t\rho)^2 ds + 2\int_{t-\tau}^{t} \partial_t\rho\partial_x(V(\rho_\tau)\rho) ds = 0 \tag{2.7}$$

for $t \geq \tau$. Thus, calculating $(2.5) + (2.4) \times \alpha + (2.7) \times \beta$, we obtain

$$\partial_t E + \partial_x F + D + R = 0, \tag{2.8}$$

where α and β are positive parameters determined later, and

$$\begin{aligned}
E &:= \frac{1}{2}\rho^2 + \alpha \int_{-\tau}^{0} e^{\varepsilon\theta}(\partial_x z)^2 d\theta, \\
F &:= \rho^2 V(\rho_\tau) - \nu\rho\partial_x\rho - \int_0^\rho V(\eta)\eta d\eta - 2\beta\nu \int_{t-\tau}^{t} \partial_t\rho(s)\partial_x\rho(s) ds, \\
D &:= ((1+\beta)\nu - \alpha)(\partial_x\rho)^2 + (\alpha e^{-\varepsilon\tau} - \beta\nu)(\partial_x\rho_\tau)^2 + 2\beta \int_{t-\tau}^{t} (\partial_t\rho)^2 ds + \alpha\varepsilon \int_{-\tau}^{0} e^{\varepsilon\theta}(\partial_x z)^2 d\theta, \\
R &:= (V(\rho) - V(\rho_\tau))\rho\partial_x\rho + 2\beta \int_{t-\tau}^{t} \partial_t\rho\partial_x(V(\rho_\tau)\rho) ds.
\end{aligned}$$

Integrating (2.8) with respect to x over $\mathbb{R}$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\rho(t)\|_{L^2}^2 + \alpha \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \right) \\
&+ ((1+\beta)\nu - \alpha)\|\partial_x\rho(t)\|_{L^2}^2 + (\alpha e^{-\varepsilon\tau} - \beta\nu)\|\partial_x\rho_\tau(t)\|_{L^2}^2 \\
&+ 2\beta \int_{t-\tau}^{t} \|\partial_t\rho(s)\|_{L^2}^2 ds + \alpha\varepsilon \int_{-\tau}^{0} e^{\varepsilon\theta}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta + \int_{\mathbb{R}} R dx = 0.
\end{aligned} \tag{2.9}$$

The next step is to estimate the remainder term $\int_{\mathbb{R}} R dx$. Employing

$$
\begin{aligned}
V(\rho(t)) - V(\rho_\tau(t)) &= \int_{t-\tau}^{t} \partial_s V(\rho(s)) ds \\
&= \int_{t-\tau}^{t} V'(\rho(s))\partial_t \rho(s) ds \leq \sqrt{\tau} \left(\int_{t-\tau}^{t} V'(\rho(s))^2 (\partial_t \rho(s))^2 ds \right)^{1/2}
\end{aligned}
$$

derived by the Schwarz inequality, we have

$$
\begin{aligned}
\int_{\mathbb{R}} (V(\rho) - V(\rho_\tau))\rho \partial_x \rho dx &\leq \|\rho\|_{L^\infty} \|\partial_x \rho\|_{L^2} \left(\int_{\mathbb{R}} |V(\rho) - V(\rho_\tau)|^2 dx \right)^{1/2} \\
&\leq K_1 \sqrt{\tau} \|\rho\|_{L^\infty} \|\partial_x \rho\|_{L^2} \left(\int_{t-\tau}^{t} \|\partial_t \rho(s)\|_{L^2}^2 ds \right)^{1/2} \\
&\leq \frac{1}{2}\sqrt{\tau} \|\rho\|_{L^\infty} \|\partial_x \rho\|_{L^2}^2 + \frac{K_1^2}{2} \sqrt{\tau} \|\rho\|_{L^\infty} \int_{t-\tau}^{t} \|\partial_t \rho(s)\|_{L^2}^2 ds,
\end{aligned}
\tag{2.10}
$$

where $K_1 := \sup_\rho |V'(\rho)|$. On the other hand, using

$$
\begin{aligned}
V(\rho_\tau)\partial_t \rho \partial_x \rho &\leq \frac{1}{4}(\partial_t \rho)^2 + V(\rho_\tau)^2 (\partial_x \rho)^2, \\
V'(\rho_\tau)\rho \partial_t \rho \partial_x \rho_\tau &\leq \frac{1}{4}(\partial_t \rho)^2 + V'(\rho_\tau)^2 \rho^2 (\partial_x \rho_\tau)^2,
\end{aligned}
$$

we estimate

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{t-\tau}^{t} \partial_t \rho \partial_x (V(\rho_\tau)\rho) ds dx \\
&\leq \frac{1}{2} \int_{t-\tau}^{t} \|\partial_t \rho(s)\|_{L^2}^2 ds + \int_{t-\tau}^{t} (K_0 + K_1 \|\rho_\tau(s)\|_{L^\infty})^2 \|\partial_x \rho(s)\|_{L^2}^2 ds \\
&\quad + K_1^2 \int_{t-\tau}^{t} \|\rho(s)\|_{L^\infty}^2 \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds.
\end{aligned}
\tag{2.11}
$$

Thus, substituting (2.10) and (2.11) into (2.9), we have

$$
\begin{aligned}
&\partial_t \left(\frac{1}{2} \|\rho(t)\|_{L^2}^2 + \alpha \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \right) + ((1+\beta)\nu - \alpha) \|\partial_x \rho(t)\|_{L^2}^2 \\
&+ (\alpha e^{-\varepsilon\tau} - \beta\nu) \|\partial_x \rho_\tau(t)\|_{L^2}^2 + \beta \int_{t-\tau}^{t} \|\partial_t \rho(s)\|_{L^2}^2 ds + \alpha\varepsilon \int_{-\tau}^{0} e^{\varepsilon\theta} \|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \\
&\leq \frac{1}{2} \sqrt{\tau} \|\rho(t)\|_{L^\infty} \|\partial_x \rho(t)\|_{L^2}^2 + \frac{K_1^2}{2} \sqrt{\tau} \|\rho(t)\|_{L^\infty} \int_{t-\tau}^{t} \|\partial_t \rho(s)\|_{L^2}^2 ds \\
&+ 2\beta \int_{t-\tau}^{t} (K_0 + K_1 \|\rho_\tau(s)\|_{L^\infty})^2 \|\partial_x \rho(s)\|_{L^2}^2 ds + 2\beta K_1^2 \int_{t-\tau}^{t} \|\rho(s)\|_{L^\infty}^2 \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds.
\end{aligned}
\tag{2.12}
$$

Integrating (2.12) with respect to t over $[\tau, t]$, and using the fact that $\|\rho(t)\|_{L^\infty} \leq N(t)$ and

$\|\rho_\tau(t)\|_{L^\infty} \le N(t)$ for any $t \ge 0$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\rho(t)\|_{L^2}^2 + \alpha \int_{-\tau}^0 e^{\varepsilon\theta}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta + ((1+\beta)\nu - \alpha)\int_\tau^t \|\partial_x \rho(s)\|_{L^2}^2 ds \\
&+ (\alpha e^{-\varepsilon\tau} - \beta\nu)\int_\tau^t \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds + \beta \int_\tau^t \int_{s-\tau}^s \|\partial_t \rho(\sigma)\|_{L^2}^2 d\sigma ds \\
&+ \alpha\varepsilon \int_\tau^t \int_{-\tau}^0 e^{\varepsilon\theta}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&\le \frac{1}{2}\|\rho(\tau)\|_{L^2}^2 + \alpha \int_{-\tau}^0 e^{\varepsilon\theta}\|\partial_x z(\tau,\theta)\|_{L^2}^2 d\theta + 2\beta K_0^2 \int_\tau^t \int_{s-\tau}^s \|\partial_x \rho(\sigma)\|_{L^2}^2 d\sigma ds + \mathcal{R}
\end{aligned}
\tag{2.13}
$$

for $t \ge \tau$, where

$$
\begin{aligned}
\mathcal{R} \coloneqq\ & \frac{1}{2}\sqrt{\tau}N(t)\int_\tau^t \|\partial_x \rho(s)\|_{L^2}^2 ds + \frac{K_1^2}{2}\sqrt{\tau}N(t)\int_\tau^t \int_{s-\tau}^s \|\partial_t \rho(\sigma)\|_{L^2}^2 d\sigma ds \\
&+ 2\beta K_1 N(t)(2K_0 + K_1 N(t)) \int_\tau^t \int_{s-\tau}^s \|\partial_x \rho(\sigma)\|_{L^2}^2 d\sigma ds \\
&+ 2\beta K_1^2 N(t)^2 \int_\tau^t \int_{s-\tau}^s \|\partial_x \rho_\tau(\sigma)\|_{L^2}^2 d\sigma ds.
\end{aligned}
$$

Here we used the notation that $N(t) \coloneqq \sup_{-\tau \le s \le t}\|\rho(s)\|_{L^\infty}$. Furthermore, (2.13) gives

$$
\begin{aligned}
&\frac{1}{2}\|\rho(t)\|_{L^2}^2 + \alpha \int_{-\tau}^0 e^{\varepsilon\theta}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta + ((1+\beta)\nu - \alpha - 2\beta\tau K_0^2)\int_\tau^t \|\partial_x \rho(s)\|_{L^2}^2 ds \\
&+ (\alpha e^{-\varepsilon\tau} - \beta\nu)\int_\tau^t \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds + \beta \int_\tau^t \int_{s-\tau}^s \|\partial_t \rho(\sigma)\|_{L^2}^2 d\sigma ds \\
&+ \alpha\varepsilon \int_\tau^t \int_{-\tau}^0 e^{\varepsilon\theta}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&\le \frac{1}{2}\|\rho(\tau)\|_{L^2}^2 + \alpha \int_{-\tau}^0 e^{\varepsilon\theta}\|\partial_x z(\tau,\theta)\|_{L^2}^2 d\theta + 2\beta\tau K_0^2 \int_0^\tau \|\partial_x \rho(s)\|_{L^2}^2 ds + \mathcal{R},
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
\mathcal{R} \le\ & \frac{1}{2}\sqrt{\tau}N(t)\int_\tau^t \|\partial_x \rho(s)\|_{L^2}^2 ds + \frac{K_1^2}{2}\sqrt{\tau}N(t)\int_\tau^t \int_{s-\tau}^s \|\partial_t \rho(\sigma)\|_{L^2}^2 d\sigma ds \\
&+ 2\beta K_1 \tau N(t)(2K_0 + K_1 N(t)) \int_0^t \|\partial_x \rho(s)\|_{L^2}^2 ds + 2\beta K_1^2 \tau N(t)^2 \int_0^t \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds.
\end{aligned}
\tag{2.15}
$$

Since the dissipation terms in (2.14), α, β and ε should be chosen as

$$
(1+\beta)\nu - \alpha - 2\beta\tau K_0^2 > 0, \qquad \alpha e^{-\varepsilon\tau} - \beta\nu > 0.
$$

Then we take

$$
\varepsilon = \frac{1}{\tau}, \qquad \alpha = \frac{\nu}{2}, \qquad \beta = \frac{\nu}{4(e\nu + 2K_0^2\tau)},
$$

and these parameters satisfy

$$
(1+\beta)\nu - \alpha - 2\beta\tau K_0^2 > \left(\frac{1}{4} + \beta\right)\nu > 0, \qquad \alpha e^{-\varepsilon\tau} - \beta\nu \ge \frac{\nu}{4e} > 0.
$$

Namely the estimate (2.14) is rewritten as

$$
\begin{aligned}
&\frac{1}{2}\|\rho(t)\|_{L^2}^2 + \frac{\nu}{2}\int_{-\tau}^{0} e^{\theta/\tau}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta + \left(\frac{1}{4}+\beta\right)\nu\int_{\tau}^{t}\|\partial_x\rho(s)\|_{L^2}^2 ds \\
&+ \frac{\nu}{4e}\int_{\tau}^{t}\|\partial_x\rho_\tau(s)\|_{L^2}^2 ds + \beta\int_{\tau}^{t}\int_{s-\tau}^{s}\|\partial_t\rho(\sigma)\|_{L^2}^2 d\sigma ds + \frac{\nu}{2\tau}\int_{\tau}^{t}\int_{-\tau}^{0} e^{\theta/\tau}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&\le \frac{1}{2}\|\rho(\tau)\|_{L^2}^2 + \frac{\nu}{2}\int_{-\tau}^{0} e^{\theta/\tau}\|\partial_x z(\tau,\theta)\|_{L^2}^2 d\theta + \frac{\nu}{4}\int_{0}^{\tau}\|\partial_x\rho(s)\|_{L^2}^2 ds + \mathcal{R}.
\end{aligned}
\tag{2.16}
$$

We estimate the remainder terms. Substituting (2.15) into (2.16), we see that $N(T)$ should satisfy

$$
\begin{aligned}
&\sqrt{\tau}N(T) \le \frac{\nu}{4}, \qquad K_1\tau N(T)\left(2K_0 + K_1N(T)\right) \le \frac{\nu}{4}, \\
&K_1^2\tau N(T)^2 \le \frac{\nu}{4}, \qquad K_1^2\sqrt{\tau}N(T) \le \frac{\nu}{4(e\nu+2K_0^2\tau)}
\end{aligned}
\tag{2.17}
$$

to get the desired estimate. The assumption (2.1) gives $\sqrt{\tau}N(T) \le \delta_0$, $K_1^2\tau N(T)^2 \le K_1^2\delta_0^2$, and

$$
\begin{aligned}
K_1\tau N(T)\left(2K_0 + K_1N(T)\right) &\le K_1(1+K_1\delta_0)\delta_0, \\
K_1^2(e\nu + 2K_0^2\tau)\sqrt{\tau}N(T) &\le (2+e\nu)K_1^2\delta_0.
\end{aligned}
$$

Thus, taking δ_0 such that

$$
\delta_0 \le \frac{\nu}{4}, \qquad K_1(1+K_1\delta_0)\delta_0 \le \frac{\nu}{4}, \qquad K_1^2\delta_0^2 \le \frac{\nu}{4}, \qquad (2+e\nu)K_1^2\delta_0 \le \frac{\nu}{4}, \tag{2.18}
$$

then (2.17) is satisfied and we obtain

$$
\begin{aligned}
&\|\rho(t)\|_{L^2}^2 + \nu\int_{-\tau}^{0} e^{\theta/\tau}\|\partial_x z(t,\theta)\|_{L^2}^2 d\theta + \frac{1}{4}\left(1+\frac{\nu}{e\nu+2K_0^2\tau}\right)\nu\int_{\tau}^{t}\|\partial_x\rho(s)\|_{L^2}^2 ds \\
&+ \frac{\nu}{4e}\int_{\tau}^{t}\|\partial_x\rho_\tau(s)\|_{L^2}^2 ds + \frac{\nu}{4(e\nu+2K_0^2\tau)}\int_{\tau}^{t}\int_{s-\tau}^{s}\|\partial_t\rho(\sigma)\|_{L^2}^2 d\sigma ds \\
&+ \frac{\nu}{\tau}\int_{\tau}^{t}\int_{-\tau}^{0} e^{\theta/\tau}\|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\
&\le \|\rho(\tau)\|_{L^2}^2 + \nu\int_{-\tau}^{0} e^{\theta/\tau}\|\partial_x z(\tau,\theta)\|_{L^2}^2 d\theta + \frac{\nu}{2}\left(1+\frac{1}{2e}\right)\int_{0}^{\tau}\|\partial_x\rho(s)\|_{L^2}^2 ds \\
&+ \frac{\nu}{4e}\int_{0}^{\tau}\|\partial_x\rho_\tau(s)\|_{L^2}^2 ds
\end{aligned}
\tag{2.19}
$$

for $t \ge \tau$.

On the other hand, to derive the energy estimate for $0 \le t \le \tau$, we treat (2.8) with $\alpha = \nu/2$, $\beta = 0$ and $\varepsilon = 1/\tau$, and this gives

$$
\begin{aligned}
&\partial_t\left\{\rho^2 + \nu\int_{-\tau}^{0} e^{\theta/\tau}(\partial_x z)^2 d\theta\right\} + \partial_x\left\{\left(2V(\rho_\tau) - V(0)\right)\rho^2 - 2\nu\rho\partial_x\rho\right\} \\
&+ \nu(\partial_x\rho)^2 + \frac{\nu}{e}(\partial_x\rho_\tau)^2 + \frac{\nu}{\tau}\int_{-\tau}^{0} e^{\theta/\tau}(\partial_x z)^2 d\theta - 2V'(\varphi\rho_\tau)\rho_\tau\rho\partial_x\rho = 0
\end{aligned}
$$

for some $\varphi \in (0,1)$. Then, integrating this equation with respect to (t,x) over $[0,t] \times \mathbb{R}$ and using the fact that

$$\begin{aligned}\int_0^t \|(V'(\varphi\rho_\tau)\rho_\tau\rho)(s)\|_{L^2}^2 ds &\le tN(t)^2 \sup_{0\le s\le t} \|(V'(\varphi\rho_\tau)\rho_\tau)(s)\|_{L^2}^2 \\ &\le K_1^2\tau N(t)^2 \sup_{-\tau\le\theta\le 0} \|\rho_0(\theta)\|_{L^2}^2 \le \frac{\nu}{4} \sup_{-\tau\le\theta\le 0} \|\rho_0(\theta)\|_{L^2}^2,\end{aligned}$$

which given by (2.1) and (2.18), we obtain

$$\begin{aligned}&\|\rho(t)\|_{L^2}^2 + \nu \int_{-\tau}^0 e^{\theta/\tau} \|\partial_x z(t,\theta)\|_{L^2}^2 d\theta \\ &+ \frac{\nu}{2} \int_0^t \|\partial_x \rho(s)\|_{L^2}^2 ds + \frac{\nu}{e} \int_0^t \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds + \frac{\nu}{\tau} \int_0^t \int_{-\tau}^0 e^{\theta/\tau} \|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \\ &\le \|\rho_0(0)\|_{L^2}^2 + \nu \int_{-\tau}^0 e^{\theta/\tau} \|\partial_x \rho_0(\theta)\|_{L^2}^2 d\theta + \frac{2}{\nu} \int_0^t \|(V'(\varphi\rho_\tau)\rho_\tau\rho)(s)\|_{L^2}^2 ds \\ &\le \|\rho_0(0)\|_{L^2}^2 + \nu \int_{-\tau}^0 e^{\theta/\tau} \|\partial_x \rho_0(\theta)\|_{L^2}^2 d\theta + \frac{1}{2} \sup_{-\tau\le\theta\le 0} \|\rho_0(\theta)\|_{L^2}^2 \le \left(\frac{3}{2} + \nu\right) \tilde{I}_0^2\end{aligned} \tag{2.20}$$

for $0 \le t \le \tau$.

Consequently, applying (2.20) with $t = \tau$ to (2.19), we derive

$$\begin{aligned}&\|\rho(t)\|_{L^2}^2 + \nu \int_{-\tau}^0 e^{\theta/\tau} \|\partial_x z(t,\theta)\|_{L^2}^2 d\theta + \frac{1}{4}\left(1 + \frac{\nu}{e\nu + 2K_0^2\tau}\right) \nu \int_\tau^t \|\partial_x \rho(s)\|_{L^2}^2 ds \\ &+ \frac{\nu}{4e} \int_\tau^t \|\partial_x \rho_\tau(s)\|_{L^2}^2 ds + \frac{\nu}{4(e\nu + 2K_0^2\tau)} \int_\tau^t \int_{s-\tau}^s \|\partial_t \rho(\sigma)\|_{L^2}^2 d\sigma ds \\ &+ \frac{\nu}{\tau} \int_\tau^t \int_{-\tau}^0 e^{\theta/\tau} \|\partial_x z(s,\theta)\|_{L^2}^2 d\theta ds \le \tilde{C}_0 \tilde{I}_0^2\end{aligned}$$

for $\tau \le t$, where

$$\tilde{C}_0 := \left(1 + \frac{1}{2e}\right)\left(\frac{3}{2} + \nu\right).$$

Furthermore, combining (2.20) and this estimate, we arrive at the desired estimate (2.3) and complete the proof. □

References

[1] Takayuki Kubo, Yoshihiro Ueda: *Existence theorem for global in time solutions to Burgers equation with a time delay*, Journal of Differential Equations **333** (2022), 184–230.